# The Pytkeev and Strong Pytkeev Properties for topological groups and topological spaces

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## Introduction

1. There are many Fréchet-Urysohn lcs which are not metrizable. For example any space $C_p(X)$ over an uncountable compact scattered $X$ is such an example.
2. On the other hand we have very recent

**Theorem 1 (Hrusak-Ramos Garcia (Malykhin problem))**

*There exists a model in ZFC where every separable Fréchet-Uryshn group is metrizable.*

The following problem have been attracted specialists from a long time:

**Problem 2**

*Describe possible good sufficient conditions under which any Fréchet-Urysohn group is metrizable.*

**Pytkeev** proved that every **sequential** space satisfies the property (so-called **Pytkeev property**) which is stronger than countable tightness.

### Definition 3

A topological space $Y$ has the ***Pytkeev property*** if for each $A \subseteq Y$ and each $y \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \dots$ of $A$ such that each neighbourhood of $y$ contains some $A_n$.

### Definition 4 (**Tsaban-Zdomsky**)

A topological space $Y$ has the ***strong Pytkeev property*** if for each $y \in Y$ there exists a countable family $\mathcal{D}$ of subsets of $Y$ such that for each neighbourhood $U$ of $y$ and each $A \subseteq Y$ with $y \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite.

More definitions and relations:

**Definition 5 (Banakh-Zdomsky)**

$Y$ has ***countable*** $cs^*$***-character*** if for each $y \in Y$ there is a countable family $\mathcal{D}$ of subsets of $Y$ such that for each non-trivial sequence in $Y$ converging to $y$ and each neighbourhood $U$ of $y$, there is $D \in \mathcal{D}$ with $D \subset U$ and $D$ contains infinitely many elements of that sequence.

**Theorem 6 (Banakh-Zdomsky)**

*A Fréchet-Urysohn topological group is metrizable iff it has countable $cs^*$-character. A Baire topological group is metrizable iff it is sequential and has countable $cs^*$-character.*

We call $X$ a $P$-***sequential*** space if $X$ is a sequential space satisfying the strong Pytkeev property.

1. Fréchet-Urysohn $\Rightarrow$ sequential $\Rightarrow$ Pytkeev property $\Rightarrow$ countable tightness.
2. First countable $\Rightarrow$ P-sequential $\Rightarrow$ strong Pytkeev property $\Rightarrow$ countable $cs^*$-character.
3. Fréchet-Urysohn $\nRightarrow$ strong Pytkeev property $\nRightarrow$k-space.

**Theorem 7 (Gabriyelyan, Kakol)**

*A Baire tvs is metrizable iff it has countable $cs^*$- character. A b-Baire-like lcs is metrizable iff it has countable $cs^*$-character.*

Second part of theorem extends a theorem of Sakai (2008) stating that the space $C_p(X)$ is metrizable iff $C_p(X)$ has countable $cs^*$-character (note that every $C_p(X)$ is b-Baire-like). Both parts use the concept of a $\mathfrak{G}$-base .

## Topological groups with a $\mathfrak{G}$-base

**Definition 8 (Cascales, Kakol, Saxon for tvs)**

Let $G$ be a topological group. A family $\mathcal{U} := \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighbourhoods of the unit $e$ is called a **$\mathfrak{G}$-base** if $\mathcal{U}$ is a base of neighbourhoods at the unit and $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$.

Every metrizable group $G$ admits a $\mathfrak{G}$-base $\{U_{\alpha_1} : (\alpha_i) \in \mathbb{N}^\mathbb{N}\}$, where $\{U_n\}_{n\in\mathbb{N}}$ - decreasing base of neighbourhoods at $e$ of $G$.

1. A topological group $G$ is metrizable iff $G$ is Fréchet-Urysohn and has a $\mathfrak{G}$-base. **(G-Ka-L)**
2. Any precompact set in a topological group $G \in \mathbf{TG}_\mathfrak{G}$ is metrizable, and hence $G$ is strictly angelic. **(G-Ka-L)**
3. Next theorem uses the concept of a $\mathfrak{G}$-base.

We say that a topological space $X$ has a **compact resolution swallowing compact sets** if $X$ admits a family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets, $K_\alpha \subset K_\beta$, whenever $\alpha \leq \beta$ and each compact set of $X$ is contained in some $K_\alpha$.

**Theorem 9 (Gabriyelyan, Kakol, Leiderman)**

*Let $X$ be space which admits a compact resolution swallowing compact sets. Then the following are equivalent:*

1. *$C_c(X)$ has the strong Pytkeev property.*
2. *$C_c(X)$ has the Pytkeev property.*
3. *$C_c(X)$ has countable tightness.*
4. *$C_p(X)$ has countable tightness.*
5. *$C_c(X)$ is barrelled.*
6. *$X$ is Lindelöf.*

1. This extends Tsaban-Zdomsky's theorem stating that $C_c(X)$ has the strong Pytkeev property for Polish $X$.
2. $C_c(X)$ has a $\mathfrak{G}$-base iff $X$ has a compact resolution swallowing compact sets (**Ferrando, Kakol** ).
3. Every topological group with a $\mathfrak{G}$-base which is a $k$-space is strongly Pytkeev (**G-Ka-L**)
4. If $E$ is a lcs with a $\mathfrak{G}$-base, then $E$ is a $k$-space iff $E$ is metrizable or $E$ is homeomorphic to $\phi$ or $\phi \times Q$, where $Q$ is the Hilbert cube (**G-Ka-L**).

### Corollary 10

*Let $X$ be a Čech-complete space. Then $C_c(X)$ has the strong Pytkeev property if and only if $X$ is Lindelöf.*

A necessary condition for topological groups satisfying the strong Pytkeev property.

## Theorem 11

*Let $G$ be a topological group with the strong Pytkeev property. Then $G$ has a base $\{U_\alpha : \alpha \in \mathbf{M}\}$ of neighbourhoods at $e$, where*

(i) *$\mathbf{M}$ is a subset of the partially ordered set $\mathbb{N}^\mathbb{N}$;*

(ii) *if $\alpha \in \mathbf{M}$ and $\beta \in \mathbb{N}^\mathbb{N}$ are such that $\beta \leq \alpha$, then $\beta \in \mathbf{M}$;*

(iii) *$U_\beta \subseteq U_\alpha$, whenever $\alpha \leq \beta$ for $\alpha, \beta \in \mathbf{M}$.*

## A sufficient condition for topological groups to have the strong Pytkeev property.

1. $\Omega$ - a set, $I$ - a partially ordered set with an order $\leq$. A family $\{A_i\}_{i\in I}$ of subsets of $\Omega$ is *$I$-decreasing* if $A_j \subseteq A_i$ for every $i \leq j$ in $I$. Example: $\mathbb{N}^{\mathbb{N}}$ endowed with the order, i.e., $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$, $i \in \mathbb{N}$, $\alpha = (\alpha_i)_{i\in\mathbb{N}}$, $\beta = (\beta_i)_{i\in\mathbb{N}}$.
2. For $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $k \in \mathbb{N}$, set $I_k(\alpha) := \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta_i = \alpha_i,\ i = 1, \dots, k\}$.
3. Let $\mathbf{M} \subseteq \mathbb{N}^{\mathbb{N}}$ and $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ be an $\mathbf{M}$-decreasing family of subsets of a set $\Omega$. Define the (countable) family $\mathcal{D}_\mathcal{U}$ of subsets of $\Omega$ by $\mathcal{D}_\mathcal{U} := \{D_k(\alpha) : \alpha \in \mathbf{M}, k \in \mathbb{N}\}$, where $D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}} U_\beta$.
4. We say that $\mathcal{U}$ satisfies the ***condition*** $(\mathbf{D})$ if $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ for every $\alpha \in \mathbf{M}$.

**Theorem 12 (Gabriyelyan, Kakol, Leiderman)**

*Let $G$ be a topological group with a $\mathfrak{G}$-base satisfying condition* (**D**)*. Then $G$ has the strong Pytkeev property.*

1. A quasibarrelled lsc with a $\mathfrak{G}$-base satisfies condition (**D**).
2. Every $(DF)$-space with countable tightness has a $\mathfrak{G}$-base with condition (**D**). (**Cascales-Kakol-Saxon**).

**Applications:** Last theorem applies to obtain the following

**Theorem 13 (Gabriyelyan, Kakol, Leiderman)**

(i) *A $(DF)$-space $E$ has countable tightness iff $E$ has the strong Pytkeev property.*

(ii) *Every strict $(LM)$-space has the strong Pytkeev property.*

(iii) *Let $(E', \beta(E', E))$ be the strong dual of a strict $(LF)$-space $E$. Then (a) $(E', \beta(E', E))$ has a $\mathfrak{G}$-base. (b) $(E', \beta(E', E))$ has countable tightness iff $(E', \beta(E', E))$ has the strong Pytkeev property.*

Any space $E$ mentioned above is *metrizable iff $E$ is Fréchet-Urysohn* (since the strong Pytkeev property + Fréchet-Urysohn $\Rightarrow$ metrizable.) Therefore, fort example, $\mathfrak{D}'(\Omega)$ has the strong Pytkeev property, and in particular, it has countable tightness.

## Topological description of cosmic and $\aleph_0$-spaces.

**Definition 14**

A topological space $X$ has a **small base** if there exists an $\mathbf{M}$-decreasing base of $\tau$ for some $\mathbf{M} \subseteq \mathbb{N}^{\mathbb{N}}$.

**Theorem 15 (Gabriyelyan, Kakol, Kubzdela, Lopez-Pellicer)**

*Let $X := (X, \tau)$ be a regular topological space. Then:*

(i) *$X$ is cosmic iff $X$ has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ with $(\mathbf{D})$. The family $\mathcal{D}_{\mathcal{U}}$ is a countable network in $X$.*

(ii) *$X$ is an $\aleph_0$-space iff $X$ has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ with $(\mathbf{D})$ such that $\mathcal{D}_{\mathcal{U}}$ is a countable k-network in $X$.*

*There is a small base $\mathcal{U}$ such that $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$ and $\mathcal{U} = \tau$, i.e. for any $W \in \tau$ there is $\alpha \in \mathbf{M}$ with $W = U_\alpha$.*

Condition (**D**) is essential: The Bohr compactification $b\mathbb{Z}$ of the discrete group $\mathbb{Z}$ has a small base and $b\mathbb{Z}$ is not cosmic as nonmetrizable.

**Theorem 16 (Gabriyelyan, Kakol)**

*A Baire topological group is metrizable iff $G$ has the strong Pytkeev property iff $G$ has a $\mathfrak{G}$-base with condition* (**D**).

**Theorem 17 (Gabriyelyan-Kakol-Zdomsky)**

*(i) A Banach space $E$ is finite-dimensional iff $E_w$ has the Pytkeev property. (ii) $B_w$ has the Pytkeev property iff $E$ does not contain $\ell_1$. (iii) $B_w$ is metrizable iff $B_w$ has the strong Pytkeev property iff $E'$ is separable.*

James tree $JT \not\supset \ell_1$, $JT^*$ is nonseparable, $JT$ has a Kadets norm under which $B_w$ is Baire with the Pytkeev property.

The last results yield

**Corollary 18**

*A Baire separable topological group $G$ is metrizable iff $G$ is cosmic.*

**Question 19**

*Let $G$ be a topological group (or even a TVS) with the strong Pytkeev property. Does $G$ admit a $\mathfrak{G}$-base?*

We do not know an answer to this question even for submetrizable groups.