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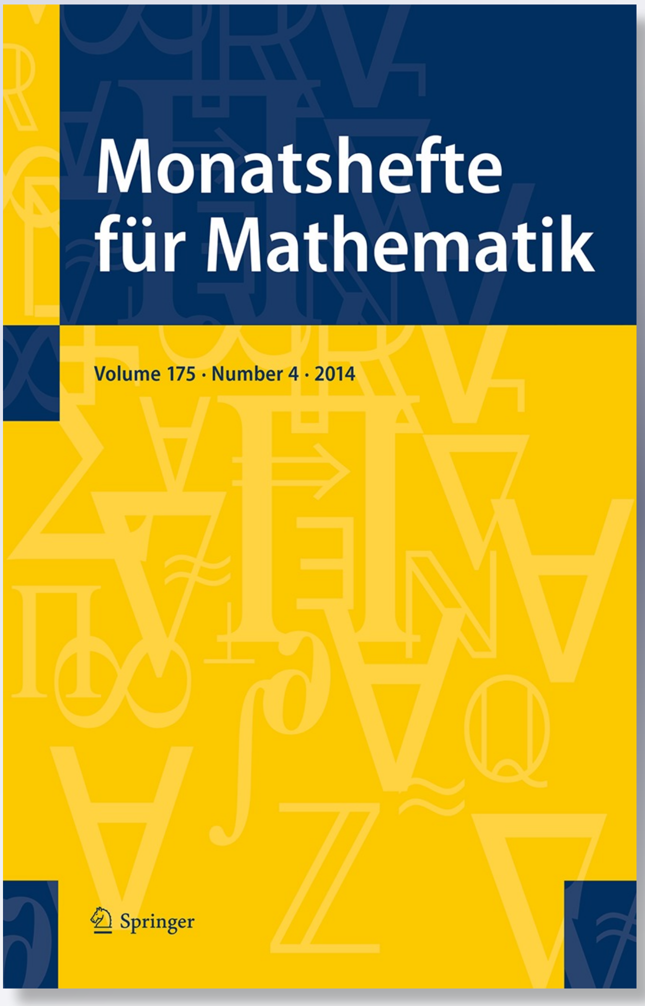

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The strong Pytkeev property for topological groups and topological vector spaces

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Abstract Generalizing the notion of metrisability, recently Tsaban and Zdomskyy (Houst J Math 35:563–571, 2009) introduced the strong Pytkeev property and proved a result stating that the space $C_c(X)$ has this property for any Polish space X. We show that the strong Pytkeev property for general topological groups is closely related to the notion of a $\mathfrak{G}$-base, investigated in Gabriyelyan et al. (On topological groups with a small base and metrizability, preprint) and Kąkol et al. (Descriptive Topology in Selected Topics of Functional Analysis. Developments in Mathematics. Springer, Berlin, 2011). Our technique leads to an essential extension of Tsaban–Zdomskyy's result. In particular, we prove that for a Čech-complete X the space $C_c(X)$ has the strong Pytkeev property if and only if X is Lindelöf. We study the strong Pytkeev property for several well known classes of locally convex spaces including (DF)-spaces and strict (LM)-spaces. Strengthening results from Cascales et al. (Proc Am Math Soc 131:3623–3631, 2003) and Dudley (Proc Am Math Soc 27:531–534, 1971) we deduce that the space of distributions $\mathfrak{D}'(\Omega)$ (which is not a k-space) has the strong

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Pytkeev property. We also show that any topological group with a $\mathfrak{G}$-base which is a k-space has already the strong Pytkeev property. We prove that, if X is an $\mathcal{MK}_\omega$-space, then the free abelian topological group $A(X)$ and the free locally convex space $L(X)$ have the strong Pytkeev property. We include various (counter) examples and pose a dozen open questions.

Keywords Topological group · Locally convex space · Fréchet–Urysohness · Sequentiality · k-space · $\mathfrak{G}$-base · The strong Pytkeev property

Mathematics Subject Classification (2010) 22A05 · 46A03 · 54C35 · 54H11

1 Introduction

All topological spaces and groups are assumed to be Hausdorff. A topological space Y is called *first countable* if it has a countable open base at each point. Any first countable topological group is metrisable. Various topological properties generalizing first countability have been studied intensively by topologists and analysts over the last half-century, especially Fréchet–Urysohness, sequentiality and countable tightness (see [10,21]). Pytkeev [33] proved that every sequential space satisfies the property which is stronger than countable tightness. Following [23], we say that a topological space Y has the *Pytkeev property* if for each $A \subseteq Y$ and each $y \in \overline{A}\backslash A$, there are infinite subsets $A_1, A_2, \ldots$ of A such that each neighbourhood of y contains some A_n. Tsaban and Zdomskyy [40] strengthened this property as follows. A topological space Y has the *strong Pytkeev property* if for each $y \in Y$, there exists a countable family $\mathcal{D}$ of subsets of Y, such that for each neighbourhood U of y and each $A \subseteq Y$ with $y \in \overline{A}\backslash A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite.

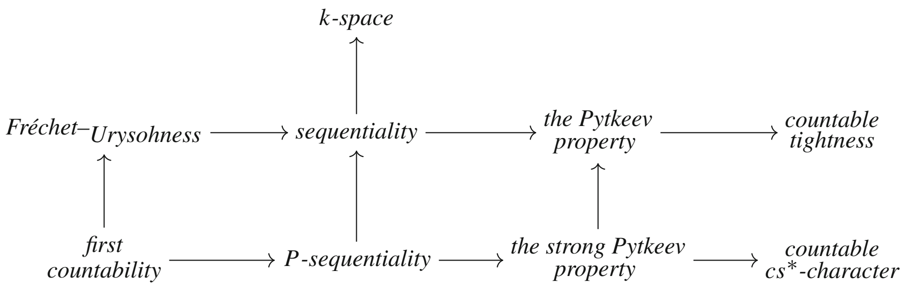
Our article is devoted to the study of topological groups with the strong Pytkeev property. It is convenient to formulate this property for topological groups as follows:

Definition 1 A topological group G has the *strong Pytkeev property* if there exists a sequence $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ of subsets of G satisfying the property

(**sp**) for each neighbourhood U of the unit e and each $A \subseteq G$ with $e \in \overline{A}\backslash A$, there is $n \in \mathbb{N}$ such that $D_n \subseteq U$ and $D_n \cap A$ is infinite.

A sequence of subsets of G which satisfies the property (**sp**) we shall call an *sp-sequence*.

Recall also (see [4]) that a topological space Y has *countable cs*-character* if for each $y \in Y$, there exists a countable family $\mathcal{D}$ of subsets of Y, such that for each non-trivial sequence in Y converging to y and each neighbourhood U of y, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and D contains infinitely many elements of that sequence. We call a topological space X to be a *P-sequential* if X is a sequential space satisfying the strong Pytkeev property. The following diagram summarizes relations between basic properties:



For a topological space X by $C(X)$ we mean the space of all continuous real-valued functions on X. The space $C(X)$ endowed with the compact-open topology we denote by $C_c(X)$. It is well known (see [26, 4.4.2]) that the space $C_c(X)$ is first countable if and only if X is hemicompact. Evidently we have:

Proposition 1 *If X is hemicompact, then the space $C_c(X)$ has the strong Pytkeev property.*

Note that a Polish space X is hemicompact if and only if X is locally compact. The following, non-trivial theorem is the main result of [40].

Theorem 1 [40] *For each Polish space X, the space $C_c(X)$ has the strong Pytkeev property.*

Sakai [36] proved that $C_p(X)$ (the space $C(X)$ endowed with the topology of pointwise convergence) has the strong Pytkeev property if and only if X is countable. Note that $C_p(X)$ over an uncountable compact scattered space X is Fréchet–Urysohn by [1, III.1.2]. Hence there are Fréchet–Urysohn abelian groups which do not have the strong Pytkeev property (see also Example 1). On the other hand, there are abelian groups having the strong Pytkeev property but which are not a k-space (see Remark 9). So, in general

$$\textit{Fréchet–Urysohness} \;\not\Longrightarrow\; the\; strong\; Pytkeev\; property \;\not\Longrightarrow\; k-space.$$

In view of Example 5, we also have

$$\begin{array}{c}\textit{Fréchet–}\\ \textit{Urysohness}\end{array} \begin{array}{c}\not\longrightarrow\\ \not\longleftarrow\end{array} P\text{-}sequentiality \begin{array}{c}\longrightarrow\\ \not\longleftarrow\end{array} \begin{array}{c}\textit{the strong Pytkeev}\\ \textit{property}\end{array}.$$

All these relations between aforementioned topological properties are valid for general topological spaces. Let us mention that Banakh and Zdomskyy [4, Theorem 1] obtained very strong results about the structure of P-sequential topological groups, though we do not use these results in our paper.

Two important closely related notions defined below in Definitions 2 and 3 play a crucial role in all our main results and their proofs.

We consider the product $\mathbb{N}^{\mathbb{N}}$ with the natural partial order, i.e., $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i \in \mathbb{N}$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$.

Definition 2 [17] Let G be a topological group. A family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighbourhoods of the unit is called a $\mathfrak{G}$-*base* if $\mathcal{U}$ is a base of neighbourhoods at the unit and $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$.

Originally, the above concept has been formally introduced in [12] in the frame of locally convex spaces (LCS in short) for studying (DF)-spaces, $C(X)$-spaces, and spaces in the class $\mathfrak{G}$ in the sense of Cascales and Orihuela (see [6] and the monograph [21]). Clearly, every metrisable group G admits a natural $\mathfrak{G}$-base $\mathcal{U} = \{U_{\alpha_1} : \alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}\}$, where $\{U_n\}_{n\in\mathbb{N}}$ forms a decreasing base of neighbourhoods at the unit e of G. Note that there are many nonmetrisable topological groups with a $\mathfrak{G}$-base (see [17]).

Surprisingly, as it will be noted in Theorem 5, every topological group G enjoying the strong Pytkeev property satisfies also a condition which seems to be "close" to have a $\mathfrak{G}$-base; namely, G admits a base of neighbourhoods at e of the form $\{U_\alpha : \alpha \in \mathbf{M}\}$, where $\mathbf{M}$ is a subset of the partially ordered set $\mathbb{N}^{\mathbb{N}}$ and $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$ for $\alpha, \beta \in \mathbf{M}$.

The following concept is due to Christensen:

Definition 3 [7] A family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets of a topological space X is called a *compact resolution* if $X = \bigcup\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and $K_\alpha \subseteq K_\beta$ for every $\alpha \le \beta$. If additionally, every compact set in X is contained in some K_α we will say that $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *swallows the compact sets* of X.

Any Polish space X has a compact resolution swallowing the compact sets of X. Moreover, Christensen [7, Theorem 3.3] proved that if X is a metrisable topological space, then X is a Polish space if and only if X has a compact resolution swallowing the compact sets of X. In particular, the space $\mathbb{N}^{\mathbb{N}}$, being Polish under the product topology of the discrete space $\mathbb{N}$, has a compact resolution swallowing the compact sets of $\mathbb{N}^{\mathbb{N}}$. It is easy to see that each hemicompact space X has a compact resolution swallowing its compact sets (see [17,39]).

It turns out (see [11] or [17]) that for a Tychonoff space X, the space $C_c(X)$ has a $\mathfrak{G}$-base if and only if X has a compact resolution that swallows the compact sets of X. In particular, for every Polish space X the space $C_c(X)$ has a $\mathfrak{G}$-base.

Let X be a topological space. Recall that a subset A of X is called *functionally bounded* if $f(A)$ is a bounded subset of $\mathbb{R}$ for every $f \in C(X)$. The space X is called a μ-*space* if every functionally bounded subset of X has compact closure. It is well known that every Lindelöf space is a μ-space (see Lemma 7 below). In particular, any Polish or hemicompact space is a μ-space. Taking into account the aforementioned Christensen's theorem, the next theorem essentially generalizes both Proposition 1 and Theorem 1.

Theorem 2 *Let X be a Tychonoff space with a compact resolution swallowing the compact sets of X. Then the following assertions are equivalent:*

(i) *$C_c(X)$ has the strong Pytkeev property.*
(ii) *$C_c(X)$ has the Pytkeev property.*
(iii) *$C_c(X)$ has countable tightness.*
(iv) *$C_p(X)$ has countable tightness.*
(v) *$C_c(X)$ is barrelled.*
(vi) *X is Lindelöf.*
(vii) *X is a μ-space.*

An important feature of our proof of this theorem is an explicit construction of an sp-sequence. In fact, as it will be explained in the proof of Theorem 2, (iii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) holds for any X. Moreover, Arhangel'skii–Pytkeev's theorem asserts that the space $C_p(X)$ has countable tightness if and only if all finite powers of X are Lindelöf [1, II.1.1]. However, (iv) does not imply in general (iii) even for a countable space X (see [25, Proposition 15]).

One cannot remove in Theorem 2 the assumption that X has a compact resolution swallowing the compact sets of X. Indeed, in Example 3 we show that there is a Lindelöf space X without a compact resolution swallowing the compact sets of X such that $C_c(X)$ has the Pytkeev property but $C_c(X)$ does not have the strong Pytkeev property (cf. [40, Corollary 3.8]).

Since any Čech-complete Lindelöf space X has a compact resolution swallowing its compact sets, Theorem 2 implies

Corollary 1 *Let X be a Čech-complete space. Then $C_c(X)$ has the strong Pytkeev property if and only if X is Lindelöf.*

Corollary 2 *Let X be a locally compact space. Then $C_c(X)$ has the strong Pytkeev property if and only if X is hemicompact.*

In the next theorem we consider some other well known classes of LCS (all relevant definitions are given in Sect. 5).

Theorem 3 (i) *If a (DF)-space E has countable tightness (in particular, E is sequential), then E has the strong Pytkeev property.*
(ii) *Every strict (LM)-space has the strong Pytkeev property.*
(iii) *Let $(E', \beta(E', E))$ be the strong dual of a strict (LF)-space E. Then the space $(E', \beta(E', E))$ has a $\mathfrak{G}$-base. If in addition $(E', \beta(E', E))$ is quasibarrelled, then $(E', \beta(E', E))$ has the strong Pytkeev property.*
(iv) *Let E be a dual metric space. If E is sequential, then E has the strong Pytkeev property.*

As a corollary of our results we deduce that the space of distributions $\mathfrak{D}'(\Omega)$ over an open set $\Omega \subset \mathbb{R}^n$ has the strong Pytkeev property and is not a k-space, that substantially strengthens results from [5,9].

Our next principal result is the following:

Theorem 4 *Let G be a topological group with a $\mathfrak{G}$-base. If G is a k-space, then G has the strong Pytkeev property. Hence G is P-sequential.*

The last theorem fails without additional assumptions on G as there exist topological groups with a $\mathfrak{G}$-base which do not have the strong Pytkeev property [see Remark 7(i)].

Theorem 4 provides several interesting corollaries. The first one partially answers Question 39 from [17].

Corollary 3 *Let a topological group G be Baire. Then G is metrisable if and only if G has a $\mathfrak{G}$-base and is a k-space.*

We extend our interest to the study of the strong Pytkeev property for several important classes of locally convex spaces. Denote by Q the Hilbert cube $[0, 1]^{\mathbb{N}}$ and by ϕ the strict inductive limit of the sequence $(\mathbb{R}^n)_n$. It is known that ϕ is a sequential non-Fréchet–Urysohn LCS (see [28]) which is an $\aleph_0$-dimensional Montel (DF)-space (as the strong dual space of the direct product $\mathbb{R}^{\mathbb{N}}$). The next corollary extends Theorems 4.5 and 4.6 from [22].

Corollary 4 *Let E be a LCS with a $\mathfrak{G}$-base. Then the following assertions are equivalent:*

(i) *E is a k-space.*
(ii) *E is metrisable or E is homeomorphic to ϕ or $\phi \times Q$.*

Since the strong dual $(E', \beta(E', E))$ of any strict (LF)-space has a $\mathfrak{G}$-base by Theorem 3(iii), Corollary 4 immediately implies:

Corollary 5 *Let E be the strong dual of a strict (LF)-space. Then the following assertions are equivalent:*

(i) *E is a k-space.*
(ii) *E is metrisable or E is homeomorphic to ϕ or $\phi \times Q$.*

Next corollary partially answers Banakh's question [2, Question].

Corollary 6 *Let E be a Montel strict (LF)-space. Then the strong dual F of E either is metrisable or is homeomorphic to a closed subspace of $\phi^{\mathbb{N}}$.*

2 A necessary condition for the strong Pytkeev property

In this section we show that topological groups with the strong Pytkeev property are close to groups with a $\mathfrak{G}$-base.

Lemma 1 *Let G be a topological group with the strong Pytkeev property and an sp-sequence $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$. Then for every neighbourhood U of the unit e there is $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that the set $N_\alpha := \bigcup_{i\in\mathbb{N}} \left(D_{\alpha_i} \cup \{e\}\right)$ is a neighbourhood of e and $N_\alpha \subseteq U$.*

Proof Set $J := \{n \in \mathbb{N} : D_n \subseteq U\}$. Then $J = \{\alpha_i\}_{i\in\mathbb{N}}$, where $\alpha_1 < \alpha_2 < \cdots$. Set $\alpha := (\alpha_i)_{i\in\mathbb{N}}$. Hence $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $N_\alpha \subseteq U$. We have to show that N_α is a neighbourhood of e.

Suppose for a contradiction that $e \in \mathrm{cl}(U\setminus N_\alpha)$. Since $e \notin U\setminus N_\alpha$, by definition, there exists $m \in \mathbb{N}$ such that $D_m \subseteq U$ and $D_m \cap (U\setminus N_\alpha)$ is infinite. But this contradicts the choice of the set J and the definition of N_α. Thus N_α is a neighbourhood of e.

Applying Lemma 1 we prove the following necessary condition for topological groups satisfying the strong Pytkeev property.

Theorem 5 *Let G be a topological group with the strong Pytkeev property. Then G has a base $\{U_\alpha : \alpha \in \mathbf{M}\}$ of neighbourhoods at e, where*

(i) $\mathbf{M}$ *is a subset of the partially ordered set* $\mathbb{N}^\mathbb{N}$;
(ii) *if* $\alpha \in \mathbf{M}$ *and* $\beta \in \mathbb{N}^\mathbb{N}$ *are such that* $\beta \leq \alpha$, *then* $\beta \in \mathbf{M}$;
(iii) $U_\beta \subseteq U_\alpha$, *whenever* $\alpha \leq \beta$ *for* $\alpha, \beta \in \mathbf{M}$.

A topological group G will be called a *quasi-$\mathfrak{G}$-group* or a *group with a quasi-$\mathfrak{G}$-base* if it has a base of neighbourhoods at e of the form $\{U_\alpha : \alpha \in \mathbf{M}\}$, where $\mathbf{M}$ satisfies conditions (i)–(iii) of Theorem 5.

Proof Let $\{D_n\}_{n\in\mathbb{N}}$ be an sp-sequence for G. Without loss of generality we suppose that $e \in D_n$ for every $n \in \mathbb{N}$. For every $k, i \in \mathbb{N}$, set

$$D_k^i := \bigcap_{l=1}^{k} D_{i-1+l}.$$

So for each $i \in \mathbb{N}$ the sequence $\{D_k^i\}_{k\in\mathbb{N}}$ is decreasing. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$, set

$$U_\alpha := \bigcup_{i\in\mathbb{N}} D_{\alpha_i}^i.$$

Clearly, $U_\alpha \subseteq U_\beta$ for each $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ with $\beta \leq \alpha$.

Take an increasing sequence $0 = n_0 < n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that $\bigcup_{k\in\mathbb{N}} D_{n_k}$ is a neighbourhood at e (which exists by Lemma 1). We claim that there exists $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ such that $U_\alpha = \bigcup_{k\in\mathbb{N}} D_{n_k}$. Indeed, if $i = n_k$ for some $k \in \mathbb{N}$ we set $\alpha_i = 1$. So $D_{\alpha_i}^i = D_{n_k}$. Now, if $n_{k-1} < i < n_k$ for some $k \in \mathbb{N}$, we set $\alpha_i := n_k - i + 1$. Then

$$D_{\alpha_i}^i = \bigcap_{l=1}^{\alpha_i} D_{i-1+l} \subseteq D_{n_k}.$$

Thus $U_\alpha = \bigcup_{k\in\mathbb{N}} D_{n_k}$.

Set $\mathbf{M} := \{\alpha \in \mathbb{N}^\mathbb{N} : U_\alpha \text{ is a neighbourhood of } e\}$. Now Lemma 1 implies that the set $\{U_\alpha : \alpha \in \mathbf{M}\}$ forms a base at e satisfying (iii). Let us show that (ii) holds.

Indeed, let $\alpha \in \mathbf{M}$ and $\beta \in \mathbb{N}^\mathbb{N}$ be such that $\beta \leq \alpha$. Since, by construction, $U_\alpha \subseteq U_\beta$ we obtain that U_β is a neighbourhoods at e. So $\beta \in \mathbf{M}$. Thus G is a quasi-$\mathfrak{G}$-group.

The character of a topological group G we denote by $\chi(G)$. As an immediate corollary of Theorem 5 we note the following:

Corollary 7 *If a topological group G has the strong Pytkeev property, then* $\chi(G) \leq \mathfrak{c}$.

In light of Theorem 5 one can suggest that each topological group with the strong Pytkeev property has a $\mathfrak{G}$-base. However, in spite of numerous efforts the following problem seems to be very interesting.

Question 1 Let G be a topological group (or a topological vector space) with the strong Pytkeev property. Does G admit a $\mathfrak{G}$-base?

Remark 1 By a very recent result of Banakh [3], the answer to this question is negative, see Remark 3 below.

The following example provides a lot of Fréchet–Urysohn groups without a $\mathfrak{G}$-base and without the strong Pytkeev property.

Example 1 Let G be a non-trivial metrisable group and κ be an arbitrary uncountable cardinal. The non-metrisable group G^κ contains a non-metrisable compact subset which is homeomorphic to $\{0,1\}^\kappa$. So G^κ does not have a $\mathfrak{G}$-base by [17, Theorem 29]. Let H be a subgroup of G^κ consisting of the elements with countable support. Then H is Fréchet–Urysohn by [10, 3.10.D]. Since H is a dense subgroup of G^κ, H does not have a $\mathfrak{G}$-base by [17, Proposition 9]. We show that the group H does not have the strong Pytkeev property. Indeed, otherwise, H would be metrisable by [4, Theorem 3]. Hence the completion G^κ of H would be also metrisable, a contradiction. Note that, if in addition, G is compact, then H is sequentially compact by [10, Example 3.10.17].

3 A sufficient condition for the strong Pytkeev property

In this section we prove a useful sufficient condition for topological groups satisfying the strong Pytkeev property and give two applications (see Theorem 7 and Proposition 2). We need some technical notations.

Let G be a topological group with a $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set

$$D_k(\alpha) := \bigcap_{\beta\in I_k(\alpha)} U_\beta, \text{ where } I_k(\alpha) := \left\{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1, \dots, k\right\}.$$

For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set $K_\alpha := \prod_{i\in\mathbb{N}}[1, \alpha_i] \subset \mathbb{N}^\mathbb{N}$, and

$$L_0(\alpha) := \mathbb{N}^\mathbb{N} \quad \text{and} \quad L_k(\alpha) := \bigcup_{\beta\in I_k(\alpha)} K_\beta = \prod_{i=1}^{k}[1, \alpha_i] \times \mathbb{N}^{\mathbb{N}\setminus\{1,\dots,k\}}.$$

The following lemma generalizes Lemma 13 of [17].

Lemma 2 *Let $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and $\beta^{jk} = (\beta_i^{jk})_{i\in\mathbb{N}} \in L_{k-1}(\alpha)\setminus L_k(\alpha)$ for every $k \in \mathbb{N}$ and each $1 \le j \le s_k < \infty$. Then there is $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha \le \gamma$ and $\beta^{jk} \le \gamma$ for every $k \in \mathbb{N}$ and each $1 \le j \le s_k$.*

Proof For every $i \in \mathbb{N}$, set

$$\gamma_i = \max\{\alpha_i, \beta_i^{jk} : 1 \le k \le i, 1 \le j \le s_k\} = \max\{\alpha_i, \beta_i^{jk} : k \in \mathbb{N}, 1 \le j \le s_k\}.$$

Clearly, $\gamma := (\gamma_i)_{i\in\mathbb{N}}$ is as desired.

Let G be a topological group with a $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha\}$. For every $\alpha \in \mathbb{N}^\mathbb{N}$ and every $k \in \mathbb{N}$, we note that $D_k(\alpha) \subset U_\alpha$. Clearly, $D_k(\alpha) \subseteq D_m(\alpha)$ for every natural numbers $k \le m$. Set $D_0(\alpha) := \{e\}$ for every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$. Finally we define

$$\mathcal{D} := \{D_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N}\}.$$

The crucial role of the family $\mathcal{D}$ has been proved by the following theorem.

Theorem 6 *Let G be a topological group with a $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Assume that the following condition holds*

(**D**) *for every $\alpha \in \mathbb{N}^\mathbb{N}$ the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighbourhood of the unit e of G.*

Then the group G has the strong Pytkeev property and the family $\mathcal{D} = \{D_k(\alpha)\}$ is an sp-sequence for G.

Proof Let $A \subseteq G$ be such that $e \in \overline{A}\setminus A$. So, for every $\alpha \in \mathbb{N}^\mathbb{N}$, the set $A \cap U_\alpha$ is infinite. By our hypothesis, the set $W := \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is a neighbourhood of the unit e. So the intersection

$$A \cap \bigcup_{k\in\mathbb{N}} D_k(\alpha) = \bigcup_{k\in\mathbb{N}} \left(A \cap \left[D_k(\alpha)\setminus D_{k-1}(\alpha)\right]\right)$$

is infinite as well. For every $k \in \mathbb{N}$, set $A_k := A \cap \left[D_k(\alpha)\setminus D_{k-1}(\alpha)\right]$. To prove the theorem it is enough to show that there exists $k \in \mathbb{N}$ such that A_k is infinite.

Suppose for a contradiction that the set A_k is finite for every $k \in \mathbb{N}$. So there exists an infinite subset I of $\mathbb{N}$ such that $A_k = \left\{a_1^k, \dots, a_{s_k}^k\right\}$ for every $k \in I$ for some natural number s_k, and $A_k = \emptyset$ if $k \notin I$. By the definition of $D_k(\alpha)$, for every $k \in I$ we select

$$\beta^{jk} = (\beta_i^{jk})_{i\in\mathbb{N}} \in I_{k-1}(\alpha)$$

such that $a_j^k \notin U_{\beta^{jk}}$ for every $1 \le j \le s_k$. By Lemma 2, there exists $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\alpha \le \gamma$ and $\beta^{jk} \le \gamma$ for every $k \in I$ and each $1 \le j \le s_k$. Hence $a_j^k \notin U_\gamma$ for every $k \in I$ and each $1 \le j \le s_k$. Since W is a neighbourhood of e, there exists $\delta \in \mathbb{N}^\mathbb{N}$, $\gamma \le \delta$, such that $U_\delta \subset W$. Hence $A \cap U_\delta$ is empty. Thus $e \notin \overline{A}\setminus A$, a contradiction.

Hence, to prove that a topological group G with a $\mathfrak{G}$-base $\mathcal{U}$ has the strong Pytkeev property it is enough to show that $\mathcal{U}$ satisfies the condition (**D**).

In the article we deal with different topological groups having a $\mathfrak{G}$-base. It appears that all those groups enjoy also the strong Pytkeev property due to the fact that $\mathfrak{G}$-bases satisfy the condition (**D**). We do not know whether the converse holds true.

Question 2 Assume that a topological group G with a $\mathfrak{G}$-base $\mathcal{U}$ has the strong Pytkeev property. Does $\mathcal{U}$ satisfy the condition (**D**)?

In what follows we need additional notations.

Let $\mathbf{0}$ denote the zero function on a topological space X. For a subset A of X and $\epsilon > 0$, we set

$$[A,\epsilon] = \left\{ f \in C(X) : \sup_{x\in A} |f(x)| < \epsilon \right\}.$$

The family of sets of the form $[K,\epsilon]$, where K is a compact subset of X and $\epsilon > 0$, forms a local base at $\mathbf{0}$ in $C_c(X)$.

Our first application of Theorem 6 is a direct and simple proof of Theorem 1 presented for the case $X = \mathbb{N}^{\mathbb{N}}$. Below we describe the topology of $C_c(\mathbb{N}^{\mathbb{N}})$ in details.

For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and each $k \in \mathbb{N}$, we set

$$U_\alpha := \left[K_\alpha, \alpha_1^{-1} \right], \quad \text{where } K_\alpha = \prod_{i\in\mathbb{N}} [1,\alpha_i] \subset \mathbb{N}^{\mathbb{N}}.$$

Clearly, each compact subset K of $\mathbb{N}^{\mathbb{N}}$ is contained in K_α for some $\alpha \in \mathbb{N}^{\mathbb{N}}$. So the family $\mathcal{U} := \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base at $\mathbf{0}$ in $C_c(\mathbb{N}^{\mathbb{N}})$.

Note that for every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and each $k \in \mathbb{N}$, we have

$$D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)} U_\beta = \left\{ f \in C(\mathbb{N}^{\mathbb{N}}) : |f(\gamma)| < \alpha_1^{-1}, \ \forall \gamma \in L_k(\alpha) \right\}.$$

We need also the following observation.

Lemma 3 *Let $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $f \in C(\mathbb{N}^{\mathbb{N}})$. If $|f(\gamma)| < \epsilon$ for every $\gamma \in K_\alpha$, then there exists $m \in \mathbb{N}$ such that*

$$|f(\gamma)| < \epsilon, \quad \forall\, \gamma \in L_m(\alpha).$$

In particular, $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$.

Proof Since f is continuous, for every $\gamma = (\gamma_i) \in K_\alpha$ we can choose $k_\gamma \in \mathbb{N}$ such that $|f(\beta)| < \epsilon$ for each $\beta \in W_\gamma$, where

$$W_\gamma := \{\gamma_1\} \times \cdots \times \{\gamma_{k_\gamma}\} \times \mathbb{N}^{\mathbb{N}\setminus\{1,\ldots,k_\gamma\}}.$$

Since K_α is compact, we can find $\gamma^1, \ldots, \gamma^s \in K_\alpha$ such that $K_\alpha \subset \bigcup_{j=1}^s W_{\gamma^j}$. Set $m := \max\{k_{\gamma^1}, \ldots, k_{\gamma^s}\}$. Let us show that m is as desired.

Take arbitrarily $\beta = (\beta_i) \in L_m(\alpha)$. Set $\beta' = (\beta_i')$, where $\beta_i' = \beta_i$ if $1 \le i \le m$, and $\beta_i' = 1$ otherwise. Clearly, $\beta' \in K_\alpha$. Choose j with $1 \le j \le s$, such that $\beta' \in W_{\gamma^j}$. This means that $\beta_i' = \beta_i = \gamma_i^j$ for every i such that $1 \le i \le k_{\gamma^j} \le m$. By the construction of W_γ, we conclude that $\beta \in W_{\gamma^j}$. Thus $|f(\beta)| < \epsilon$.

Tsaban and Zdomskyy [40] proved Theorem 1 firstly for the (most important and difficult) case $X = \mathbb{N}^{\mathbb{N}}$: The space $C_c(\mathbb{N}^{\mathbb{N}})$ has the strong Pytkeev property. Using a different argument, we propose below a simple direct proof of this fact even in a more general form describing explicitly an sp-sequence for $C_c(\mathbb{N}^{\mathbb{N}})$.

Theorem 7 *The space $C_c(\mathbb{N}^{\mathbb{N}})$ has the strong Pytkeev property. More precisely, the family $\mathcal{D} := \{D_k(\alpha)\}_{k\in\mathbb{N}}$ is an sp-sequence for $C_c(\mathbb{N}^{\mathbb{N}})$ and $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ for every $\alpha \in \mathbb{N}^{\mathbb{N}}$.*

Proof By Lemma 3, the space $C_c(\mathbb{N}^{\mathbb{N}})$ with the $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha\}$ satisfies condition (**D**). Thus $C_c(\mathbb{N}^{\mathbb{N}})$ has the strong Pytkeev property by Theorem 6.

Our second application of Theorem 6 refers to the restricted direct product of metrisable groups endowed with the box topology. Let $(G_n, \tau_n)_{n\in\mathbb{N}}$ be a sequence of topological groups. The family of subsets of the form $\prod_{n\in\mathbb{N}} U_n$, where $U_n \in \mathcal{N}(G_n)$, forms a base for a group topology on the direct product $G := \prod_{n\in\mathbb{N}} G_n$. This topology is called *the box topology* and is denoted by τ_b. The *restricted direct product* G_0 of $\{(G_n, \tau_n)\}_{n\in\mathbb{N}}$ is the subgroup of G consisting of all sequences with finite support. Clearly, G_0 is a closed subgroup of G.

In order to prove Proposition 2 we need two simple lemmas.

Lemma 4 *Let G be a topological group with the strong Pytkeev property and an sp-sequence $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$. Then*

(i) *The sequence $\overline{\mathcal{D}} = \{\overline{D_n}\}_{n\in\mathbb{N}}$ is also an sp-sequence for G.*
(ii) *Every subgroup H of G has the strong Pytkeev property.*

Proof (i) Let A be a subset of G such that $e \in \overline{A}\setminus A$. Fix a neighbourhood U of e. Take an open neighborhood V of e such that $\overline{V} \subseteq U$. Choose D_n such that $D_n \subseteq V$ and $D_n \cap A$ is infinite. Clearly, $\overline{D_n} \subseteq U$ and $\overline{D_n} \cap A$ is infinite.
(ii) It is clear that the strong Pytkeev property of H is witnessed by the sequence $\{D_n \cap H\}_{n\in\mathbb{N}}$.

Lemma 5 *Let G be a topological group with a $\mathfrak{G}$-base satisfying the condition* (**D**). *Then each subgroup H of G also has a $\mathfrak{G}$-base satisfying the condition* (**D**).

Proof Clearly, H has a $\mathfrak{G}$-base $\{U_\alpha \cap H\}$. This base satisfies the condition (**D**) by the following

$$\bigcup_{k\in\mathbb{N}} D_k^H(\alpha) = \bigcup_{k\in\mathbb{N}} \left(\bigcap_{\beta\in I_k(\alpha)} (U_\alpha \cap H) \right) = H \cap \left(\bigcup_{k\in\mathbb{N}} D_k(\alpha) \right).$$

Next proposition shows in particular that there are topological groups with a $\mathfrak{G}$-base which do not satisfy the condition (**D**) (see also Example 6 below).

Proposition 2 *Let $\{G_i\}_{i\in\mathbb{N}}$ be a sequence of metrisable groups such that infinitely many of G_i are not discrete, and let G be the direct product and G_0 the restricted direct product of this sequence endowed with the box topology τ_b. Then G has a natural $\mathfrak{G}$-base. For a subgroup H of G the following assertions are equivalent:*

(i) *H has the strong Pytkeev property.*
(ii) *H has the Pytkeev property.*
(iii) *H has countable tightness.*
(iv) *$H \cap G_0$ is an open subgroup of H.*
(v) *H has a $\mathfrak{G}$-base which satisfies the condition* (**D**).

In particular, G does not have countable tightness.

Proof We note first that G, and hence also H, has a natural open $\mathfrak{G}$-base which is defined as follows (see [17, Proposition 12]). Let r_i be a metric in G_i and e_i be the unit in G_i. For every $n \in \mathbb{N}$, set $U^i_n := \{g_i \in G_i : r_i(g_i, e_i) < \frac{1}{n}\}$. Then the family $\mathcal{U}$ of the sets

$$U_\alpha := \prod_{i\in\mathbb{N}} U^i_{\alpha_i}, \quad \forall\, \alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}},$$

forms a natural open $\mathfrak{G}$-base in G. Note that, for every $\alpha \in \mathbb{N}^{\mathbb{N}}$, we have

$$D_k(\alpha) = \bigcap_{\beta\in I_k(\alpha)} U_\beta = \prod_{i=1}^{k} U_{\alpha_i} \times \prod_{i>k}\{e_i\}. \tag{1}$$

(i)⇒(ii) and (ii)⇒(iii) are clear.

(iii)⇒(iv). Suppose for a contradiction that $H \cap G_0$ is not open in H. Set $A := H\backslash(H \cap G_0)$. Then $e \in \overline{A}\backslash A$. To obtain a contradiction it is enough to show that for each countable subset $B = \{\mathbf{g}^n\}_{n\in\mathbb{N}}$ of A there is a neighbourhood U of e such that $B \cap U = \emptyset$.

Let $\mathbf{g}^n = (g^n_k)_{k\in\mathbb{N}}$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, set

$$J_n := \{i : 1 \le i \le n \text{ and } g^i_n \ne e_n\}.$$

If J_n is not empty we choose $U_n \in \mathcal{N}(G_n)$ such that $g^i_n \notin U_n$ for every $i \in J_n$, and if $J_n = \emptyset$ we put $U_n := G_n$. Set $U := \prod_{n\in\mathbb{N}} U_n$. Then U is an open neighbourhood of e. As each $\mathbf{g}^n$ has an infinite support, we obtain that $\mathbf{g}^n \notin U$ for every $n \in \mathbb{N}$. Hence $B \cap U = \emptyset$. So H does not have countable tightness. This contradiction shows that $H \cap G_0$ is an open subgroup of H.

(iv)⇒(v). Since $H \cap G_0$ is open in H, Lemma 5 shows that it is enough to prove that G_0 with the $\mathfrak{G}$-base $\{G_0 \cap U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the condition (**D**). But this follows from (1) since

$$G_0 \cap \left(\bigcup_{k\in\mathbb{N}} D_k(\alpha)\right) = G_0 \cap U_\alpha.$$

(v)⇒(i) immediately follows from Theorem 6.

Example 2 It is known that the space ϕ is a sequential non-Fréchet–Urysohn space. Since ϕ is the restricted direct product of countably many $\mathbb{R}$, the space ϕ has the strong Pytkeev property by Proposition 2. So ϕ is a P-sequential space.

4 The strong Pytkeev property for $C_c(X)$

In this section we prove Theorem 2 and give some applications. In what follows we shall use more than once the following facts. For a subset B of a TVS E and a natural number k we set $kB := \{kx : x \in B\}$.

Lemma 6 [17] *Let E be a TVS and $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base in E consisting of closed and symmetric subsets. Then, for each bounded subset $B \subset E$, there exists $k \in \mathbb{N}$ such that $B \subseteq kD_k(\alpha)$. In particular, $E = \bigcup_k kD_k(\alpha)$ for every $\alpha \in \mathbb{N}^\mathbb{N}$.*

Proposition 3 [17] *Let G be a topological group with a $\mathfrak{G}$-base. Then G is a k-space if and only if it is sequential.*

Theorem 8 [11,17] *For a Tychonoff space X the following assertions are equivalent.*

(i) *$C_c(X)$ has a $\mathfrak{G}$-base.*
(ii) *X has a compact resolution that swallows the compact sets.*

Recall that a LCS E is called *quasibarrelled* if every closed absolutely convex subset of E which absorbs all bounded sets is a neighbourhood of zero (see [29]). Clearly, every metrisable or barrelled (in particular, Baire) LCS is quasibarrelled. Our proof of Theorem 2 relies on the following general fact.

Theorem 9 *Let E be a quasibarrelled LCS with a $\mathfrak{G}$-base. Then E has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ satisfying the condition* (**D**). *In particular, E has the strong Pytkeev property.*

Proof Let $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base. We may assume that the sets V_α are absolutely convex. For every $\alpha \in \mathbb{N}^\mathbb{N}$ we set $U_\alpha := \mathrm{cl}(V_\alpha)$. Then the family of the sets U_α forms a $\mathfrak{G}$-base of symmetric absolutely convex and absorbing closed sets in E [11,17].

Fix $\alpha \in \mathbb{N}^\mathbb{N}$ and set $B := \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ and $A := \mathrm{cl}(B)$. Then A is a symmetric closed absolutely convex set which absorbs all bounded sets in E by Lemma 6. So A is a quasibarrell. By assumption on E the set A is a neighbourhood of $\mathbf{0}$.

We claim that $A \subseteq 2B$. Indeed, take $x \notin 2B$. Then $\frac{1}{2}x \notin B$. Hence, for every $k \in \mathbb{N}$, there exists a $\beta^k \in I_k(\alpha)$ such that $\frac{1}{2}x \notin U_{\beta^k}$. By Lemma 2 we can choose $\beta \in \mathbb{N}^\mathbb{N}$ such that $\beta^k \le \beta$ and $\alpha \le \beta$. Then $\frac{1}{2}x \notin U_\beta$. To prove the claim it is enough to show that $x \notin A$. Suppose for a contradiction that x belongs to A. Then $x \in B + U_\beta$. So $x \in D_k(\alpha) + U_\beta$ for some $k \in \mathbb{N}$. Hence

$$x \in U_{\beta^k} + U_\beta \subseteq 2U_{\beta^k}.$$

Thus $\frac{1}{2}x \in U_{\beta^k}$. This contradiction proves the claim. Hence the set $B = \frac{1}{2}(2B)$ is a neighbourhood of $\mathbf{0}$, and so the $\mathfrak{G}$-base $\mathcal{U}$ satisfies the condition (**D**).

Now Theorem 6 implies that E has the strong Pytkeev property.

For convenience of the reader and for the sake of completeness of the paper we include a proof of the following well known result.

Lemma 7 *Any Lindelöf space X is a μ-space.*

Proof Let A be a functionally bounded subset of X. Since X is Lindelöf, the closure $\bar{A}$ is also Lindelöf. Hence $\bar{A}$ is realcompact [10, 3.11.12] and normal [10, 3.8.2]. If $f \in C(\bar{A})$, then Tietze–Urysohn's theorem implies that f can be extended to a continuous function $\bar{f}$ on X. So $f = \bar{f}|_{\bar{A}}$ is bounded. Hence $\bar{A}$ is pseudocompact. It follows from [10, 3.11.1] that $\bar{A}$ is compact. Thus X is a μ-space.

An open cover γ of X is called *finite-open* (*compact-open*) if every finite (compact) subset of X is contained in some member of γ. The next proposition was proved by Pytkeev [34, 1.13]. We add its simple proof for the sake of completeness.

Proposition 4 *Let X be a Tychonoff space. If $C_c(X)$ has countable tightness, then $C_p(X)$ has countable tightness as well.*

Proof We shall use Pytkeev's lemma [34, 1.12] (see also [21, Lemma 16.4]), namely:

(1) The space $C_p(X)$ has countable tightness if only if every finite-open cover of X has a countable finite-open subcover.
(2) The space $C_c(X)$ has countable tightness if only if every compact-open cover of X has a countable compact-open subcover.

Let γ be a finite-open cover of X. Denote by $\tilde{\gamma}$ a family which is obtained by taking all finite unions of members from γ. It is easy to see that $\tilde{\gamma}$ is a compact-open cover of X. By Pytkeev's lemma, there is a countable compact-open subcover $\tilde{\gamma}_0 \subset \tilde{\gamma}$. Then all those members of the family γ which are involved in the construction of members of $\tilde{\gamma}_0$ form a countable finite-open subcover. Applying Pytkeev's lemma once again we obtain that $C_p(X)$ has countable tightness.

Now we are in position to prove Theorem 2.

Proof of Theorem 2 Note that since X has a compact resolution swallowing the compact sets of X, the space $C_c(X)$ has a $\mathfrak{G}$-base by Theorem 8.

(i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear.
(iii) $\Rightarrow$ (iv) follows from Proposition 4.
(iv) $\Rightarrow$ (vi) follows from [1, II.1.1].
(vi) $\Rightarrow$ (vii) follows from Lemma 7.
(vii) $\Rightarrow$ (v) follows from [29, Theorem 10.1.20].
(v) $\Rightarrow$ (i) follows from Theorem 9. □

Theorem 2 applies, for example, to all Čech-complete Lindelöf spaces X (which was kindly reminded to us by Zdomskyy). In fact, every Čech-complete Lindelöf space X has a compact resolution swallowing its compact sets. This easily follows from the well known fact stating that X is a Čech-complete Lindelöf space if and only if it is a pre-image of a Polish space under a perfect surjective map, see [19, Corollary 3.7] and [7, Theorem 3.3]. Therefore this proves Corollary 1. Trivially, locally compact Lindelöf space is hemicompact, this proves Corollary 2.

Remark 2 Tkachuk [39] gave a surprising example of a locally compact, countably compact non-compact space X with a compact resolution swallowing its compact sets. Hence, X is not Lindelöf. So $C_c(X)$ does not have the strong Pytkeev property by Theorem 2.

Example 3 We show that the Pytkeev property does not imply the strong Pytkeev property in the class of spaces $C_c(X)$. For the class of spaces $C_p(X)$ such result is known [40, Corollary 3.8]. Let X_0 be an uncountable discrete space and let $X := X_0 \cup \{e\}$ be a one-point Lindelöfication of X_0. Since each compact subset of X is finite we obtain that $C_c(X) = C_p(X)$. Now Corollary 2.16 of [36] implies that $C_c(X)$ does not have the strong Pytkeev property. On the other hand, it is known (and easy to check) that the space $C_p(X)$ is homeomorphic to the Σ-product of $|X|$ copies of $\mathbb{R}$. So $C_p(X)$ is Fréchet–Urysohn by [10, 3.10.D]. Thus $C_c(X)$ has the Pytkeev property. Note also that the following fact easily follows from this example and Theorem 2: Any uncountable discrete space X_0 does not have a compact resolution swallowing its compact sets [7, Theorem 3.3].

For a Tychonoff space X we denote by $C_c^b(X)$ the subspace of $C_c(X)$ consisting of all bounded functions. It is easy to see that, if X is not compact, then the subset

$$\{f \in C(X) : |f(x)| \le 1, \ \forall x \in X\}$$

is a barrell in $C_c^b(X)$ which is not a neighbourhood of $\mathbf{0}$. Thus $C_c^b(X)$ is not barrelled.

If X is as in Theorem 2 the space $C_c(X)$ has the strong Pytkeev property if and only if $C_c(X)$ is barrelled. This motivates the following example.

Example 4 There is a LCS with the strong Pytkeev property but which is non-barrelled and non-sequential. Indeed, let $X := \mathbb{N}^{\mathbb{N}}$. Then $C_c^b(X)$ has the strong Pytkeev property by Lemma 4 and Theorem 2. The space $C_c^b(X)$ is not barrelled since X is not compact. The space $C_c^b(X)$ is not sequential, since it contains a closed subset $C_c(X, \{0, 1\})$ which is non-sequential by [20, Corollary 3.11]. Note also that $C_c^b(X)$ has a $\mathfrak{G}$-base and is not a k-space by Theorem 8 and Proposition 3.

Question 3 Let X be a non-separable metrisable space such that $C_c(X)$ has the Pytkeev property. Does $C_c(X)$ have also the strong Pytkeev property?

Let us note that the space of rationals $\mathbb{Q}$, although is σ-compact, does not admit a compact resolution swallowing the compact sets by [7, Theorem 3.3]. Hence $C_c(\mathbb{Q})$ does not have a $\mathfrak{G}$-base by Theorem 8.

Question 4 Does the space $C_c(\mathbb{Q})$ have the (strong) Pytkeev property?

Remark 3 Question 4 has been asked in the original version of the paper. Our paper was submitted for publication and, meanwhile, we continued discussion of this crucial problem with many mathematicians. Being inspired by our private conversation, Taras Banakh solved Question 4 by showing the following theorem (see [3]): for every separable metrisable space X, the space $C_c(X)$ has the strong Pytkeev property. This remarkable result gives also a negative answer to Question 1: the space $C_c(\mathbb{Q})$ has the strong Pytkeev property but does not have a $\mathfrak{G}$-base. Let us mention also that Banakh kindly informed us that Question 4 was already posed by L. Zdomskyy in 2007 on L'viv's topological seminar.

Now we apply the obtained results to the important classes of free LCS and free abelian topological groups. The following concept is due to Markov [24], see also Graev [18].

Definition 4 Let X be a Tychonoff space. An abelian topological group $A(X)$ is called *the (Markov) free abelian topological group* over X if $A(X)$ satisfies the following conditions:

(i) There is a continuous mapping $i : X \to A(X)$ such that $i(X)$ algebraically generates $A(X)$.
(ii) If $f : X \to G$ is a continuous mapping to an abelian topological group G, then there exists a continuous homomorphism $\bar{f} : A(X) \to G$ such that $f = \bar{f} \circ i$.

The topological group $A(X)$ always exists and is essentially unique. Note that the mapping i is a topological embedding [18,24].

Analogously we can define free LCS.

Definition 5 [13,14,24,35,41] Let X be a Tychonoff space. The *free LCS* $L(X)$ on X is a pair consisting of a LCS $L(X)$ and a continuous mapping $i : X \to L(X)$ such that every continuous mapping f from X to a LCS E gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$.

Also the free LCS $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding [13,14,35,41]. The identity map $id_X : X \to X$ extends to a canonical homomorphism $id_{A(X)} : A(X) \to L(X)$. It is known that $id_{A(X)}$ is an embedding of topological groups [38,42].

Recall that a topological space X is called an *$\mathcal{MK}_\omega$-space* if there exists an increasing sequence of compact metrisable subspaces $\{C_n\}_{n\in\mathbb{N}}$ covering X such that a subset $U \subseteq X$ is open in X if and only if $U \cap C_n$ is open in C_n for every $n \in \mathbb{N}$.

Proposition 5 *Let X be an $\mathcal{MK}_\omega$-space. Then $A(X)$ and $L(X)$ have a $\mathfrak{G}$-base satisfying the condition* (**D**) *and therefore have the strong Pytkeev property.*

Proof Since X is a hemicompact k-space, $C_c(X)$ is completely metrisable by [26, 5.8.1]. Moreover, X has a countable k-network, therefore, the space $C_c(X)$ has also a countable network by [26, 4.1.3]. So $C_c(X)$ is separable, and hence it is a Polish space. Therefore $C_c(C_c(X))$ has a $\mathfrak{G}$-base $\mathcal{U}$ by Theorem 8. The $\mathfrak{G}$-base $\mathcal{U}$ satisfies the condition (**D**) by Theorem 9. Further, $A(X)$ and $L(X)$ are closed subgroups of $C_c(C_c(X))$ by [13,14,41]. Thus $A(X)$ and $L(X)$ have $\mathfrak{G}$-bases satisfying the condition (**D**) by Lemma 5. Finally, $A(X)$ and $L(X)$ have the strong Pytkeev property by Theorem 6 (or alternatively by Theorem 2 and Lemma 4).

Example 5 Denote by $\mathbf{e} = \{e_n\}_{\mathbb{N}}$ the sequence in the direct sum $\mathbb{Z}^{(\mathbb{N})}$ with $e_1 = (1, 0, 0, \ldots)$, $e_2 = (0, 1, 0, \ldots), \ldots$. Then $\mathbf{e}$ converges to zero in the topology induced on $\mathbb{Z}^{(\mathbb{N})}$ by the product topology on $(\mathbb{Z}_d)^{\mathbb{N}}$, where $\mathbb{Z}_d$ is the group of integers $\mathbb{Z}$ endowed with the discrete topology. Denote by $\tau_{\mathbf{e}}$ the finest group topology on $\mathbb{Z}^{(\mathbb{N})}$ in which $e_n \to 0$ (the topology $\tau_{\mathbf{e}}$ is described in [15]). If $\mathbf{s} = \{\frac{1}{n}\} \cup \{0\}$ with the usual topology induced from $\mathbb{R}$, then $A(\mathbf{s}) = \left(\mathbb{Z}^{(\mathbb{N})}, \tau_{\mathbf{e}}\right)$ (see [15]). Hence $A(\mathbf{s})$ is a countable sequential non-Fréchet–Urysohn and hemicompact group by [31, 2.3.1, 2.3.9 and 4.1.5]. The group $A(\mathbf{s})$ has a $\mathfrak{G}$-base and the strong Pytkeev property by Proposition 5. So $A(\mathbf{s})$ is P-sequential.

5 Applications to LCS

Recall definitions of some classes of LCS which are used in what follows. A LCS E is called a *Fréchet space* if it is metrisable and complete. We say that a LCS E is *Montel* if it is barrelled and each closed bounded subset of E is compact. A LCS E is called a *strict (LM)-space* (respectively, *strict* (LF)*-space* or *strict (LB)-space*) if E is a strict inductive limit of a sequence $E_1 \subseteq E_2 \subseteq \cdots$ of metrisable (respectively, Fréchet or Banach) spaces. It is well known that the sets of the form

$$U_\alpha := \bigcup_{k\in\mathbb{N}} \left(U^1_{\alpha_1} + U^2_{\alpha_2} + \cdots + U^k_{\alpha_k} \right), \qquad (2)$$

where $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, form a base at zero in E. Grothendieck's *(DF)-spaces* are those $\aleph_0$-quasibarrelled spaces having a fundamental sequence of bounded sets. Recall that the strong dual $(E', \beta(E', E))$ of a LCS E is the topological dual E' of E endowed with the strong topology $\beta(E', E)$ being the topology of the uniform convergence on all *bounded* sets of E, see [29]. For example, ϕ is the strong dual of $\mathbb{R}^{\mathbb{N}}$ and vice versa, $\mathbb{R}^{\mathbb{N}}$ is the strong dual of ϕ. It is well-known that E is a Montel (DF)-space if and only if E is the strong dual of a Fréchet–Montel space. A LCS E is said to be a *quasi-(LB)-space* if E admits a resolution consisting of Banach discs (a subset A of E is called a *Banach disk* if it is a bounded absolutely convex set in E such that $E_A := \bigcup_n nA$, endowed with the norm $\|\cdot\|_A$ given by the Minkowski gauge of A, is a Banach space). Every (LF)-space (in particular, every metrisable and complete LCS) is a quasi-(LB)-space as well as its strong dual [44, Propositions 5,6]. Valdivia [45] generalized the class of (DF)-spaces by selecting the class of *dual metric* spaces. A LCS E is *dual metric* if it has a fundamental sequence of bounded sets, and in addition E is ℓ^∞-quasibarrelled (i.e., every $\beta(E', E)$-bounded set in the topological dual E' is equicontinuous).

Now we are in position to prove Theorem 3. We note that item (i) of this theorem is also motivated by Example 6 (dealing with the (DF)-space $C_c[0, \omega_1)$).

Proof of Theorem 3 (i) First we note that E is quasibarrelled by [21, Proposition 16.4]. Since E is a (DF)-space, the strong dual F of E is metrisable. Let $\{U_n\}_n$ be a decreasing base of neighbourhoods of zero in F. Set $A_\alpha := \bigcap_i \alpha_i U_i$ for each $\alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}}$. Then $\{A_\alpha : \alpha \in \mathbb{N}\}$ is a resolution consisting of bounded sets in F swallowing the bounded sets of F. Recall that the polar U° of a neighbourhood U of zero in E is a bounded subset of F. So, taking into account that E is quasibarrelled, the family of polars $\{A^\circ_\alpha : \alpha \in \mathbb{N}\}$ in E forms a $\mathfrak{G}$-base in E (see [29, Proposition 3.1.4]). Now Theorem 9 applies.

(ii) It immediately follows from (2) that each strict (LM)-space E has a $\mathfrak{G}$-base. Since E is quasibarrelled and has a $\mathfrak{G}$-base, the space E has the strong Pytkeev property by Theorem 9.

(iii) Since E is a quasi-(LB)-space by [44, Proposition 5], Proposition 22 of [44] implies that there is a resolution $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in E consisting of Banach discs such that every Banach disc B of E is contained in some A_α. Since E is complete [37, II.6.6], the closure of any bounded set in E is a Banach disc [29, Theorem

5.1.6]. Hence $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows all bounded sets in E. Thus the polars in $(E', \beta(E', E))$ of the sets A_α form a $\mathfrak{G}$-base in $(E', \beta(E', E))$. The last assertion follows from Theorem 9.

(iv) Every sequential ℓ^∞-quasibarrelled space is quasibarrelled by [22, Theorem 4.4]. Now item (i) completes the proof. □

If $\Omega \subset \mathbb{R}^n$ is an open set, then the space of test functions $\mathfrak{D}(\Omega)$ is a complete Montel (LF)-space. As usual, $\mathfrak{D}'(\Omega)$ denotes its strong dual, the space of distributions. Dudley [9] proved that $\mathfrak{D}'(\Omega)$ is not a sequential space. It has been shown (see [5]) that the space of distributions $\mathfrak{D}'(\Omega)$ has countable tightness.

Applying our technique we are able to strengthen substantially both aforementioned results.

Corollary 8 *$\mathfrak{D}'(\Omega)$ is not a k-space. $\mathfrak{D}'(\Omega)$ has the strong Pytkeev property, and in particular, it has countable tightness.*

Proof Since $\mathfrak{D}'(\Omega)$ is the strong dual space of the strict (LF)-space $\mathfrak{D}(\Omega)$, it has a $\mathfrak{G}$-base by Theorem 3(iii). Now Proposition 3 and Dudley's theorem [9] imply that $\mathfrak{D}'(\Omega)$ is not a k-space. It is well known that the space $\mathfrak{D}'(\Omega)$ is a Montel space and hence is barrelled. Applying Theorem 3(iii) once again we obtain that $\mathfrak{D}'(\Omega)$ has the strong Pytkeev property.

Fréchet spaces E whose strong dual are quasibarrelled (equivalently bornological or barrelled [27, Proposition 25.12]) are called *distinguished* spaces. This important class of Fréchet spaces has been intensively studied, see [21,29] and [43] for additional references and properties of such spaces. Theorem 3 yields the following sequential characterization of distinguished spaces.

Corollary 9 *Let E be a Fréchet space and F be its strong dual. Then the following are equivalent:*

(i) *E is distinguished.*
(ii) *F has countable tightness.*
(iii) *F has the strong Pytkeev property.*

Proof It is well known that the strong dual of a metrisable LCS is a (DF)-space. So F is a (DF)-space.

(i)⇒(iii) follows from Theorem 3(iii).
(iii)⇒(ii) is clear.
(ii)⇒(i) follows from [21, Proposition 16.4].

Remark 4 We note that in general the converse in item (iv) does not hold. Indeed, let E be a non-Montel distinguished Fréchet space and let E' be its strong dual space. Then E' has the strong Pytkeev property by Theorem 3(iii). Let us show that E' is not sequential. Since E is Fréchet, E' is a (DF)-space and hence a dual metric space. As E is non-Montel, we obtain that E' is not Montel. Since E' is not normable (otherwise, E' is finite-dimensional, and hence E is Montel), the space E' is not sequential by [22, Theorem 4.5].

Remark 5 Assume that the space $C_c(X)$ has countable tightness. Then the space X is Lindelöf by [25, Proposition 5]. Hence $C_c(X)$ is both barrelled and bornological by applying [29, Theorem 10.1.12]. In a particular case that X has a compact resolution swallowing the compact sets of X we obtain additionally that $C_c(X)$ has the strong Pytkeev property by Theorem 2. It is natural to ask whether each LCS E having the (strong) Pytkeev property is bornological. The next example shows that there is a LCS E with the strong Pytkeev property which is not even quasibarrelled. Indeed, [29, Example 6.4.3] provides a strict (LF)-space W containing a dense subspace E which is not quasibarrelled. However, since W has the strong Pytkeev property by Theorem 3(iii), E also has the strong Pytkeev property. Note that every Fréchet–Urysohn LCS is bornological by [21, 14.6], so the spaces W and E are not Fréchet–Urysohn.

Having in mind Question 1 we consider the important class of locally convex spaces endowed with the weak topology. If E is a LCS, we denote by $(E, \sigma(E, E'))$ the space E endowed with the weak topology.

Theorem 10 *For any LCS E the following are equivalent:*

(i) $(E, \sigma(E, E'))$ *has a $\mathfrak{G}$-base.*
(ii) *E is finite-dimensional or $(E, \sigma(E, E'))$ is isomorphic to a dense linear subspace of $\mathbb{R}^{\mathbb{N}}$.*

Proof (i)⇒(ii) Set $F := (E, \sigma(E, E'))$. Fix arbitrarily a Hamel base Γ in the dual E'. Define

$$T := F \to \mathbb{R}^{\Gamma}, \quad (Tx)(\gamma) := \gamma(x), \ \forall x \in F.$$

It is well known (and easy to check) that T is a linear embedding of F into $\mathbb{R}^{\Gamma}$. By [8, Corollary 1.2.2], for every finite collection $\gamma_1, \dots, \gamma_n$ in Γ and any finite collection of real numbers $r_1, \dots, r_n$ there is $x \in E$ such that

$$\gamma_i(x) = r_i, \quad \forall\, i = 1, \dots, n.$$

This means that $T(F)$ is a dense subspace of $\mathbb{R}^{\Gamma}$.

Suppose that E is infinite-dimensional. Since F has a $\mathfrak{G}$-base, then $\mathbb{R}^{\Gamma}$ also has a $\mathfrak{G}$-base by [17, Proposition 9]. Hence every precompact subset of $\mathbb{R}^{\Gamma}$ is metrisable by [17, Theorem 29]. Thus Γ is countable.

(ii)⇒(i) is clear.

This theorem inspires the next question:

Question 5 Does there exist a LCS E such that E' has uncountable algebraic dimension and (E, w) has the (strong) Pytkeev property?

If this question has a positive answer, then character of the space E is less or equal than $\mathfrak{c}$ by Corollary 7.

It is known that for any Banach space E the space (E, w) embeds into $C_p(B_{E^*}, w^*)$. Since the space (B_{E^*}, w^*) is compact, $C_p(B_{E^*}, w^*)$ has countable tightness by Pytkeev's theorem.

Question 6 Does there exist an infinite-dimensional Banach space E with separable dual such that (E, w) has the (strong) Pytkeev property?

Theorem 2 and Theorem 3 suggest the following question.

Question 7 Let X be a metrisable space such that $C_c(X)$ has the (strong) Pytkeev property. Is then X a Polish space?

Remark 6 The answer to this question is negative. We have already mentioned in Remark 3 that the space $C_c(\mathbb{Q})$ has the strong Pytkeev property, but clearly $\mathbb{Q}$ is not Polish.

6 Proofs of Theorem 4 and Corollaries 3–6

To prove Theorem 4 we need to show that a sequential group with a $\mathfrak{G}$-base satisfies condition (**D**).

Lemma 8 *Let G have an open $\mathfrak{G}$-base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. If G is a k-space, then the set $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ is an open neighbourhood of the unit e for any $\alpha \in \mathbb{N}^\mathbb{N}$.*

Proof Proposition 3 implies that G is sequential.

Set $A := \bigcup_{k\in\mathbb{N}} D_k(\alpha)$. Since G is sequential, it is enough to prove that A is sequentially open. Let $x \in A$ and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to x in G. We have to show that there exists $N \in \mathbb{N}$ such that $x_n \in A$ for every $n > N$. Suppose this fails. Let m be the minimal index such that $x \in D_m(\alpha)$. So $x \in U_\beta$ for every $\beta \in I_m(\alpha)$. By assumption, there exists an index n_1 such that $x_{n_1} \notin A$. So x_{n_1} does not belong to $D_{m+1}(\alpha)$. Hence $x_{n_1} \notin U_{\beta^1}$ for some $\beta^1 \in I_{m+1}(\alpha)$. For n_1 there exists an index $n_2, n_2 > n_1$, such that $x_{n_2} \notin A$. So x_{n_2} does not belong to $D_{m+2}(\alpha)$. Hence $x_{n_2} \notin U_{\beta^2}$ for some $\beta^2 \in I_{m+2}(\alpha)$. Continuing this process we obtain a subsequence $\{x_{n_k}\}_k$ of $\{x_n\}_n$ and a sequence $\{\beta^k\}_k$ in $\mathbb{N}^\mathbb{N}$ such that

$$x_{n_k} \notin U_{\beta^k} \text{ and } \beta^k \in I_{m+k}(\alpha) \quad \text{for every } k \in \mathbb{N}.$$

Set $\gamma = (\gamma_i)_{i\in\mathbb{N}}$, where $\gamma_i = \alpha_i$ if $1 \le i \le m$, and $\gamma_i = \max\{\beta_i^1, \beta_i^2, \dots, \beta_i^{i-m}\}$ if $i > m$. Then $x \in D_m(\alpha) \subseteq U_\gamma$. On the other hand, since $U_\gamma \subseteq U_{\beta^k}$, we conclude that $x_{n_k} \notin U_\gamma$ for every $k \in \mathbb{N}$. As U_γ is open and contains x we conclude $x_n \not\to x$, a contradiction. Hence A is sequentially open. Thus A is open.

We are ready to prove Theorem 4.

Proof Take an open $\mathfrak{G}$-base $\mathcal{U}$ in G. Lemma 8 shows that G satisfies the condition (**D**). Thus G satisfies the strong Pytkeev property by Theorem 6. The last claim follows from Proposition 3.

Remark 7 (i) As it was noticed in [28, Footnote 2], van Douwen has shown that if even one of the factors in the sequence in Proposition 2 is not locally compact and all groups G_i are abelian, then the group G is not sequential. By [17, Proposition

19], the group G is not a k-space. So Proposition 2 shows that there exists a topological group G having a $\mathfrak{G}$-base and the strong Pytkeev property such that the sets of the form $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$ are open for any $\alpha \in \mathbb{N}^{\mathbb{N}}$ but which is not even a k-space. This conclusion holds also for the space $C_c(\mathbb{N}^{\mathbb{N}})$ by Lemma 3 and [25, Corollary 9]. In particular, these examples show that Proposition 2 does not follow from Theorem 4.

(ii) As Proposition 2 shows, Lemma 8 may fail if the group G is not a k-space. In other words, if G has a $\mathfrak{G}$-base and is not a k-space, then the closure of the sets of the form $\bigcup_{k\in\mathbb{N}} D_k(\alpha)$, $\alpha \in \mathbb{N}^{\mathbb{N}}$, may not have interior points.

As a consequence of Theorem 4 we obtain an alternative proof of an important result [17, Theorem 2]:

Corollary 10 *A topological group G is metrisable if and only if G has a $\mathfrak{G}$-base and is Fréchet–Urysohn.*

Proof Clearly, if G is metrisable, then the group G has a $\mathfrak{G}$-base and is Fréchet–Urysohn. Conversely, assume that G has a $\mathfrak{G}$-base and is Fréchet–Urysohn. Theorem 4 implies that G has the strong Pytkeev property. So G has the countable cs^*-character. Thus G is metrisable by [4, Theorem 3].

Let G be an abelian topological group. Denote by $G^\wedge$ the group of all continuous characters of G endowed with the compact open topology.

Remark 8 Banakh and Zdomskyy [4, Theorem 3] proved the next equivalence:

$$countable\, cs^* - character \wedge \text{Fréchet–Urysohness} \iff metrizability.$$

However we cannot weaken Fréchet–Urysohness to sequentiality in this equivalence. Moreover, the space ϕ and the group $A(\mathbf{s})$ (see Examples 2 and 5) show that in general even

$$P\text{-}sequentiality \not\Longrightarrow \text{Fréchet–Urysohness.}$$

Note also that $A(\mathbf{s})^\wedge$ is a Polish group [15]. Hence both $A(\mathbf{s})$ and $A(\mathbf{s})^\wedge$ have the strong Pytkeev property. We do not know whether there exists an abelian non-metrisable and non-locally precompact abelian topological group G such that both G and $G^\wedge$ are Fréchet–Urysohn.

This remark inspires the next question.

Question 8 Let X be a topological group of countable cs^*-character and with the Pytkeev property. Does X have the strong Pytkeev property?

Remark 9 It should be mentioned that Pytkeev [32] proved that, for any Tychonoff space X, $C_c(X)$ is Fréchet–Urysohn if and only if $C_c(X)$ is sequential if and only if $C_c(X)$ is a k-space. Further, for a separable metric space X, $C_c(X)$ is a k-space if and only if X is locally compact by [30] if and only if $C_c(X)$ is a Polish space by [26, 5.7.6]. So, if X is a Polish non-locally compact space, then $C_c(X)$ has the strong Pytkeev property but is not a k-space.

Now we prove Corollaries 3–6.

Proof of Corollary 3 Clearly, if G is metrisable, then G has a $\mathfrak{G}$-base and is a k-space. Conversely, assume that G has a $\mathfrak{G}$-base and is a k-space. Then G is sequential by Proposition 3. Theorem 4 implies that G has the strong Pytkeev property. So G has countable cs^*-character. Thus G is metrisable by [4, Theorem 3]. □

Corollary 3 inspires the next question.

Question 9 Let G be a Baire topological group with a $\mathfrak{G}$-base. Is G a k-space?

Remark 10 This question has a positive answer in the class of TVS by [17, Theorem 4]. Recently, we obtained a partial positive result for topological groups: a Baire topological group with a $\mathfrak{G}$-base satisfying the condition (**D**) is metrisable, hence a k-space. We suppose that Question 9 in general has a negative answer.

Taking into account Lemmas 1 and 4 one can ask:

Question 10 Let G be a Baire topological group with the strong Pytkeev property. Is G metrisable?

Remark 11 After the paper was submitted for publication we obtained a positive solution of Question 10: a Baire topological group with the strong Pytkeev property is metrisable. We decided to provide a proof of this result in a separate paper because it is lengthy and requires extra results and concepts which are a little bit far from the main picture of the present paper.

Proof of Corollary 4 (i)⇒(ii) If E is a k-space, Theorem 4 implies that E has the strong Pytkeev property. So E has countable cs^*-character. The space E is sequential by Proposition 3. If E is nonmetrisable, then E is homeomorphic either to ϕ or $\phi \times Q$ by [4, Corollary 1]. The implication (ii)⇒(i) is clear. □

Note that the Baire property of G in Corollary 4 can be replaced by any condition from [4, Theorem 3].

Proof of Corollary 6 Let $(E_n)_n$ be a sequence of Fréchet spaces which witnesses E being a Montel (LF)-space. So E_n is Montel and the strong dual E_n' of E_n is a Montel (DF)-space for every $n \in \mathbb{N}$. Thus E_n' is sequential by [22, Theorem 4.5]. Note that E_n' is metrisable if and only if E_n is finite-dimensional. Since E_n' are complete, F is complete by [37, II.5.3]. We distinguish between two cases.

(i) Assume that E is not isomorphic to ϕ. So we can assume that E_n is infinite-dimensional, and hence E_n' is sequential non-metrisable space for every $n \in \mathbb{N}$. By Corollary 4 each space E_n' is homeomorphic either to ϕ or to $\phi \times Q$. The space F is isomorphic to a subspace of the product $\prod_n E_n'$, this subspace is closed because F is complete. Consequently, $\prod_n E_n'$ is homeomorphic either to $\phi^{\mathbb{N}}$ or to $\phi^{\mathbb{N}} \times Q$. Note that the latter space is homeomorphic to $\phi^{\mathbb{N}}$. Thus E is homeomorphic to $\phi^{\mathbb{N}}$.

(ii) If E is isomorphic to ϕ, then F is metrisable because it is isomorphic to $\mathbb{R}^{\mathbb{N}}$. □

It is known (see [2]) that $\mathfrak{D}'(\Omega)$ is isomorphic to $\phi^{\mathbb{N}}$. This justifies the next problem:

Question 11 Find necessary and sufficient conditions on a Montel (LF)-space E such that the strong dual F of E is homeomorphic to the product $\phi^{\mathbb{N}}$.

Recall that a topological space X is *submetrisable* if it admits a weaker metric topology. It is known that each countable topological group is submetrisable.

Example 6 Let $X := [0, \omega_1)$ be the interval of all countable ordinals. Since X is not a Lindelöf space, $C_c(X)$ does not have countable tightness. Hence $C_c(X)$ does not have the strong Pytkeev property. By [26, Corollary 4.3.2], for a Tychonoff space Y, $C_c(Y)$ is submetrisable if and only if Y contains a dense σ-compact subset. Recall that every compact set in X is countable, so any σ-compact subset of X is also countable and cannot be dense. Thus $C_c(X)$ is not submetrisable.

Under hypothesis $\mathfrak{d} = \aleph_1$, the space $C_c(X)$ has a $\mathfrak{G}$-base $\mathcal{U}$ by [17, Proposition 48] answering negatively [17, Question 18]. Note also that $\mathcal{U}$ does not satisfy the property (**D**) by Theorem 6. It is independent of ZFC whether $C_c(X)$ has a $\mathfrak{G}$-base, since Tkachuk [39] showed that under MA + ¬CH the space X does not have a compact resolution swallowing its compact sets.

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