Selected topics on the weak topology of Banach spaces

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Introduction. Certain topological properties of function spaces have been intensively studied from many years. Particularly, various topological properties generalizing metrizability attracted specialists both from topology and analysis. One should mention, for example, Fréchet-Urysohn property, sequentiality, k-space property, $k_{\mathbb{R}}$-space property.

1. It is well known that: **metric** $\Rightarrow$ **first countable** $\Rightarrow$ **Fréchet-Urysohn** $\Rightarrow$ **sequential** $\Rightarrow$ k-**space** $\Rightarrow$ $k_{\mathbb{R}}$-**space** .
2. None of these implications is reversible.

1. A topological space X is *sequential* if every sequentially closed subset of X is closed. Trivially every metrizable space is sequential.
2. A topological space X is a k-space if for each space Y any map $f : X \to Y$ is continuous whenever the restrictions $f|K$ for all compact sets $K \subset X$ are continuous.
3. A topological space X is a $k_{\mathbb{R}}$*-space* if every $f : X \to \mathbb{R}$, whose restriction to every compact $K \subset X$ is continuous, is continuous on X.
4. The space $C_c(X)$ is complete iff X is a $k_{\mathbb{R}}$-space (Nachbin).
5. A Banach space E in the weak topology is metrizable iff E is finite-dimensional.

1. A topological space X has *countable tightness at a point* $x \in X$ if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$; X has *countable tightness* if it has countable tightness at each point $x \in X$.
2. Corson 1961 started a systematic study of certain topological properties of the weak topology of Banach spaces.
3. This line of research provided more general classes such as reflexive Banach spaces, Weakly Compactly Generated Banach spaces and the class of weakly K-analytic and weakly K-countably determined Banach spaces.
4. Another line of such a research was essentially related with the concept of networks and k-networks.

Recall the following classical

Theorem 1 (Kaplansky)

If E is a metrizable lcs, $E_w := (E, \sigma(E, E'))$ has countable tight.

The Kaplansky theorem can be strengthened by using other sequential concepts which are of great importance for the study of function spaces.

1. A topological space X has *countable fan tightness at a point* $x \in X$ if for each sets $A_n \subset X$, $n \in \mathbb{N}$, with $x \in \bigcap_n \overline{A_n}$ there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\cup_n F_n}$; X has *countable fan tightness* if X has countable fan tightness at each point $x \in X$ (Arhangel'skii). Clearly, **countable fan tightness**, $\Rightarrow$ **countable tightness**.

1. Pytkeev proved that every sequential space satisfies the following property, now known as the Pytkeev property, which is stronger than having countable tightness: A topological space X has the *Pytkeev property* if for any sets $A \subset X$ and each $x \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \dots$ of A such that each neighborhood of x contains some A_n.
2. A topological space X has the *Reznichenko property* (or is a *weakly Fréchet–Urysohn* space) if $x \in \overline{A} \setminus A$ and $A \subset X$ imply the existence of a countable infinite disjoint family $\mathcal{N}$ of finite subsets of A such that for every neighborhood U of x the family $\{N \in \mathcal{N} : N \cap U = \emptyset\}$ is finite.
3. **sequential** $\Rightarrow$ **Pytkeev** $\Rightarrow$ **Reznichenko** $\Rightarrow$ **countable tight**, and none of these implications is reversible

For Tychonoff X we denote by $C_c(X)$ and $C_p(X)$ the space of all continuous real-valued functions on X endowed with the compact-open topology and the topology of pointwise convergence, respectively. If X is a σ-compact space, then $C_p(X)$ has countable fan tightness by Arkhangell'ski and has the Reznichenko property by Kočinac, Scheepers. If E is a metrizable lcs, then $X := (E', \sigma(E', E))$ is σ-compact. Since E_w embeds into $C_p(X)$, we notice the following generalization of the Kaplansky theorem.

Theorem 2 (G-K-Z)

Let E be a metrizable lcs (in particular, a Banach space). Then E_w has countable fan tightness and the Reznichenko property.

On the other hand, infinite dimensional Banach spaces in the weak topology are never k-spaces.

Theorem 3 (S-W)

If E is a Banach space, then E_w is a k-space (in particular, sequential) if and only if E is finite dimensional.

We prove another result of this type having in mind that the concepts of being a k-space and having the Pytkeev property are independent in general.

Theorem 4 (G-K-Z)

If E is a normed space, then E_w has the Pytkeev property if and only if E is finite dimensional.

1. Hence the question when E_w is homeomorphic to a certain fixed model space from the infinite-dimensional topology is very restrictive and motivated specialists to detect above conditions only for some natural classes of subsets of E, e.g., balls of E.
2. By B_w we denote the closed unit ball B of a Banach space E endowed with the weak topology. It is well known that B_w is (separable) metrizable if and only if the dual space E' is norm separable.

Theorem 5 (S-W)

The following conditions on a Banach space E are equivalent: (a) *B_w is Fréchet–Urysohn;* (b) *B_w is sequential;* (c) *B_w is a k-space;* (d) *E contains no isomorphic copy of ℓ_1.*

Ascoli spaces.

1. A Tychonoff (Hausdorff) space X is called an *Ascoli space* if each compact subset K of $C_c(X)$ is evenly continuous.
2. For a topological space X, denote by $\psi : X \times C_c(X) \to \mathbb{R}$, $\psi(x, f) := f(x)$, the evaluation map. Recall that a subset K of $C_c(X)$ is *evenly continuous* if the restriction of ψ onto $X \times K$ is jointly continuous, i.e. for any $x \in X$, each $f \in K$ and every neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset K$ of f and $O_x \subset X$ of x such that $U_f(O_x) := \{g(y) : g \in U_f,\ y \in O_x\} \subset O_{f(x)}$.
3. A space X is Ascoli iff the canonical valuation map $X \hookrightarrow C_c(C_c(X))$ is an embedding.

Schlüchtermann and Wheeler showed that an infinite-dimensional Banach space is never a k-space in the weak topology. We strengthen this result as follows.

Theorem 6 (G-K-Pl)

A Banach space E in the weak topology is Ascoli if and only if E is finite-dimensional.

Therefore it seems to be natural to verify whether there exists a Banach space E containing a copy of ℓ_1 and such that B_w is Ascoli or a $k_{\mathbb{R}}$-space. We answer such a question in the negative.

Theorem 7 (G-K-Pl)

Let E be a Banach space and B_w its closed unit ball with the weak topology. Then the following assertions are equivalent:

(i) *B_w is an Ascoli space;*

(ii) *B_w is a $k_{\mathbb{R}}$-space;*

(iii) *every sequentially continuous real-valued map on B_w is continuous;*

(iv) *E does not contain a copy of ℓ_1.*

The proof of (i)⇒(iv) uses basic properties of stochastically independent measurable functions and measure-theoretic properties of ℓ_1. We also provide a canonical example of a sequentially continuous but not continuous function on B_w related with (iii). Our construction builds on measure-theoretic properties of ℓ_1-sequences of continuous functions.

Let E be a Banach space containing a copy of ℓ_1 and let B_w denote the unit ball in E equipped with the weak topology.

1. We know already that B_w is not a $k_{\mathbb{R}}$-space iff there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous. We construct such a function.
2. Recall that a (normalized) sequence (x_n) in a Banach space E is said to be equivalent to the standard basis of ℓ_1, or simply called an θ-ℓ_1-sequence, if for some $\theta > 0$

$$\left\| \sum_{i=1}^{n} c_i x_i \right\| \geq \theta \cdot \sum_{i=1}^{n} |c_i|,$$

for any natural number n and any scalars $c_i \in \mathbb{R}$.

Proposition 8 (G-K-Pl)

Let K be a compact space and let (g_n) be a normalized θ-ℓ_1-sequence in the Banach space $C(K)$. Then there exists a regular probability measure μ on K such that

$$\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2 \text{ whenever } n \neq k.$$

Example 9 (G-K-Pl)

Suppose that E is a Banach space containing an isomorphic copy of ℓ_1. Then there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous.

1. Let K denote the dual unit ball B_{E^*} equipped with the *weak** topology. Let Ix be the function on K given by $Ix(x^*) = x^*(x)$ for $x^* \in K$. Then $I : E \to C(K)$ is an isometric embedding.
2. Since E contains a copy of ℓ_1, there is a normalized sequence (x_n) in E which is a θ-ℓ_1-sequence for some $\theta > 0$. Then the functions $g_n = Ix_n$ form a θ-ℓ_1-sequence in $C(K)$. There is a probability measure μ on K such that $\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2$ whenever $n \neq k$.
3. Define a function Φ on E by $\Phi(x) = \int_K |Ix| \, \mathrm{d}\mu$. If $y_j \to y$ weakly in E then $Iy_j \to Iy$ weakly in $C(K)$, i.e. $(Iy_j)_j$ is a uniformly bounded sequence converging pointwise to Iy. Consequently, $\Phi(y_j) \to \Phi(y)$ by the Lebesgue dominated convergence theorem. Thus Φ is sequentially continuous.

1. We now check that Φ is not weakly continuous at 0 on B_w. Consider a basic weak neighbourhood of $0 \in B_w$ of the form

$$V = \{x \in B_w : |x_j^*(x)| < \varepsilon \text{ for } j = 1, \ldots, r\}.$$

2. Then there is an infinite set $N \subset \mathbb{N}$ such that $\big(x_j^*(x_n)\big)_{n \in N}$ is a converging sequence for every $j \leq r$. Hence there are $n \neq k$ such that $|x_j^*(x_n - x_k)| < \varepsilon$ for every $j \leq r$, which means that $(x_n - x_k)/2 \in V$. On the other hand, $\Phi\big((x_n - x_k)/2\big) \geq \theta/4$ which demonstrates that Φ is not continuous at 0.

What is the situation for the spaces $C_c(X, Y)$? For $I = [0, 1]$ Pol proved the following remarkable result

Theorem 10 (Pol)

Let X be a first countable paracompact space. Then the space $C_c(X, I)$ is a k-space if and only if $X = L \cup D$ is the topological sum of a locally compact Lindelöf space L and a discrete space D.

Theorem easily implies the following result noticed by S. Gabriyelyan.

Corollary 11

For a metric space X, the space $C_c(X)$ is a k-space if and only if $C_c(X)$ is a Polish space if and only if X is a Polish locally compact space.

Finally we have

Theorem 12 (G-K-Pl)

For a metrizable space X, $C_c(X)$ is Ascoli if and only if $C_c(X)$ is a $k_{\mathbb{R}}$-space if and only if X is locally compact.

On the weak topology of Banach spaces; networks and renormings

Recall that a Banach space E is called *Weakly Compactly Generated* (WCG) if E admits a weakly compact set whose linear span is dense in E. All separable, as well as, all reflexive Banach spaces are WCG.

1. The space $c_0(\Gamma)$ is WCG not being separable nor reflexive for uncountable Γ.
2. If E is WCG, then there exists a continuous linear injective map from E into $c_0(\Gamma)$ (Amir-Lindenstrauss).

Theorem 13 (T-P, A-L)

If E is a WCG Banach space, then E_w is a Lindelöf space. Moreover, dens E_w = dens E'_{w^*}.

Theorem 14 (Gull'ko, Reznichenko)

The following conditions for a Banach space E are equivalent:

1. *E_w is Lindelöf.*
2. *E_w is normal.*
3. *E_w is paracompact.*

A WCG Banach space E is separable iff every compact set in E_w is metrizable. What about nonseparable Banach spaces E for which all compact sets in E_w are metrizable?

1. A space X is a continuous image under a compact-covering map from a metrizable space Y iff every compact set in X is metrizable (Michael).
2. A regular space X has a countable k-network iff X is a continuous image under a compact-covering map from a metrizable separable space Y (Michael).

1. For many classes of separable Banach spaces E, the space E_w is a generalized metric space of some type. Such types of spaces are defined by different types of networks. The concept of network, coming from the pure set-topology, which turned out to be of great importance to study successfully renorming theory in Banach spaces, see the survey paper Cascales-Orihuela; especially for E_w-slicely networks.
2. "Classes of generalized metric or metrizable spaces are those which possess some of the useful structure of metrizable spaces. They have had many applications in the theory of topological groups, in function space theory, dimension theory, and other areas. Even some applications in theoretical computer science are appearing see, e.g., the article by G.M. Reed in this volume" (G. Gruenhage)."

1. A regular space X is an $\aleph_0$-*space* if it has a countable k-network, i.e. there is a countable family $\mathcal{D}$ of subsets of X if whenever $K \subset U$ with K compact and U open in X, there exists $D \in \mathcal{D}$ with $K \subset D \subset U$.
2. O'Meara generalized the concept of $\aleph_0$-spaces as follows: A topological space X is called an $\aleph$-*space* if it is regular and has a σ-locally finite k-network. Any metrizable space is an $\aleph$-space and all compact sets in $\aleph$-spaces are metrizable, see papers of Gruenhage.
3. A regular space X is a σ-space if it has a σ-locally finite network.
4. A regular space X is an $\aleph_0$-space (resp. a σ-space) iff X is Lindelöf and is an $\aleph$-space (resp. has a countable network) [S-K-Ku-M].

Theorem 15 (G-K-Ku-M)

The Banach space $\ell_1(\Gamma)$ is an $\aleph$-space in the weak topology iff the cardinality of Γ does not exceed the continuum.

So, the nonseparable Banach space $\ell_1(\mathbb{R})$ endowed with the weak topology is an $\aleph$-space but is not an $\aleph_0$-space. Moreover, the space $\ell_1(\mathbb{R})$ in the weak topology is not normal. $C_c(X)$ is metrizable iff X is *hemicompact* (Arens). Moreover, $C_c(X)$ is complete iff X is a k_R-space. Note that $C_p(X)$ is an $\aleph$-space iff X is countable (Sakai). If K is compact, the Banach space $C(K)$ is a weakly $\aleph_0$-space if and only if K is countable (Corson). The following general theorem extends these results.

Theorem 16 (G-K-Ku-M)

A Fréchet lcs $C_c(X)$ is a weakly $\aleph$-space if and only if $C_c(X)$ is a weakly $\aleph_0$-space if and only if X is countable.

Last Theorem combined with some recent results of Pol-Marciszewski provides concrete Banach spaces $C(K)$ which under the weak topology are σ-spaces but is not $\aleph$-spaces.

Corollary 17 (G-K-Ku-M)

Let K be an uncountable separable compact space. If K is a linearly ordered space, or a dyadic space, then $C(K)$ endowed with the weak topology is a σ-space but not an $\aleph$-space. If additionally K is metrizable, then $C(K)$ endowed with the weak topology is a cosmic space but is not an $\aleph$-space.

If X is a countable and *locally compact* space, last theorem guarantees that $C_c(X)$ is even a weakly $\aleph_0$-space.

1. Question: Is $C_c(X)$ a weakly $\aleph$-space for any countable Tychonoff space X?
2. Having in mind that the weak topology of $C_c(X)$ lies between the compact open topology and the pointwise one, the question is especially interesting for the case X is an $\aleph_0$-space. Recall that for such X the spaces $C_c(X)$ and $C_p(X)$ are $\aleph_0$-spaces by Michael, and $C_p(X)$ is even separable and metrizable.

Theorem 18 (G-K-Ku-M)

Let E be a Banach space not containing a copy of ℓ_1. Then E is a weakly $\aleph$-space if and only if E is a weakly $\aleph_0$-space if and only its strong dual E' is separable.

Networks and renormings. Let E be a Banach space endowed with a norm $\|.\|$ and let S be the unit sphere. Then the norm is said to be

1. *locally uniformly rotund* (LUR), if, whenever $x, x_k \in E$ are such that $\|x\| = \|x_k\| = 1$ and $\|x_k + x\| \to 2$, then $\|x - x_k\| \to 0$
2. *Kadec* , if $\sigma(E, E')$ and the norm topology coincide on S.

Questions concerning renormings in Banach spaces have been of particular importance to provide smooth functions and tools for optimization theory. An excellent monograph of renorming theory up to 1993 by R. Deville, G. Godefroy, and V. Zizler.

1. The famous result of Asplund states that if a Banach space E admits a (LUR) norm as well as another norm whose dual norm is (LUR), then E admits a third norm which is simultaneously C^1-smooth and (LUR).
2. Recently Hajek proved that if E is a WCG Banach space admitting a C^k-smooth norm where $k \in \mathbb{N} \cup \{\infty\}$, then E admits an equivalent norm which is simultaneously, C^1-smooth, (LUR), and the limit of a sequence of C^k-smooth norms.
3. Raja proved that a dual Banach space E' has an equivalent $(W^* LUR)$ norm if and only if the weak*-topology has a σ-isolated network.
4. The study of the existence of equivalent (LUR) norms plays a central role in the geometric theory of Banach spaces.

1. One of the deepest results on (LUR) renorming is given by Molto, Orihuela and Troyanski, who characterized the existence of an equivalent (LUR) norm in a Banach space in terms of a variant of a certain topological property introduced by Jayne, Namioka and Rogers.
2. Every Banach space with a (LUR) norm has a Kadec norm.
3. Is every Banach space with a (LUR) norm a weakly σ-space? Does every non-separable Banach space which is a weakly $\aleph$-space have a LUR norm?
4. Let E be a Banach space which is a weakly $\aleph$-space and admitting a (LUR) norm. What can be said additionally about properties of this (LUR) norm?

1. It seems that nothing is known about the relation between the property of being a weakly $\aleph$-space and the existence of a (LUR) norm on a Banach space. Another motivation for above problems might be related with the following result that answered a long standing open question.

Recall that E has a $\sigma(E, E')$-LUR norm $\|.\|$ if $x_k \to x$ in $\sigma(E, E')$, whenever $(2\|x\|^2 + 2\|x_k\|^2 - \|x + x_k\|^2) \to 0$.

Theorem 19 (M-Or-Tr-Val)

If E is a Banach space, then E admits an equivalent $\sigma(E, E')$-lower semicontinuous and $\sigma(E, E')$-LUR norm iff $\sigma(E, E')$ admits a network $\mathcal{N} = \bigcup_n N_n$, where N_n is $\sigma(E, E')$-slicely isolated. A Banach space E has a $\sigma(E, E')$-LUR norm iff and it has an equivalent (LUR) norm.