Selected topics on the weak topology of Banach spaces

JERZY KĄKOL

A. MICKIEWICZ UNIVERSITY, POZNAŃ

Function Spaces 2015

Introduction.

1. Let E be a Banach space and E_w the space E endowed with the weak topology $w := \sigma(E, E')$. E'_{w^*} denotes the weak*-dual with the topology $w^* := \sigma(E', E)$. Let B_w be the closed unit ball in E with the weak topology.
2. Let X be a completely regular Hausdorff space. Let $C_p(X)$ and $C_c(X)$ be the spaces of all real-valued continuous functions on X endowed with the pointwise topology and the compact-open topology, respectively.
3. Clearly
$$E_w \hookrightarrow C_p(E'_{w^*}), \quad E'_{w^*} \hookrightarrow C_p(E_w).$$
4. **Corson** 1961 started a systematic study of certain topological properties of the weak topology of Banach spaces.

1. This line of research provided more general classes such as reflexive Banach spaces, **Weakly Compactly Generated Banach spaces** ((WCG) shortly) and the class of weakly K-analytic and weakly K-countably determined Banach spaces. (**Corson, Amir, Lindenstrauss, Talagrand, Preiss,**....). Still active area of research, for example the modern renorming theory deals also with spaces E_w.
2. Another line of such a research was essentially related with the concept of networks and k-networks for the spaces E_w.
3. E is (WCG) if E admits a weakly compact set with dense linear span in E.
4. $c_0(\Gamma)$ is (WCG) not being separable nor reflexive for uncountable Γ.
5. If E is (WCG), there exists a continuous linear injective map from E into $c_0(\Gamma)$ (**Amir-Lindenstrauss**).

Theorem 1 (Reznichenko)

The following conditions for a Banach space E are equivalent:

1. E_w *is Lindelöf.*
2. E_w *is normal.*
3. E_w *is paracompact.*

Theorem 2 (Talagrand-Preiss)

E_w *is Lidelöf for every (WCG) Bnach space* E. *Moreover,* $dens\ E_w = dens\ E'_{w^*}$.

Generalized metric concepts. Certain topological properties of function spaces have been intensively studied from many years. Particularly, various topological properties generalizing metrizability attracted specialists both from topology and analysis. One should mention, for example, Fréchet-Urysohn property, sequentiality, k-space property, $k_{\mathbb{R}}$-space property.

1. It is well known that: **metric** $\Rightarrow$ **first countable** $\Rightarrow$ **Fréchet-Urysohn** $\Rightarrow$ **sequential** $\Rightarrow$ k-**space** $\Rightarrow$ $k_{\mathbb{R}}$-**space** .
2. None of these implications is reversible.

1. A topological space X is sequential if every sequentially closed subset of X is closed. Trivially every metrizable space is sequential.
2. A topological space X is a k-space if for each space Y any map $f : X \to Y$ is continuous whenever the restrictions $f|K$ for all compact sets $K \subset X$ are continuous.
3. A topological space X is a $k_{\mathbb{R}}$-space if every $f : X \to \mathbb{R}$, whose restriction to every compact $K \subset X$ is continuous, is continuous on X.
4. The space $C_c(X)$ is complete iff X is a $k_{\mathbb{R}}$-space (Nachbin).

1. A topological space X has countable tightness if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$.

The space E_w is metrizable iff E is finite-dimensional BUT B_w is metrizable iff E' is separable (well-known!). Nevertheless, we have the following classical

Theorem 3 (Kaplansky)

If E is a metrizable lcs, $E_w := (E, \sigma(E, E'))$ has countable tight.

Theorem 4 (Schluchtermann-Wheller)

If E is a Banach space, then E_w is a k-space (in particular, sequential) if and only if E is finite-dimensional.

Hence the question when E_w is homeomorphic to a certain fixed model space from the infinite-dimensional topology is very restrictive and motivated specialists to detect above conditions only for some natural classes of subsets of E, e.g., balls of E.

It is well known that B_w is metrizable iff the dual E' is separable.

Theorem 5 (Keller-Klee)

Let E be an infinite-dimensional separable reflexive Banach space. Then B_w is homeomorphic to $[0,1]^{\aleph_0}$.

Theorem 6 (Schluchtermann-Wheller)

The following conditions on a Banach space E are equivalent: (a) B_w is Fréchet–Urysohn; (b) B_w is sequential; (c) B_w is a k-space; (d) E contains no isomorphic copy of ℓ_1.

Problem 7

Characterize those separable Banach spaces E such that E_w are homeomorphic.

Ascoli spaces

1. A Tychonoff (Hausdorff) space X is called an **Ascoli space** if each compact subset K of $C_c(X)$ is evenly continuous.
2. Clearly: k-**space** $\Rightarrow$ $k_{\mathbb{R}}$-**space** $\Rightarrow$ **Ascoli space.**
3. For a topological space X, denote by $\psi : X \times C_c(X) \to \mathbb{R}$, $\psi(x, f) := f(x)$, the evaluation map. Recall that a subset K of $C_c(X)$ is evenly continuous if the restriction of ψ onto $X \times K$ is jointly continuous, i.e. for any $x \in X$, each $f \in K$ and every neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset K$ of f and $O_x \subset X$ of x such that $U_f(O_x) := \{g(y) : g \in U_f,\ y \in O_x\} \subset O_{f(x)}$.
4. A space X is Ascoli iff the canonical valuation map $X \hookrightarrow C_c(C_c(X))$ is an embedding.

Theorem 8 (Gabriyelyan-Kakol-Plebanek)

A Banach space E in the weak topology is Ascoli if and only if E is finite-dimensional.

Problem 9

Does there exist a Banach space E containing a copy of ℓ_1 such that B_w is Ascoli or a $k_{\mathbb{R}}$-space?

Theorem 10 (Gabriyelyan-Kakol-Plebanek)

The following are equivalent for a Banach space E.

(i) B_w *embeds into* $C_c(C_c(B_w))$;

(ii) B_w *is a* k*-space;*

(iii) B_w *is a* $k_{\mathbb{R}}$*-space;*

(iv) *a sequentially continuous real map on* B_w *is continuous;*

(v) E *does not contain a copy of* ℓ_1.

Let E be a Banach space containing a copy of ℓ_1 and let B_w denote the unit ball in E equipped with the weak topology.

1. We know already that B_w is not a $k_{\mathbb{R}}$-space iff there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous. We construct such a function.
2. Recall that a (normalized) sequence (x_n) in a Banach space E is said to be equivalent to the standard basis of ℓ_1, or simply called an θ-ℓ_1-sequence, if for some $\theta > 0$

$$\left\|\sum_{i=1}^{n} c_i x_i\right\| \geq \theta \cdot \sum_{i=1}^{n} |c_i|,$$

for any natural number n and any scalars $c_i \in \mathbb{R}$.

Proposition 11 (Gabriyelyan-Kakol-Plebanek)

Let K be a compact space and let (g_n) be a normalized θ-ℓ_1-sequence in the Banach space $C(K)$. Then there exists a regular probability measure μ on K such that

$$\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2 \text{ whenever } n \neq k.$$

Example 12 (Gabriyelyan-Kakol-Plebanek)

Suppose that E is a Banach space containing an isomorphic copy of ℓ_1. Then there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous.

1. Let K denote the dual unit ball B_{E^*} equipped with the *weak** topology. Let Ix be the function on K given by $Ix(x^*) = x^*(x)$ for $x^* \in K$. Then $I : E \to C(K)$ is an isometric embedding.
2. Since E contains a copy of ℓ_1, there is a normalized sequence (x_n) in E which is a θ-ℓ_1-sequence for some $\theta > 0$. Then the functions $g_n = Ix_n$ form a θ-ℓ_1-sequence in $C(K)$. There is a probability measure μ on K such that $\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2$ whenever $n \neq k$.
3. Define a function Φ on E by $\Phi(x) = \int_K |Ix| \, \mathrm{d}\mu$. If $y_j \to y$ weakly in E then $Iy_j \to Iy$ weakly in $C(K)$, i.e. $(Iy_j)_j$ is a uniformly bounded sequence converging pointwise to Iy. Consequently, $\Phi(y_j) \to \Phi(y)$ by the Lebesgue dominated convergence theorem. Thus Φ is sequentially continuous.

1. We now check that Φ is not weakly continuous at 0 on B_w. Consider a basic weak neighbourhood of $0 \in B_w$ of the form

$$V = \{x \in B_w : |x_j^*(x)| < \varepsilon \text{ for } j = 1, \dots, r\}.$$

2. Then there is an infinite set $N \subset \mathbb{N}$ such that $\left(x_j^*(x_n)\right)_{n \in N}$ is a converging sequence for every $j \leq r$. Hence there are $n \neq k$ such that $|x_j^*(x_n - x_k)| < \varepsilon$ for every $j \leq r$, which means that $(x_n - x_k)/2 \in V$. On the other hand, $\Phi\big((x_n - x_k)/2\big) \geq \theta/4$ which demonstrates that Φ is not continuous at 0.

Theorem 13 (Pol)

For a metric separable space X the space $C_c(X)$ is a k-space iff X is locally compact.

Theorem 14 (Gabriyelyan-Kakol-Plebanek)

For a metrizable space X, $C_c(X)$ is Ascoli if and only if $C_c(X)$ is a $k_{\mathbb{R}}$-space if and only if X is locally compact.

On the weak topology of Banach spaces; networks and renormings

A (WCG) Banach space E is separable iff every compact set in E_w is metrizable. What about nonseparable Banach spaces E for which all compact sets in E_w are metrizable?

1. A space X is a continuous image under a compact-covering map from a metrizable space Y iff every compact set in X is metrizable (**Michael**).
2. A regular space X has a countable k-network iff X is a continuous image under a compact-covering map from a metrizable separable space Y (**Michael**)

1. For many classes of (separable) Banach spaces E, the space E_w is a generalized metric space of some type. Such types of spaces are defined by different types of networks. The concept of network, coming from the pure set-topology, which turned out to be of great importance to study successfully renorming theory in Banach spaces, see the survey paper **Cascales-Orihuela**.
2. A regular space X is an $\aleph_0$-space (**Michael**) if it has a countable k-network, i.e. there is a countable family $\mathcal{D}$ of subsets of X if whenever $K \subset U$ with K compact and U open in X, there exists $D \in \mathcal{D}$ with $K \subset D \subset U$. Any separable metric space is an $\aleph_0$-space (by **Urysohn's** theorem).

1. **O'Meara** generalized the concept of $\aleph_0$-spaces as follows: A topological space X is called an $\aleph$-space if it is regular and has a σ-locally finite k-network. Any metrizable space is an $\aleph$-space (by **Nagata-Smirnov's** theorem) and all compact sets in $\aleph$-spaces are metrizable, see papers of **Gruenhage**.
2. A regular space X is a σ-space if it has a σ-locally finite network.
3. A regular space X is an $\aleph_0$-space (resp. a σ-space) iff X is Lindelöf and is an $\aleph$-space (resp. has a countable network) [S-K-Ku-M].

Theorem 15 (Gabriyelyan-Kakol-Kubis-Marciszewski)

The Banach space $\ell_1(\Gamma)$ is an $\aleph$-space in the weak topology iff the cardinality of Γ does not exceed the continuum.

Hence $\ell_1(\mathbb{R})$ endowed with the weak topology is an $\aleph$-space but it is not an $\aleph_0$-space and $\ell_1(\mathbb{R})$ in the weak topology is not normal. If K is compact, the Banach space $C(K)$ is a weakly $\aleph_0$-space if and only if K is countable (**Corson**). The following general theorem extends this result.

Theorem 16 (Gabriyelyan-Kakol-Kubis-Marciszewski)

A Fréchet lcs $C_c(X)$ is a weakly $\aleph$-space if and only if $C_c(X)$ is a weakly $\aleph_0$-space if and only if X is countable.

Last Theorem combined with some recent results of Pol-Marciszewski provides concrete Banach spaces $C(K)$ which under the weak topology are σ-spaces but is not $\aleph$-spaces.

Corollary 17 (Gabriyelyan-Kakol-Kubis-Marciszewski)

Let K be an uncountable separable compact space. If K is a linearly ordered space, or a dyadic space, then $C(K)$ endowed with the weak topology is a σ-space but not an $\aleph$-space. If additionally K is metrizable, then $C(K)$ endowed with the weak topology is a cosmic space but is not an $\aleph$-space.

Theorem 18 (Gabriyelyan-Kakol-Kubis-Marciszewski)

Let E be a Banach space not containing a copy of ℓ_1. Then E is a weakly $\aleph$-space if and only if E is a weakly $\aleph_0$-space if and only its strong dual E' is separable.

Networks and renormings. Let E be a Banach space endowed with a norm $\|.\|$ and let S be the unit sphere. Then the norm is said to be

1. *locally uniformly rotund* (LUR), if, whenever $x, x_k \in E$ are such that $\|x\| = \|x_k\| = 1$ and $\|x_k + x\| \to 2$, then $\|x - x_k\| \to 0$
2. *Kadec* , if $\sigma(E, E')$ and the norm topology coincide on S.
3. The space $\ell_1([0,1])$ has in the weak topology a σ-locally finite k-network and yet has an equivalent (LUR)-norm.

Questions concerning renormings in Banach spaces have been of particular importance to provide smooth functions and tools for optimization theory. An excellent monograph of renorming theory up to 1993 by R. Deville, G. Godefroy, and V. Zizler.

1. The famous result of Asplund states that if a Banach space E admits a (LUR) norm as well as another norm whose dual norm is (LUR), then E admits a third norm which is simultaneously C^1-smooth and (LUR).
2. Recently Hajek proved that if E is a (WCG) Banach space admitting a C^k-smooth norm where $k \in \mathbb{N} \cup \{\infty\}$, then E admits an equivalent norm which is simultaneously, C^1-smooth, (LUR), and the limit of a sequence of C^k-smooth norms.
3. Raja proved that a dual Banach space E' has an equivalent (W^*LUR) norm if and only if the weak*-topology has a σ-isolated network.
4. The study of the existence of equivalent (LUR) norms plays a central role in the geometric theory of Banach spaces.

1. A Banach space with a (LUR)-norm has a Kadec norm.
2. It seems that nothing is known about the relation between the property of being a weakly $\aleph$-space and the existence of a (LUR) norm on a Banach space. Another motivation for above problems might be related with the following result.

Recall that E has a $\sigma(E, E')$-LUR norm $\|.\|$ if $x_k \to x$ in $\sigma(E, E')$, whenever $(2\|x\|^2 + 2\|x_k\|^2 - \|x + x_k\|^2) \to 0$.

Theorem 19 (Molto-Orihuela-Troyanski-Valdivia)

If E is a Banach space, then E admits an equivalent $\sigma(E, E')$-lower semicontinuous and $\sigma(E, E')$-LUR norm iff $\sigma(E, E')$ admits a network $\mathcal{N} = \bigcup_n N_n$, where N_n is $\sigma(E, E')$-slicely isolated. A Banach space E has a $\sigma(E, E')$-LUR norm iff it has an equivalent (LUR) norm.