The Pytkeev and Strong Pytkeev Properties for topological groups and topological spaces

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Introduction

1. There are many Fréchet-Urysohn lcs which are not metrizable. For example any space $C_p(X)$ over an uncountable compact scattered $X$ is such an example.

2. On the other hand we have very recent

Theorem 1 (Hrusak-Ramos Garcia (Malykhin problem))

There exists a model in ZFC where every separable Fréchet-Uryshn group is metrizable.

The following problem have been attracted specialists from a long time:

Problem 2

Describe possible good sufficient conditions under which any Fréchet-Urysohn group is metrizable.
Pytkeev proved that every **sequential** space satisfies the property (so-called **Pytkeev property**) which is stronger than countable tightness.

**Definition 3**

A topological space $Y$ has the **Pytkeev property** if for each $A \subseteq Y$ and each $y \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \ldots$ of $A$ such that each neighbourhood of $y$ contains some $A_n$.

**Definition 4 (Tsaban-Zdomsky)**

A topological space $Y$ has the **strong Pytkeev property** if for each $y \in Y$ there exists a countable family $\mathcal{D}$ of subsets of $Y$ such that for each neighbourhood $U$ of $y$ and each $A \subseteq Y$ with $y \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite.
More definitions and relations:

**Definition 5 (Banakh-Zdomsky)**

Y has *countable cs*-character if for each $y \in Y$ there is a countable family $D$ of subsets of $Y$ such that for each non-trivial sequence in $Y$ converging to $y$ and each neighbourhood $U$ of $y$, there is $D \in D$ with $D \subset U$ and $D$ contains infinitely many elements of that sequence.

**Theorem 6 (Banakh-Zdomsky)**

A Fréchet-Urysohn topological group is metrizable iff it has countable cs*-character. A Baire topological group is metrizable iff it is sequential and has countable cs*-character.

We call $X$ a *P-sequential* space if $X$ is a sequential space satisfying the strong Pytkeev property.
1. Fréchet-Urysohn $\Rightarrow$ sequential $\Rightarrow$ Pytkeev property $\Rightarrow$ countable tightness.

2. First countable $\Rightarrow$ P-sequential $\Rightarrow$ strong Pytkeev property $\Rightarrow$ countable cs*-character.

3. Fréchet-Urysohn $\not\Rightarrow$ strong Pytkeev property $\not\Rightarrow k$-space.

**Theorem 7 (Gabriyelyan, Kakol)**

A Baire tvs is metrizable iff it has countable cs*-character. A b-Baire-like lcs is metrizable iff it has countable cs*-character.

Second part of theorem extends a theorem of Sakai (2008) stating that the space $C_p(X)$ is metrizable iff $C_p(X)$ has countable cs*-character (note that every $C_p(X)$ is b-Baire-like). Both parts use the concept of a $G$-base.
Topological groups with a $\mathcal{G}$-base

**Definition 8 (Cascales, Kakol, Saxon for tvs)**

Let $G$ be a topological group. A family $\mathcal{U} := \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighbourhoods of the unit $e$ is called a $\mathcal{G}$-base if $\mathcal{U}$ is a base of neighbourhoods at the unit and $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$.

Every metrizable group $G$ admits a $\mathcal{G}$-base $\{U_{\alpha_1} : (\alpha_i) \in \mathbb{N}^\mathbb{N}\}$, where $\{U_n\}_{n \in \mathbb{N}}$ - decreasing base of neighbourhoods at $e$ of $G$.

1. A topological group $G$ is metrizable iff $G$ is Fréchet-Urysohn and has a $\mathcal{G}$-base. (G-Ka-L)

2. Any precompact set in a topological group $G \in TG_{\mathcal{G}}$ is metrizable, and hence $G$ is strictly angelic. (G-Ka-L)

3. Next theorem uses the concept of a $\mathcal{G}$-base.
We say that a topological space $X$ has a compact resolution swallowing compact sets if $X$ admits a family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact sets, $K_\alpha \subset K_\beta$, whenever $\alpha \leq \beta$ and each compact set of $X$ is contained in some $K_\alpha$.

**Theorem 9 (Gabriyelyan, Kakol, Leiderman)**

Let $X$ be space which admits a compact resolution swallowing compact sets. Then the following are equivalent:

1. $C_c(X)$ has the strong Pytkeev property.
2. $C_c(X)$ has the Pytkeev property.
3. $C_c(X)$ has countable tightness.
4. $C_p(X)$ has countable tightness.
5. $C_c(X)$ is barrelled.
6. $X$ is Lindelöf.
This extends Tsaban-Zdomsky’s theorem stating that $C_c(X)$ has the strong Pytkeev property for Polish $X$.

$C_c(X)$ has a $\mathcal{G}$-base iff $X$ has a compact resolution swallowing compact sets (Ferrando, Kakol). 

Every topological group with a $\mathcal{G}$-base which is a $k$-space is strongly Pytkeev (G-Ka-L).

If $E$ is a lcs with a $\mathcal{G}$-base, then $E$ is a $k$-space iff $E$ is metrizable or $E$ is homeomorphic to $\phi$ or $\phi \times Q$, where $Q$ is the Hilbert cube (G-Ka-L).

**Corollary 10**

Let $X$ be a Čech-complete space. Then $C_c(X)$ has the strong Pytkeev property if and only if $X$ is Lindelöf.
A necessary condition for topological groups satisfying the strong Pytkeev property.

**Theorem 11**

Let $G$ be a topological group with the strong Pytkeev property. Then $G$ has a base $\{U_\alpha : \alpha \in \mathcal{M}\}$ of neighbourhoods at $e$, where

(i) $\mathcal{M}$ is a subset of the partially ordered set $\mathbb{N}^\mathbb{N}$;

(ii) if $\alpha \in \mathcal{M}$ and $\beta \in \mathbb{N}^\mathbb{N}$ are such that $\beta \leq \alpha$, then $\beta \in \mathcal{M}$;

(iii) $U_\beta \subseteq U_\alpha$, whenever $\alpha \leq \beta$ for $\alpha, \beta \in \mathcal{M}$. 
A sufficient condition for topological groups to have the strong Pytkeev property.

1. Ω - a set, I - a partially ordered set with an order ≤. A family \( \{A_i\}_{i \in I} \) of subsets of Ω is \( I\text{-decreasing} \) if \( A_j \subseteq A_i \) for every \( i \leq j \) in I. Example: \( \mathbb{N}^\mathbb{N} \) endowed with the order, i.e., \( \alpha \leq \beta \) if \( \alpha_i \leq \beta_i, \ i \in \mathbb{N}, \ \alpha = (\alpha_i)_{i \in \mathbb{N}}, \ \beta = (\beta_i)_{i \in \mathbb{N}} \).

2. For \( \alpha = (\alpha_i)_{i \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}, \ k \in \mathbb{N} \), set \( I_k(\alpha) := \{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i, \ i = 1, \ldots, k\} \).

3. Let \( M \subseteq \mathbb{N}^\mathbb{N} \) and \( \mathcal{U} = \{U_\alpha : \alpha \in M\} \) be an \( M \)-decreasing family of subsets of a set Ω. Define the (countable) family \( \mathcal{D}_\mathcal{U} \) of subsets of Ω by \( \mathcal{D}_\mathcal{U} := \{D_k(\alpha) : \alpha \in M, k \in \mathbb{N}\} \), where \( D_k(\alpha) = \bigcap_{\beta \in I_k(\alpha) \cap M} U_\beta \).

4. We say that \( \mathcal{U} \) satisfies the condition (D) if \( U_\alpha = \bigcup_{k \in \mathbb{N}} D_k(\alpha) \) for every \( \alpha \in M \).
Theorem 12 (Gabriyelyan, Kakol, Leiderman)

Let $G$ be a topological group with a $\mathcal{G}$-base satisfying condition (D). Then $G$ has the strong Pytkeev property.

1. A quasibarrelled lsc with a $\mathcal{G}$-base satisfies condition (D).
2. Every $(DF)$-space with countable tightness has a $\mathcal{G}$-base with condition (D). (Cascales-Kakol-Saxon).
Applications: Last theorem applies to obtain the following

**Theorem 13 (Gabriyelyan, Kakol, Leiderman)**

(i) A $(DF)$-space $E$ has countable tightness iff $E$ has the strong Pytkeev property.

(ii) Every strict $(LM)$-space has the strong Pytkeev property.

(iii) Let $(E', \beta(E', E))$ be the strong dual of a strict $(LF)$-space $E$. Then (a) $(E', \beta(E', E))$ has a $\mathfrak{G}$-base. (b) $(E', \beta(E', E))$ has countable tightness iff $(E', \beta(E', E))$ has the strong Pytkeev property.

Any space $E$ mentioned above is metrizable iff $E$ is Fréchet-Urysohn (since the strong Pytkeev property $+$ Fréchet-Urysohn $\Rightarrow$ metrizable.) Therefore, for example, $\mathcal{D}'(\Omega)$ has the strong Pytkeev property, and in particular, it has countable tightness.
Topological description of cosmic and $\aleph_0$-spaces.

**Definition 14**

A topological space $X$ has a **small base** if there exists an $M$-decreasing base of $\tau$ for some $M \subseteq \mathbb{N}^\mathbb{N}$.

**Theorem 15 (Gabriyelyan, Kakol, Kubzdela, Lopez-Pellicer)**

Let $X := (X, \tau)$ be a regular topological space. Then:

(i) $X$ is cosmic iff $X$ has a small base $\mathcal{U} = \{ U_\alpha : \alpha \in M \}$ with (D). The family $\mathcal{D}_\mathcal{U}$ is a countable network in $X$.

(ii) $X$ is an $\aleph_0$-space iff $X$ has a small base $\mathcal{U} = \{ U_\alpha : \alpha \in M \}$ with (D) such that $\mathcal{D}_\mathcal{U}$ is a countable $k$-network in $X$.

There is a small base $\mathcal{U}$ such that $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$ and $\mathcal{U} = \tau$, i.e. for any $W \in \tau$ there is $\alpha \in M$ with $W = U_\alpha$. 
Condition (D) is essential: The Bohr compactification $b\mathbb{Z}$ of the discrete group $\mathbb{Z}$ has a small base and $b\mathbb{Z}$ is not cosmic as nonmetrizable.

**Theorem 16 (Gabriyelyan, Kakol)**

A Baire topological group is metrizable iff $G$ has the strong Pytkeev property iff $G$ has a $\mathfrak{S}$-base with condition (D).

**Theorem 17 (Gabriyelyan-Kakol-Zdomsky)**

(i) A Banach space $E$ is finite-dimensional iff $E_w$ has the Pytkeev property. (ii) $B_w$ has the Pytkeev property iff $E$ does not contain $\ell_1$. (iii) $B_w$ is metrizable iff $B_w$ has the strong Pytkeev property iff $E'$ is separable.

James tree $JT \not\cong \ell_1$, $JT^*$ is nonseparable, $JT$ has a Kadets norm under which $B_w$ is Baire with the Pytkeev property.
The last results yield

**Corollary 18**

A Baire separable topological group $G$ is metrizable iff $G$ is cosmic.

**Question 19**

Let $G$ be a topological group (or even a TVS) with the strong Pytkeev property. Does $G$ admit a $\mathcal{G}$-base?

We do not know an answer to this question even for submetrizable groups.