On the weak topology of Banach spaces

JERZY KĄKOL

A. MICKIEWICZ UNIVERSITY, POZNAŃ

Dedicated to professor Jose Bonet on the occasion of his 60th birthday

Valencia 2015

Certain topological properties of function spaces have been intensively studied from many years. Particularly, various topological properties generalizing metrizability attracted specialists both from topology and analysis. One should mention, for example, Fréchet-Urysohn property, sequentiality, k-space property, $k_{\mathbb{R}}$-space property.

1. **metric $\Rightarrow$ first countable $\Rightarrow$ Fréchet-Urysohn $\Rightarrow$ sequential $\Rightarrow$ k-space $\Rightarrow$ $k_{\mathbb{R}}$-space .**
2. **sequential $\Rightarrow$ countable tightness.**

Theorem 1 (McCoy-Pytkeev)

For a completely regular Hausdorff space X the space $C_p(X)$ is Fréchet-Urysohn iff $C_p(X)$ is sequential iff $C_p(X)$ is a k-space iff $C_p(X)$ is sequential. If X is a compact space, $C_p(X)$ is Fréchet-Urysohn iff X is scattered.

1. X is **sequential** if every sequentially closed subset of X is closed.
2. X is a k-**space** if for any Y any map $f : X \to Y$ is continuous, whenever $f|K$ for any compact K is continuous. X is a $k_{\mathbb{R}}$-**space** if the same holds for $Y = \mathbb{R}$.
3. X has **countable tightness** if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$.
4. X is **Fréchet-Urysohn** if whenever $x \in \overline{A}$ and $A \subseteq X$, there exists a sequence in A converging to x.

B_w the closed unit ball of a Banach space E with the weak topology $w := \sigma(E, E')$, $E_w := (E, \sigma(E, E'))$.

Problem 2

Characterize those Banach spaces E for which E_w (B_w, resp.) is a $k_{\mathbb{R}}$-space.

1. The space $C_c(X)$ is complete iff X is a $k_{\mathbb{R}}$-space.
2. Corson 1961 started a systematic study of certain topological properties of the weak topology of Banach spaces.
3. This line of research provided more general classes such as reflexive Banach spaces, Weakly Compactly Generated Banach spaces and the class of weakly K-analytic and weakly K-countably determined Banach spaces (works of **Corson, Amir, Lindenstrauss, Talagrand, Preiss,**....). Still active area of research, for example the modern renorming theory deals also with spaces E_w.
4. For a Banach space E the space E_w is Lindelöf iff E_w is paracompact iff E_w is normal (**Reznichenko**).

The space E_w is metrizable iff E is finite-dimensional BUT B_w is metrizable iff E' is separable (well-known!). Nevertheless, we have the following classical

Theorem 3 (Kaplansky)

If E is a metrizable lcs, E_w has countable tight.

Extension to class $\mathfrak{G}$ done by Cascales-Kakol-Saxon.

Theorem 4 (Schluchtermann-Wheller)

If E is a Banach space, then E_w is a k-space iff E is finite-dimensional.

The question when E_w is homeomorphic to a fixed model space from the infinite-dimensional topology is very restrictive and motivated specialists to detect above conditions only for some natural classes of subsets of E, e.g., ball B_w.

Theorem 5 (Schluchtermann-Wheller)

The following conditions are equivalent for a Banach space E: (a) B_w is Fréchet–Urysohn; (b) B_w is sequential; (c) B_w is a k-space; (d) E contains no isomorphic copy of ℓ_1.

Compare with the $C_p(X)$ case above!

Theorem 6 (Keller-Klee)

Let E be an infinite-dimensional separable reflexive Banach space. Then B_w is homeomorphic to $[0,1]^{\aleph_0}$.

Problem 7

Characterize those separable Banach spaces E for which E_w are homeomorphic.

Having in mind Problem 1 we consider also the following concept.

1. A Tychonoff (Hausdorff) space X is called an **Ascoli space** if each compact subset K of $C_c(X)$ is evenly continuous.
2. For a topological space X, denote by $\psi : X \times C_c(X) \to \mathbb{R}$, $\psi(x, f) := f(x)$, the valuation map. Recall that a subset K of $C_c(X)$ is *evenly continuous* if the restriction of ψ onto $X \times K$ is jointly continuous, i.e. for any $x \in X$, each $f \in K$ and every neighborhood $O_{f(x)} \subset Y$ of $f(x)$ there exist neighborhoods $U_f \subset K$ of f and $O_x \subset X$ of x such that $U_f(O_x) := \{g(y) : g \in U_f,\ y \in O_x\} \subset O_{f(x)}$.
3. k**-space** $\Rightarrow$ $k_{\mathbb{R}}$**-space** $\Rightarrow$ **Ascoli space.** (by **Noble**)
4. A space X is Ascoli iff the canonical valuation map $X \hookrightarrow C_c(C_c(X))$ is an embedding.

Theorem 8 (Gabriyelyan-Kakol-Plebanek)

A Banach space E in the weak topology is Ascoli if and only if E is finite-dimensional.

Theorem 9 (Gabriyelyan-Kakol-Plebanek)

The following are equivalent for a Banach space E.

(i) B_w *embeds into* $C_c(C_c(B_w))$.

(ii) B_w *is a* $k_{\mathbb{R}}$*-space.*

(iii) B_w *is a* k*-space.*

(iv) *A sequentially continuous real map on* B_w *is continuous.*

(v) E *does not contain a copy of* ℓ_1.

Let E be a Banach space containing a copy of ℓ_1 and again let B_w denote the unit ball in E equipped with the weak topology. We know already that B_w is not a $k_{\mathbb{R}}$-space iff there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous. How to construct such a function for E containing a copy of ℓ_1?

Recall that a (normalized) sequence (x_n) in a Banach space E is said to be equivalent to the standard basis of ℓ_1, or simply called an θ-ℓ_1-sequence, if for some $\theta > 0$

$$\left\| \sum_{i=1}^{n} c_i x_i \right\| \geq \theta \cdot \sum_{i=1}^{n} |c_i|,$$

for any natural number n and any scalars $c_i \in \mathbb{R}$.

Lemma 10 (Gabriyelyan-Kakol-Plebanek)

Let K be a compact space and let (g_n) be a normalized θ-ℓ_1-sequence in the Banach space $C(K)$. Then there exists a regular probability measure μ on K such that

$$\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2 \text{ whenever } n \neq k.$$

Example 11 (Gabriyelyan-Kakol-Plebanek)

Suppose that E is a Banach space containing an isomorphic copy of ℓ_1. Then there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous.

1. Let K denote the dual unit ball B_{E^*} equipped with the *weak** topology. Let Ix be the function on K given by $Ix(x^*) = x^*(x)$ for $x^* \in K$. Then $I : E \to C(K)$ is an isometric embedding.
2. Since E contains a copy of ℓ_1, there is a normalized sequence (x_n) in E which is a θ-ℓ_1-sequence for some $\theta > 0$. Then the functions $g_n = Ix_n$ form a θ-ℓ_1-sequence in $C(K)$. There is a probability measure μ on K such that $\int_K |g_n - g_k| \, \mathrm{d}\mu \geq \theta/2$ whenever $n \neq k$.
3. Define a function Φ on E by $\Phi(x) = \int_K |Ix| \, \mathrm{d}\mu$. If $y_j \to y$ weakly in E then $Iy_j \to Iy$ weakly in $C(K)$, i.e. $(Iy_j)_j$ is a uniformly bounded sequence converging pointwise to Iy. Consequently, $\Phi(y_j) \to \Phi(y)$ by the Lebesgue dominated convergence theorem. Thus Φ is sequentially continuous.

1. We now check that Φ is not weakly continuous at 0 on B_w. Consider a basic weak neighbourhood of $0 \in B_w$ of the form

$$V = \{x \in B_w : |x_j^*(x)| < \varepsilon \text{ for } j = 1, \ldots, r\}.$$

2. Then there is an infinite set $N \subset \mathbb{N}$ such that $\left(x_j^*(x_n)\right)_{n \in N}$ is a converging sequence for every $j \leq r$. Hence there are $n \neq k$ such that $|x_j^*(x_n - x_k)| < \varepsilon$ for every $j \leq r$, which means that $(x_n - x_k)/2 \in V$. On the other hand, $\Phi\left((x_n - x_k)/2\right) \geq \theta/4$ which demonstrates that Φ is not continuous at 0.

Theorem 12 (Pol)

For a metric separable space X the space $C_c(X)$ is a k-space iff X is locally compact.

Theorem 13 (Gabriyelyan-Kakol-Plebanek)

For a metrizable space X, $C_c(X)$ is Ascoli if and only if $C_c(X)$ is a $k_{\mathbb{R}}$-space if and only if X is locally compact.

Problem 14

Is the ball B_w a stratifiable space (in sense of Borges) in the space ℓ_1?

Conjecture: B_w is stratifiable iff B_w is metrizable. Avilles and Marciszewski (very recently, preprint) proved that if H is a nonseparable Hilbert space then B_w is not stratifiable.