Valdivia's version of Nikodym property

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Outline

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Notations

$\mathcal{B}$ subset of set-algebra $\mathcal{A}$.
$L(\mathcal{B}) := span\{\chi_C : C \in \mathcal{B}\}$ with supremum norm $\|\cdot\|$
$ba(\mathcal{A})$ variation norm $|\cdot| := |\cdot|(\Omega)$.
$\mu \in ba(\mathcal{A})$
$\mu \in (L(\mathcal{A}))'$ defined by $\mu(\chi_C) := \mu(C)$, for each $C \in \mathcal{A}$.
Polar sets in $< L(\mathcal{A}), ba(\mathcal{A}) >$
$\tau_s(\mathcal{B})$ topology in $ba(\mathcal{A})$ of pointwise convergence in $\mathcal{B}$

Nikodym-Dieudonné-Grothendieck theorem

$\mathcal{B}$ ($\subset \mathcal{A}$) has N-property if $\mathcal{B}$-*pointwise bounded* $\Longrightarrow$ *bounded in* $ba(\mathcal{A})$

$\mathcal{B}$ ($\subset \mathcal{A}$) has sN-property if for each increasing covering $(\mathcal{B}_m)_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has N-property.

Theorem (Nikodym-Dieudonné-Grothendieck theorem)

Each σ-algebra $\mathcal{S}$ of subsets of Ω has N-property.

Theorem (Valdivia theorem)

Each σ-algebra $\mathcal{S}$ of subsets of Ω has sN-property.

Problem (Valdivia 2013)

Let $\mathcal{A}$ be an algebra of subsets of Ω. Is it true that N-property of $\mathcal{A}$ implies sN-property?

Web version of N-D-G theorem

$\{B_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$, $B_{m_1 m_2 \ldots m_p} \subset A$ is an *increasing web in A* if:

- $(B_{m_1})_{m_1}$ is an increasing covering of A
- and $(B_{m_1, m_2, \ldots, m_p m_{p+1}})_{m_{p+1}}$ is an increasing covering of $B_{m_1 m_2 \ldots m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p+1$.

Theorem

If $\{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ is an increasing web in a σ-algebra $\mathcal{S}$ of subsets of Ω, then there exists a sequence $(n_i)_i$ in $\mathbb{N}$ such that each $\mathcal{B}_{n_1 n_2 \ldots n_i}$ has sN-property, for each $i \in \mathbb{N}$.

Outline

2. Nikodym property and deep unbounded sets
   - Deep unbounded sets
   - NV-sets of indexes
   - Deep unbounded families and NV-sets

Simple equivalences

Proposition

Let $\mathcal{A}$ be an algebra of subsets of Ω and let M be an absolutely convex $\tau_s(\mathcal{A})$-closed subset of $ba(\mathcal{A})$. The following properties are equivalent:

1. *For each finite subset $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$ the set $M \cap \mathcal{Q}^\circ$ is an unbounded subset of $ba(\mathcal{A})$.*
2. *For each finite subset $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$ such that $span\{M^\circ\} \cap span\{\mathcal{Q}\} = \{0\}$ the set $M \cap \mathcal{Q}^\circ$ is an unbounded subset of $ba(\mathcal{A})$.*
3. *M° is not a neighborhood of zero in $span\{M^\circ\}$ or the codimension of $span\{M^\circ\}$ in $L(\mathcal{A})$ is infinite.*

If M is unbounded and $\overline{span}\{M^\circ\} = L(\mathcal{A})$ then M verifies the previous properties.

Simple equivalences

Proof.

If there exists $m_1 \in M^\circ$ such that $\chi_{Q_1} = h_1 m_1 + \Sigma\{h_i \chi_{Q_i} : 2 \leqslant i \leqslant r\}$ and if $h := 2 + \Sigma\{|h_i| : 1 \leqslant i \leqslant r\}$ then

$$\text{absco}(M^\circ \cup \mathcal{Q}) \subset h\,\text{absco}(M^\circ \cup \{\mathcal{Q} \backslash \{\chi_{Q_1}\}\}).$$

Taking polar sets

$$M \cap \{\mathcal{Q} \backslash \{\chi_{Q_1}\}\}^\circ \subset h(M \cap \mathcal{Q}^\circ),$$

$M \cap \{\mathcal{Q} \backslash \{\chi_{Q_1}\}\}^\circ$ unbounded $\Longrightarrow M \cap \mathcal{Q}^\circ$ unbounded. □

Deep unbounded sets

Definition

Let B be an element of the algebra $\mathcal{A}$ of subsets of Ω. A subset M of $ba(\mathcal{A})$ is deep B-unbounded if each finite subset $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$ verifies that

$$\sup\{|\mu(C)| : \mu \in M \cap \mathcal{Q}^\circ,\ C \in \mathcal{A},\ C \subset B\} = \infty.$$

or, equivalently, $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} = \infty$.

M is deep Ω-unbounded if $M \cap \mathcal{Q}^\circ$ is unbounded in $ba(\mathcal{A})$ for each $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leqslant i \leqslant r\}$.
M, absco $\tau_s(\mathcal{A})$-closed, is deep Ω-unbounded $\iff$ M verifies (2) or (3) in Proposition.
If, additionally, $\overline{span}\{M^\circ\} = L(\mathcal{A})$ then M is deep Ω-unbounded $\iff$ M is unbounded.

Sequences of deep Ω-unbounded subsets

Proposition

Let $\mathcal{A}$ be an algebra of subsets of Ω and let $(\mathcal{B}_m)_m$ be an increasing sequence of subsets of $ba(\mathcal{A})$ such that $\mathcal{B}_m$ does not have N-property for each $m \in \mathbb{N}$. If $\overline{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\} = L(\mathcal{A})$ there exists $n_0 \in \mathbb{N}$ and a sequence $(M_m : m \geqslant n_0)$ of deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subsets of $ba(\mathcal{A})$ such that

$$\sup\{|\mu(C)| : \mu \in M_m\} < \infty,$$

for each $m \geqslant n_0$ and each $C \in \mathcal{B}_m$. In particular this proposition holds if $\cup_m \mathcal{B}_m = \mathcal{A}$ (Valdivia) or if $\cup_m \mathcal{B}_m$ has N-property.

Sequences of deep Ω-unbounded subsets

Proof.

If $\forall m$, cod $H_m := \overline{span}\{\chi_C : C \in \mathcal{B}_m\} = \infty$ and $P_m := \overline{absco}\{\chi_C : C \in \mathcal{B}_m\}$ then $M_m := P_m^\circ$ is deep Ω-unbounded and $\sup\{|\mu(C)| : \mu \in M_m\} \leqslant 1$, for each $C \in \mathcal{B}_m$. If cod $\overline{span}\{\chi_C : C \in \mathcal{B}_p\} \in \mathbb{N}$ then $\overline{span}\{\chi_C : C \in \mathcal{B}_m\} = L(\mathcal{A})$, for each $m \geqslant n_0$. Apply Proposition. □

Notations

Let $\{t := (t_1, t_2, \ldots, t_p),\ s := (s_1, s_2, \ldots, s_q)\} \cup T \cup U \subset \cup_s \mathbb{N}^s$.
For $1 \leqslant i \leqslant p$, $t(i) := (t_1, t_2, \ldots, t_i)$ is a *section of t of length i*,
$t(j) := \emptyset$ if $j > p$,
$T(m) := \{t(m) : t \in T\}$, $m \in \mathbb{N}$,
$t \times s := (t_1, t_2, \ldots, t_p, t_{p+1}, t_{p+2}, \ldots, t_{p+q})$, with $t_{p+j} := s_j$, for $1 \leqslant j \leqslant q$,
$T \times U := \{t \times u : t \in T,\ \ u \in U\}$.
We simplify (t_1), (n) and $T \times \{(n)\}$ by t_1, n and $T \times n$.
$(t^n)_n$, $t^n = (t_1^n, t_2^n, \ldots, t_n^n, \ldots) \in \cup_s \mathbb{N}^s$, is an *infinite chain* if $\emptyset \neq t^n(n) = t^{n+1}(n)$, $\forall n \in \mathbb{N}$.
$|C|$ is the cardinal of C.

NV-set of indexes

Definition

Let $T \subset \cup_{s\in\mathbb{N}}\mathbb{N}^s$, T without infinite chains. T is an *NV*-set if $|T(1)| = \infty$, and
$-$ $T = T(1)$
$-$ or $T \cap T(1) = \emptyset$ and then each $t := (t_1, t_2, \ldots, t_p) \in T$ has the following properties:

1. If $1 < i < p$ then $\{(t_1, t_2, \ldots, t_{i-1}) \times \mathbb{N}\} \cap T(i)$ is an infinite set disjoint with T.
2. The set $\{(t_1, t_2, \ldots, t_{p-1}) \times (\cup_{s\in\mathbb{N}}\mathbb{N}^s)\} \cap T$ is an infinite subset of $T(p)$.

An *NV*-set T is *a trivial NV-set* if $T = T(1)$.
The sets $\mathbb{N}^i$, $i \in \mathbb{N}\backslash\{1\}$, and $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are non trivial *NV*-sets.

Example

Example

Let $\mathcal{B} := \{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in an algebra $\mathcal{A}$ of subsets of Ω with the property that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $q \in \mathbb{N}$ such that $\mathcal{B}_{m_1 m_2 \ldots m_q}$ does not have *sN*-property.
Then there exists an increasing web $\mathcal{C} := \{\mathcal{C}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ in $\mathcal{A}$ and a *NV*-set T such that for each $t = (t_1, t_2, \ldots, t_p) \in T$ there exists a deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absolutely convex subset $M_{t_1 t_2 \ldots t_p}$ of $ba(\mathcal{A})$ which is pointwise bounded in $\mathcal{C}_{t_1 t_2 \ldots t_p}$, i.e.,

$$\sup\{|\mu(C)| : \mu \in M_{t_1 t_2 \ldots t_p}\} < \infty,$$

for each $C \in \mathcal{C}_{t_1 t_2 \ldots t_p}$.

Example (proof part I)

Proof.

If $\mathcal{B}_{m_1}$, $m_1 \in \mathbb{N}$, does not have N-property then $\mathcal{C} := \mathcal{B}$ and $T := \mathbb{N} \backslash \{1, 2, \ldots, n_0 - 1\}$, with n_0 the natural number given by Proposition applied to the increasing covering $(\mathcal{B}_{m_1})_{m_1}$ of $\mathcal{A}$.
Hence we may suppose that there exists m_1' such that $\mathcal{B}_{t_1}$ has N-property for each $t_1 \geqslant m_1'$.
Now we define $Q_1 := \emptyset$ and $Q_1' := \{t_1 \in \mathbb{N} : t_1 \geqslant m_1'\}$ if $\mathcal{B}_{t_1}$ does not have sN-property for each $t_1 \in \mathbb{N}$.
If there exists $m_1 \in \mathbb{N}$ such that $\mathcal{B}_{t_1}$ has sN-property for each $t_1 \geqslant m_1$ then we define $Q_1 := \{t_1 \in \mathbb{N} : t_1 \geqslant m_1\}$ and $Q_1' := \emptyset$.
.... □

Example (proof part II

Proof.

If $t := (t_1, t_2, \ldots, t_i) \in Q_i$ then, by induction, $\mathcal{B}_{t_1 t_2 \ldots t_i}$ has sN-property and from $\mathcal{B}_{t_1 t_2 \ldots t_i} = \cup_n \mathcal{B}_{t_1 t_2 \ldots t_i n}$ it follows that:

(i) Or $\mathcal{B}_{t_1 t_2 \ldots t_i n}$ has sN-property for each $n \geqslant m_{i+1}$ and then we define $S_{t_1 t_2 \ldots t_i} := \{n \in \mathbb{N} : m_{i+1} \leqslant n\}$ and $S'_{t_1 t_2 \ldots t_i} := \emptyset$.

(ii) Or $\mathcal{B}_{t_1 t_2 \ldots t_i n}$ does not have sN-property for each $n \in \mathbb{N}$. As $\mathcal{B}_{t_1 t_2 \ldots t_i}$ has sN-property then there exists $m'_{i+1} \in \mathbb{N}$ such that $\mathcal{B}_{t_1 t_2 \ldots t_i n}$ has N-property for each $n \geqslant m'_{i+1}$. In this case let $S_{t_1 t_2 \ldots t_i} := \emptyset$ and $S'_{t_1 t_2 \ldots t_i} := \{n \in \mathbb{N} : m'_{i+1} \leqslant n\}$.

$Q_{i+1} := \cup\{t \times S_t : t \in Q_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q_i\}$.
$T_0 := \cup\{Q'_j : j \in \mathbb{N}\}$ .... □

Increasing property

Let $\{s := (s_1, s_2, \ldots, s_p)\} \cup S \subset \cup_n \mathbb{N}^n$.
S has the *increasing property* at s if S contains p elements $s^1 = (s_1^1, s_2^1, \ldots)$ and $s^i = (s_1, s_2, \ldots, s_{i-1}, s_i^i, s_{i+1}^i, \ldots)$, $1 < i \leqslant p$, such that $s_i < s_i^i$, for each $1 \leqslant i \leqslant p$.
S has the *increasing property* (*increasing property with respect to a subset V of $\cup_s \mathbb{N}^s$*) if S has the increasing property at each $s \in S$ (at each $v \in V$).
If an NV-set T contains a subset S which has the increasing property then S is an NV-set. From this observation it follows the Claim 1.

Claim

If $(S_n)_n$ is a sequence of non-void subsets of an NV-set T such that for each $n \in \mathbb{N}$ the set S_{n+1} verifies the increasing property with respect to S_n, then $S := \cup_n S_n$ is an NV-set.

NV-extension

$U \subset$ (non-trivial *NV*-set) T.
t ($\in T$) *admits NV extension in U if* $\{v \in \cup_n \mathbb{N}^n : t \times v \in U\}$ *contains an NV-set.*
Elementary facts (T non-trivial):

(a) If U does not contain an *NV*-set $\Longrightarrow \exists m_1 \in \mathbb{N}$ such that each $n \in T(1)$, with $m_1 \leqslant n$, does not admit *NV* extension in U.

If $t \in T(i)$ does not admit *NV*-extension in U then:

(b1) if $\{v \in \cup_s \mathbb{N}^s : t \times v \in T\}$ is non-trivial *NV*-set, $\exists m_{i+1} \in \mathbb{N}$ such that each $t \times n \in T(i+1)$, with $m_{i+1} < n$, does not admit an *NV*-extension in U.

(b2) if $\{v \in \cup_s \mathbb{N}^s : t \times v \in T\}$ is a trivial *NV*-set, $\exists m'_{i+1} \in \mathbb{N}$ such that $\left(t \times \{\mathbb{N} \setminus \{1, 2, \ldots, m'_{i+1}\}\right) \cap T(i+1)$ is an infinite subset of $T \setminus U$.

NV-subsets

Proposition

Let U be a subset of an NV-set T. If U does not contain an NV-set then $T \setminus U$ contains an NV-set.

Proof.

We may suppose that T is a non-trivial NV-set.
By (a), $\exists m_1 \in \mathbb{N}$ such that each $n \in Q_1 := \{n \in T(1) : m_1 \leqslant n\}$ $(\subset T(1) \setminus T)$ does not admit NV-extension in U. Define $Q_1' := \emptyset$.
.... □

NV-subsets. Proof part II

Proof.

If $t \in Q_i$ then t does not admit *NV*-extension in U and the following two cases may happen:

i. Or $\{v \in \cup_s \mathbb{N}^s : t \times v \in T\}$ is a non-trivial *NV*-set and, by (b1), there exists $m_{i+1} \in \mathbb{N}$ such that if $t \times n \in T(i+1)$, $n > m_{i+1}$, $t \times n$ does not admit *NV*-extension in U. $S_t := \{n > m_{i+1}, t \times n \in T(i+1)\}$, $S'_t := \emptyset$.

ii. Or $\{v \in \cup_s \mathbb{N}^s : t \times v \in T\}$ is a trivial an *NV*-set and then, by (b2), there exists $m'_{i+1} \in \mathbb{N}$ such that $t \times n \subset T \backslash U$ if $m'_{i+1} \leqslant n$. $S'_t := \{n \in \mathbb{N} : m'_{i+1} \leqslant n, t \times n \in T(i+1)\}$ and $S_t := \emptyset$.

$Q_{i+1} := \cup\{t \times S_t : t \in Q_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q_i\}$.
..............$W := \cup\{Q'_j : j \in \mathbb{N}\}$ is a *NV*-subset of $T \backslash U$. □

Preservation of deep-unboundedness.

Proposition

Let B an element of an algebra $\mathcal{A}$, $\{C_1, C_2, \ldots, C_q\}$ be a finite measurable partition of B and let $\{M_t : t \in T\}$ be a family of deep B-unbounded subsets of $ba(\mathcal{A})$ indexed by an NV-set T. Suppose that $\{t^1, t^2, \ldots, t^k\}$ is a finite subset of T. Then

1. *for each t^j, $1 \leqslant j \leqslant k$, there exists $i^j \in \{1, 2, \ldots, q\}$ such that M_{t^j} is deep C_{i^j}-unbounded (Valdivia),*
2. *exists $i_0 \in \{1, 2, \ldots, q\}$ and a NV-set $T_{i_0} \subset T$ such that $\{M_t : t \in T_{i_0}\}$ is a family of deep C_{i_0}-unbounded subsets.*

Hence, if $t^j := (t^j_1, t^j_2, \ldots, t^j_{p_j})$, for each $1 \leqslant j \leqslant k$, and $q = 2 + \Sigma\{p_j : 1 \leqslant j \leqslant k\}$ then there exists $h \in \{1, 2, \cdots, q\}$ and a NV-set T_1 such that $\{t^1, t^2, \ldots, t^k\} \subset T_1 \subset T$ and $\{M_t : t \in T_1\}$ is a family of deep $B \backslash C_h$-unbounded subsets.

Preservation of deep-unboundedness (proof).

Proof.

(1) $\sup\{|\mu|(C_i) : \mu \in M_{tj} \cap (\mathcal{Q}^i)^\circ\} < H_i$, $i \in \{1, 2, \ldots, q\}$, imply

$\sup\{|\mu|(C_1 \cup \cdots \cup C_q) : \mu \in M_{tj} \cap (\mathcal{Q}^1 \cup \cdots \cup \mathcal{Q}^q)^\circ\} < H_1 + \cdots + H_q$.

(2) $V_j := \{t \in T : M_t$ is deep C_j-unbounded$\}$, $1 \leqslant j \leqslant q$,
$\Longrightarrow T = \cup\{V_j : 1 \leqslant j \leqslant q\} \Longrightarrow \exists i_0$, with $1 \leqslant i_0 \leqslant q$, such that V_{i_0} contains an *NV*-set T_{i_0}. □

Valdivia proposition.

Proposition

Let $\{B, Q_1, \ldots, Q_r\}$ be a subset of the algebra $\mathcal{A}$ of subsets of Ω and let M be a deep B-unbounded absolutely convex subset of ba($\mathcal{A}$).
Then given a positive real number α and a natural number $q > 1$ there exists a measurable partition $\{C_1, C_2, \ldots, C_q\}$ of B and a subset $\{\mu_1, \mu_2, \ldots, \mu_q\}$ of M such that

- $|\mu_i(B_i)| > \alpha$
- *and* $\Sigma\{\left|\mu_i(Q_j)\right| : 1 \leqslant j \leqslant r\} \leqslant 1$, *for* $i = 1, 2, \ldots, q$.

Deep unboundedness and inequalities.

Proposition

Let $\{B, Q_1, \ldots, Q_r\}$ be a subset of an algebra $\mathcal{A}$ of subsets of Ω and $\{M_t : t \in T\}$ a family of deep B-unbounded absolutely convex subsets of $ba(\mathcal{A})$, indexed by an NV-set T. Then for each finite subset $\{t^j : 1 \leqslant j \leqslant k\}$ of T and a positive real number α there exist k pairwise disjoint measurable subsets B_j of B, $1 \leqslant j \leqslant k$, k finite additive measures $\mu_j \in M_{t^j}$, $1 \leqslant j \leqslant k$, and a NV-set T^* such that:

1. $\left|\mu_j(B_j)\right| > \alpha$ *and* $\Sigma\{\left|\mu_j(Q_i)\right| : 1 \leqslant i \leqslant r\} \leqslant 1$, *for* $j = 1, 2, \ldots, k$,
2. $\{t^j : 1 \leqslant j \leqslant k\} \subset T^* \subset T$ *and* $\{M_t : t \in T^*\}$ *is a family of deep* $B \setminus \cup \{B_j : 1 \leqslant j \leqslant k\}$*-unbounded sets.*

Deep unboundedness and inequalities (proof).

Proof.

Let $t^j := (t^j_1, \ldots, t^j_{p_j})$, for $1 \leqslant j \leqslant k$, $q = 2 + \Sigma\{p_j : 1 \leqslant j \leqslant k\}$. Valdivia Prop. (for B, α, q and M_{t^1}) gives a meas. partition $\{C^1_1, C^1_2, \cdots, C^1_q\}$ of B and $\{\lambda_1, \lambda_2, \cdots, \lambda_q\} \subset M_{t^1}$ such that:

$$\left|\lambda_k(C^1_k)\right| > \alpha \quad \text{and} \quad \Sigma\{|\lambda_k(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1, \quad \text{for} \quad k = 1, 2, \ldots, q.$$

DUP Proposition (for $\{C^1_1, C^1_2, \cdots, C^1_q\}$, $\{M_t : t \in T\}$ and $\{t^j : 1 \leqslant j \leqslant k\}$ gives $h \in \{1, 2, \cdots, q\}$, an *NV*-set T_1 such that

$$\{t^1, t^2, \ldots, t^k\} \subset T_1 \subset T$$

and a family $\{M_t : t \in T_1\}$ of deep $B \backslash C^1_h$-unbounded subsets. If $B_1 := C^1_h$ and $\mu_1 := \lambda_h$ then we repeat with $B \backslash B_1$, α, q and M_{t^2}

Outline

3 Proof of web N-D-G Theorem
- Determination of some sequences
- Coverings and NV-sets
- A contradiction

Construction of some sequences

Proposition

Let $\{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be a increasing web in an algebra $\mathcal{A}$ with the property that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $h \in \mathbb{N}$ such that $\mathcal{B}_{m_1 m_2 \ldots m_h}$ does not have sN-property and let $(i_n)_n = (1, 1, 2, 1, 2, 3, \ldots)$. Then there exist

1. *a strictly increasing sequence $(j_n)_n$ in $\mathbb{N}$, a sequence $(B_{i_n j_n})_n$ of pairwise disjoints measurable sets and a sequence $(\mu_{i_n j_n})_n$ in ba$(\mathcal{A})$ such that, for each $n \in \mathbb{N}$,*

$$\Sigma_s\{\left|\mu_{i_{n+1} j_{n+1}}(B_{i_s j_s})\right| : 1 \leqslant s \leqslant n\}) < 1,$$

$$\left|\mu_{i_n j_n}(B_{i_n j_n})\right| > j_n,$$

$$\left|\mu_{i_n j_n}(\cup_s\{B_{i_s j_s} : n < s\})\right| < 1,$$

2. *and an increasing web $\{\mathcal{C}_{m_1 m_2 \ldots m_q} : q, m_1, m_2, \ldots, m_q \in \mathbb{N}\}$*

Construction of some sequences II

Proposition

Let $\{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be a increasing web in an algebra $\mathcal{A}$ with the property that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $h \in \mathbb{N}$ such that $\mathcal{B}_{m_1 m_2 \ldots m_h}$ does not have sN-property and let $(i_n)_n = (1, 1, 2, 1, 2, 3, \ldots)$. Then there exist

1. *and an increasing web $\{\mathcal{C}_{m_1 m_2 \ldots m_q} : q, m_1, m_2, \ldots, m_q \in \mathbb{N}\}$ in $\mathcal{A}$ and a countable NV-set $\{t^i : i \in \mathbb{N}\}$ such that for each $r \in \mathbb{N}$, each $H \in \mathcal{C}_{t^r}$ and each strictly increasing sequence $(n_p)_p$, with $i_{n_p} = r$ for each $p \in \mathbb{N}$, we have that*

$$\sup\{\left|\mu_{i_{n_p} j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty.$$

First step of the proof.

Proof.

By Example $\exists\ \{\mathcal{C}_t : t \in \cup_s \mathbb{N}^s\}$ in $\mathcal{A}$, a NV-set T and $\{M_t \subset ba(\mathcal{A}) : t \in T\}$ (deep Ω-unbounded $\tau_s(\mathcal{A})$-closed absco): $\forall H \in \mathcal{C}_t$

$$\sup\{|\mu(H)| : \mu \in M_t\} < \infty.$$

(Induction) $\exists\ (k_j \in \mathbb{N})_j \uparrow$, NV-set $\{t^r : r \in \mathbb{N}\} \subset T$, $\{B_{ij} : (i,j) \in \mathbb{N}^2,\ i \leqslant k_j\}$ (pairwise disjoint measurable) and $\{\mu_{ij} \in M_{t^i} : (i,j) \in \mathbb{N}^2, i \leqslant k_j\} : \forall (i,j) \in \mathbb{N}^2$, with $i \leqslant k_j$,

$$\Sigma_{s,v}\{\left|\mu_{ij}(B_{sv})\right| : s \leqslant k_v,\ 1 \leqslant v < j\}) < 1,$$

$$\left|\mu_{ij}(B_{ij})\right| > j,$$

□

First step of the proof II

Proof.

and for each $r \in \mathbb{N}$ and each $H \in \mathcal{C}_{t^r}$ we have

$$\sup_w \{|\mu_{rw}(H)| : r \leqslant w\} < \infty.$$

Fix $t^1 \in T$. By Prop DUI ($B := \Omega$, $\alpha = 1$, $\{Q_1, \ldots, Q_r\} := \emptyset$ and $\{t^1\}$) $\exists B_{11} \in \mathcal{A}$, $\mu_{11} \in M_{t^1}$ and a NV-set T_1 such that

1. $|\mu_{11}(B_{11})| > 1$,
2. $t^1 \in T_1 \subset T$, and $\{M_t : t \in T_1\}$ is a family of deep $\Omega \backslash B_{11}$-unbounded subsets.

....

$r \leqslant w \Longrightarrow r \leqslant k_w \Longrightarrow \{\mu_{rw} : r \leqslant w\} \subset M_{t^r}$, giving the last inequality. □

Second step of the proof (to find $(j_n)_n$).

Proof.

Let $j_1 := 1$. If $\left|\mu_{i_1 j_1}\right|(\Omega) < s_1$, let $\{N_u^1, 1 \leqslant u \leqslant s_1\}$ a partition of $\{m \in \mathbb{N} : m > j_1\}$ in s_1 infinite subsets.
Let $B_u^1 := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N_u^1,\ s \leqslant k_t\}$, $1 \leqslant u \leqslant s_1$. From

$$\Sigma\{\left|\mu_{i_1 j_1}\right|(B_u^1) : 1 \leqslant u \leqslant s_1\} < s_1$$

$\exists u'$ such that $\left|\mu_{i_1 j_1}\right|(B_{u'}^1) < 1$. $N^{(1)} := N_{u'}^1$ and $B^1 := B_{u'}^1$ verify: $N^{(1)} \subset \{m \in \mathbb{N} : m > j_1\}$ and

$$\left|\mu_{i_1 j_1}\right|(B^1) < 1. \qquad \ldots\ldots\ldots\ldots$$

□

Second step of the proof (to find $(j_n)_n$). Part II

Proof.

Let j_{l+1} be the first element in $N^{(l)}$ $(j_l < j_{l+1})$. If $\left|\mu_{i_{l+1}j_{l+1}}\right|(\Omega) < s_{l+1}$, let $\{N_r^{l+1}, 1 \leqslant r \leqslant s_{l+1}\}$ a partition of $\{m \in \mathbb{N}^{(l)} : m > j_{l+1}\}$ in s_{l+1} infinite disjoint subfamilies then the subsets $B_r^{l+1} := \cup\{B_{st} : (s,t) \in \mathbb{N} \times N_r^{l+1}, s \leqslant k_t\}$, $1 \leqslant r \leqslant s_{l+1}$. From

$$\Sigma\{\left|\mu_{i_{l+1}j_{l+1}}\right|(B_r^{l+1}) : 1 \leqslant r \leqslant s_{l+1}\} < s_{l+1}, \text{ it follows } ....$$

□

Coverings indexed by *NV*-sets.

Proposition

If $\{Y_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ is an increasing web in Y and T is an NV-set then $Y = \cup\{Y_u : u \in T\}$.

Proof.

$y \in Y \setminus \cup_{t \in T} Y_t = \cup_{t \in T(1)} Y_t \setminus \cup_{t \in T} Y_t \Longrightarrow$
$\exists t^1 := (t_1^1, t_2^1, \ldots) \in T$ s t $y \in Y_{t^1(1)} \setminus \cup_{t \in T} Y_t$....
$y \in Y_{t^n(n)} \setminus \cup_{t \in T} Y_t = \cup_{(t_1^n, t_2^n, \ldots, t_n^n, s) \in T(n+1)} Y_{t_1^n, t_2^n, \ldots, t_n^n, s} \setminus \cup_{t \in T} Y_t,$
$\Longrightarrow \exists t^{n+1} \in T,\ t^n(n) = t^{n+1}(n)$ and

$$y \in Y_{t^{n+1}(n+1)} \setminus (\cup\{Y_t : t \in T\}).$$

Whence T contains the infinite chain $(t^n)_n$ (contradiction $\Longrightarrow Y = \cup\{Y_u : u \in T\}$). □

Proof of the web version of *N-D-G* theorem (I).

Proof of web *NDG* theeorem.

Assume Theorem fails. $\exists\ (j_n)_n \uparrow$ in $\mathbb{N}$, $(B_{i_n j_n})_n$ of pairwise disjoints measurable sets, $(\mu_{i_n j_n})_n$ in $ba(\mathcal{A})$, web $\{\mathcal{C}_{m_1 m_2 \ldots m_q} : q, m_1, m_2, \ldots, m_q \in \mathbb{N}\} \uparrow$ in $\mathcal{A}$ and a countable NV-set $\{t^i : i \in \mathbb{N}\}$ such that,
for each $n \in \mathbb{N}\backslash\{1\}$, $r \in \mathbb{N}$, strictly $(n_p)_p \uparrow$ with $i_{n_p} = r$ and each $H \in \mathcal{C}_{t^r}$:

$$\Sigma_s\{\left|\mu_{i_n j_n}(B_{i_s j_s})\right| : \ s < n\}) < 1,$$

$$\left|\mu_{i_n j_n}(B_{i_n j_n})\right| > j_n,$$

$$\left|\mu_{i_n j_n}(\cup_s\{B_{i_s j_s} : \ n < s\})\right| < 1,$$

$$\sup\{\left|\mu_{i_{n_p} j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty.$$

□

Proof of the web version of *N-D-G* theorem (II).

Proof.

As $H_0 := \cup\{B_{i_s j_s} : s = 1, 2, \ldots\} \in \mathcal{S}$ and $\{t^i : i \in \mathbb{N}\}$ is an *NV*-set then $\exists\ r' \in \mathbb{N}$ such that $H_0 \in \mathcal{C}_{t^{r'}}$. Fix a strictly $(n_q)_q \uparrow$ with $i_{n_q} = r'$, $\forall q \in \mathbb{N}$.

$$\sup\{\left|\mu_{i_{n_q} j_{n_q}}(H_0)\right| : q \in \mathbb{N}\} < \infty. \tag{1}$$

$C := \cup_s\{B_{i_s j_s} : s < n_q\}$, $B_{i_{n_q} j_{n_q}}$ and $D := \cup_s\{B_{i_s j_s} : n_q < s\}$ (H_0-partition).
$\left|\mu_{i_{n_q} j_{n_q}}(C)\right| < 1$, $\mu_{i_{n_q} j_{n_q}}(B_{i_{n_q} j_{n_q}}) > n_q$ and $\left|\mu_{i_{n_q} j_{n_q}}(D)\right| < 1$, $\forall q \in \mathbb{N} \backslash \{1\}$. Hence we get the contradiction

$$\left|\mu_{i_{n_q} j_{n_q}}(H_0)\right| > n_q - 2 \Longrightarrow \lim_p \left|\mu_{i_{n_p} j_{n_p}}(H_0)\right| = \infty.$$

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