On the weak topology of Banach spaces

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Dedicated to professor Jose Bonet on the occasion of his 60th birthday

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Certain topological properties of function spaces have been intensively studied from many years. Particularly, various topological properties generalizing metrizability attracted specialists both from topology and analysis. One should mention, for example, Fréchet-Urysohn property, sequentiality, $k$-space property, $k_\mathbb{R}$-space property.

1. metric $\Rightarrow$ first countable $\Rightarrow$ Fréchet-Urysohn $\Rightarrow$ sequential $\Rightarrow$ $k$-space $\Rightarrow$ $k_\mathbb{R}$-space.

2. sequential $\Rightarrow$ countable tightness.

**Theorem 1 (McCoy-Pytkeev)**

For a completely regular Hausdorff space $X$ the space $C_p(X)$ is Fréchet-Urysohn iff $C_p(X)$ is sequential iff $C_p(X)$ is a $k$-space iff $C_p(X)$ is sequential. If $X$ is a compact space, $C_p(X)$ is Fréchet-Urysohn iff $X$ is scattered.
1. $X$ is **sequential** if every sequentially closed subset of $X$ is closed.

2. $X$ is a **$k$-space** if for any $Y$ any map $f : X \rightarrow Y$ is continuous, whenever $f|K$ for any compact $K$ is continuous. $X$ is a $k_R$-space if the same holds for $Y = \mathbb{R}$.

3. $X$ has **countable tightness** if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$.

4. $X$ is **Fréchet-Urysohn** if whenever $x \in \overline{A}$ and $A \subseteq X$, there exists a sequence in $A$ converging to $x$.

$B_w$ the closed unit ball of a Banach space $E$ with the weak topology $w := \sigma(E, E')$, $E_w := (E, \sigma(E, E'))$.

**Problem 2**

*Characterize those Banach spaces $E$ for which $E_w$ ($B_w$, resp.) is a $k_R$-space.*
The space $C_c(X)$ is complete iff $X$ is a $k^R$-space.

Corson 1961 started a systematic study of certain topological properties of the weak topology of Banach spaces.

This line of research provided more general classes such as reflexive Banach spaces, Weakly Compactly Generated Banach spaces and the class of weakly K-analytic and weakly K-countably determined Banach spaces (works of Corson, Amir, Lindenstrauss, Talagrand, Preiss,....). Still active area of research, for example the modern renorming theory deals also with spaces $E_w$.

For a Banach space $E$ the space $E_w$ is Lindelöf iff $E_w$ is paracompact iff $E_w$ is normal (Reznichenko).
The space $E_w$ is metrizable iff $E$ is finite-dimensional BUT $B_w$ is metrizable iff $E'$ is separable (well-known!). Nevertheless, we have the following classical

**Theorem 3 (Kaplansky)**

*If $E$ is a metrizable lcs, $E_w$ has countable tight.*

Extension to class $\mathcal{G}$ done by Cascales-Kakol-Saxon.

**Theorem 4 (Schluchtermann-Wheller)**

*If $E$ is a Banach space, then $E_w$ is a $k$-space iff $E$ is finite-dimensional.*

The question when $E_w$ is homeomorphic to a fixed model space from the infinite-dimensional topology is very restrictive and motivated specialists to detect above conditions only for some natural classes of subsets of $E$, e.g., ball $B_w$. 

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Theorem 5 (Schluchtermann-Wheller)

The following conditions are equivalent for a Banach space $E$:
(a) $B_w$ is Fréchet–Urysohn; (b) $B_w$ is sequential; (c) $B_w$ is a $k$-space; (d) $E$ contains no isomorphic copy of $\ell_1$.

Compare with the $C_p(X)$ case above!

Theorem 6 (Keller-Klee)

Let $E$ be an infinite-dimensional separable reflexive Banach space. Then $B_w$ is homeomorphic to $[0, 1]^{\aleph_0}$.

Problem 7

Characterize those separable Banach spaces $E$ for which $E_w$ are homeomorphic.
Having in mind Problem 1 we consider also the following concept.

1. A Tychonoff (Hausdorff) space $X$ is called an **Ascoli space** if each compact subset $K$ of $C_c(X)$ is evenly continuous.

2. For a topological space $X$, denote by $\psi : X \times C_c(X) \to \mathbb{R}$, $\psi(x, f) := f(x)$, the valuation map. Recall that a subset $K$ of $C_c(X)$ is *evenly continuous* if the restriction of $\psi$ onto $X \times K$ is jointly continuous, i.e. for any $x \in X$, each $f \in K$ and every neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset K$ of $f$ and $O_x \subset X$ of $x$ such that $U_f(O_x) := \{g(y) : g \in U_f, y \in O_x\} \subset O_{f(x)}$.

3. **$k$-space $\Rightarrow k_{\mathbb{R}}$-space $\Rightarrow$ Ascoli space.** (by Noble)

4. A space $X$ is Ascoli iff the canonical valuation map $X \hookrightarrow C_c(C_c(X))$ is an embedding.
Theorem 8 (Gabriyelyan-Kakol-Plebanek)

A Banach space $E$ in the weak topology is Ascoli if and only if $E$ is finite-dimensional.

Theorem 9 (Gabriyelyan-Kakol-Plebanek)

The following are equivalent for a Banach space $E$.

(i) $B_w$ embeds into $C_c(C_c(B_w))$.
(ii) $B_w$ is a $k_R$-space.
(iii) $B_w$ is a $k$-space.
(iv) A sequentially continuous real map on $B_w$ is continuous.
(v) $E$ does not contain a copy of $\ell_1$. 
Let $E$ be a Banach space containing a copy of $\ell_1$ and again let $B_w$ denote the unit ball in $E$ equipped with the weak topology. We know already that $B_w$ is not a $k_\mathbb{R}$-space iff there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous. How to construct such a function for $E$ containing a copy of $\ell_1$?

Recall that a (normalized) sequence $(x_n)$ in a Banach space $E$ is said to be equivalent to the standard basis of $\ell_1$, or simply called an $\theta$-$\ell_1$-sequence, if for some $\theta > 0$

$$\left\| \sum_{i=1}^{n} c_i x_i \right\| \geq \theta \cdot \sum_{i=1}^{n} |c_i|,$$

for any natural number $n$ and any scalars $c_i \in \mathbb{R}$. 
Lemma 10 (Gabriyelyan-Kakol-Plebanek)

Let $K$ be a compact space and let $(g_n)$ be a normalized $\theta$-$\ell_1$-sequence in the Banach space $C(K)$. Then there exists a regular probability measure $\mu$ on $K$ such that

$$\int_K |g_n - g_k| \, d\mu \geq \theta/2 \text{ whenever } n \neq k.$$ 

Example 11 (Gabriyelyan-Kakol-Plebanek)

Suppose that $E$ is a Banach space containing an isomorphic copy of $\ell_1$. Then there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous.
Let $K$ denote the dual unit ball $B_{E^*}$ equipped with the weak* topology. Let $Ix$ be the function on $K$ given by $Ix(x^*) = x^*(x)$ for $x^* \in K$. Then $I : E \to C(K)$ is an isometric embedding.

Since $E$ contains a copy of $\ell_1$, there is a normalized sequence $(x_n)$ in $E$ which is a $\theta$-$\ell_1$-sequence for some $\theta > 0$. Then the functions $g_n = Ix_n$ form a $\theta$-$\ell_1$-sequence in $C(K)$. There is a probability measure $\mu$ on $K$ such that $\int_K |g_n - g_k| \, d\mu \geq \theta/2$ whenever $n \neq k$.

Define a function $\Phi$ on $E$ by $\Phi(x) = \int_K |Ix| \, d\mu$. If $y_j \to y$ weakly in $E$ then $Iy_j \to ly$ weakly in $C(K)$, i.e. $(Iy_j)_j$ is a uniformly bounded sequence converging pointwise to $ly$. Consequently, $\Phi(y_j) \to \Phi(y)$ by the Lebesgue dominated convergence theorem. Thus $\Phi$ is sequentially continuous.
We now check that $\Phi$ is not weakly continuous at 0 on $B_w$. Consider a basic weak neighbourhood of $0 \in B_w$ of the form

$$V = \{ x \in B_w : |x_j^*(x)| < \varepsilon \text{ for } j = 1, \ldots, r \}.$$

Then there is an infinite set $N \subset \mathbb{N}$ such that $(x_j^*(x_n))_{n \in N}$ is a converging sequence for every $j \leq r$. Hence there are $n \neq k$ such that $|x_j^*(x_n - x_k)| < \varepsilon$ for every $j \leq r$, which means that $(x_n - x_k)/2 \in V$. On the other hand, $\Phi((x_n - x_k)/2) \geq \theta/4$ which demonstrates that $\Phi$ is not continuous at 0.
Theorem 12 (Pol)

For a metric separable space $X$ the space $C_c(X)$ is a $k$-space iff $X$ is locally compact.

Theorem 13 (Gabriyelyan-Kakol-Plebanek)

For a metrizable space $X$, $C_c(X)$ is Ascoli if and only if $C_c(X)$ is a $k_R$-space if and only if $X$ is locally compact.

Problem 14

Is the ball $B_w$ a stratifiable space (in sense of Borges) in the space $\ell_1$?

Conjecture: $B_w$ is stratifiable iff $B_w$ is metrizable. Avilés and Marciszewski (very recently, preprint) proved that if $H$ is a nonseparable Hilbert space then $B_w$ is not stratifiable.