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Research Article

Some Topological Properties of $C_b(X)$

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If X is a completely regular space, first we characterize those spaces $C_b(X)$ whose compact sets are metrizable. Then we use this result to provide a general condition for X to ensure the metrizability of compact sets in $C_b(X)$. Finally, we characterize those spaces $C_b(X)$ that have a $\mathfrak{G}$-basis.

1. Preliminaries

We start by recalling that a subset A of a topological space X is called (functionally) *bounded* [1, Chapter 0] if $f(A)$ is a bounded set in $\mathbb{R}$ for every real-valued continuous function f on X. In what follows, unless otherwise stated, X will be a Hausdorff completely regular space. We will denote by $C_p(X)$, $C_c(X)$, or $C_b(X)$ the ring $C(X)$ of all real-valued continuous functions defined on X provided with the pointwise convergence topology, the compact-open topology, or the bounded-open topology, that is, the locally convex topology on $C(X)$ of uniform convergence on the bounded sets of X, respectively. An account of the different uniform convergence topologies on $C(X)$ can be found in [2]. We will denote by $L(X)$ the topological dual of $C_p(X)$ and by $L_p(X)$ the linear space $L(X)$ equipped with the weak* topology [1]. From now onwards a covering $\{A_\alpha : \alpha \in \Sigma\}$ of X with $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}^{\mathbb{N}}$ is equipped with the product (discrete) topology and $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$, will be referred to as an *ordered* Σ-*covering*, although if $\Sigma = \mathbb{N}^{\mathbb{N}}$ we will speak of *ordered covering* rather than ordered $\mathbb{N}^{\mathbb{N}}$-covering. Every K-analytic space X contains an ordered covering consisting of compact sets [3]. In [4, Definition 2.1] an unbounded subspace Σ of $\mathbb{N}^{\mathbb{N}}$ is called σ-*complete* if each bounded sequence $\{\alpha_n\}_{n=1}^{\infty}$ in Σ satisfies that $\sup_{n\in\mathbb{N}}\alpha_n \in \Sigma$. Clearly $\Sigma = \mathbb{N}^{\mathbb{N}}$ is σ-complete. The main result of [4] is the following.

Theorem 1 (see [4, Theorem 2.8]). *If there exists a dense subspace Y of X that admits an ordered Σ-covering consisting of Y-bounded subsets indexed by a σ-complete set, then every compact set in $C_b(X)$ is metrizable. Particularly, if Y has an ordered covering made up of Y-bounded sets, every compact set in $C_b(X)$ is metrizable.*

A uniform space $(X, \mathcal{N})$ is called *transseparable* if for each vicinity N of $\mathcal{N}$ there is a countable set Q in X such that $N[Q] = \bigcup_{x\in Q} U_N(x) = X$, where $U_N(x) = \{y \in X : (x, y) \in N\}$; see [5, Section 6.4]. Equivalently, $(X, \mathcal{N})$ is transseparable if each uniform cover of X admits a countable subcover [6]. A family $\mathcal{F}$ of functions from a uniform space $(X, \mathcal{N})$ into a uniform space $(Y, \mathcal{M})$ is called *uniformly equicontinuous* [7, X.2.1 Definition 2] if for each $V \in \mathcal{M}$ there is $U \in \mathcal{N}$ such that $(f(x), f(y)) \in V$, whenever $f \in \mathcal{F}$ and $(x, y) \in U$. Here we have a characterization of transseparable spaces $(X, \mathcal{N})$ in terms of uniformly equicontinuous sets of $C(X, \tau_{\mathcal{N}})$, where $\tau_{\mathcal{N}}$ stands for the uniform topology on X.

Theorem 2 (see [8, Theorem 1]). *A uniform space $(X, \mathcal{N})$ is transseparable if and only if every pointwise bounded uniformly equicontinuous set of functions from $(X, \mathcal{N})$ into $\mathbb{R}$ is metrizable in $C_c(X, \tau_{\mathcal{N}})$ or, equivalently, in $C_p(X, \tau_{\mathcal{N}})$.*

Particularly, by Ascoli's theorem, if every compact set of $C_c(X, \tau_{\mathcal{N}})$ is metrizable, then $(X, \mathcal{N})$ is transseparable. Separable uniform spaces as well as Lindelöf uniform spaces are transseparable and for uniform pseudometrizable spaces transseparability is equivalent to separability. The class of transseparable uniform spaces is hereditary, productive, and closed under uniform continuous images. Some applications of transseparable spaces can be found in [5, Section 6.4]. Regarding transseparability, the following result is useful.

Theorem 3 (see [9, Theorem]). *Let $(X, \mathscr{N})$ be a uniform space. If X has an ordered covering consisting of $\mathscr{N}$-precompact sets, then $(X, \mathscr{N})$ is transseparable.*

This result is easily generalized for ordered Σ-coverings indexed by σ-complete sets.

Theorem 4 (see [4, Lemma 2.7]). *Let $(X, \mathscr{N})$ be a uniform space. If X has an ordered Σ-covering consisting of $\mathscr{N}$-precompact sets such that Σ is σ-complete, then $(X, \mathscr{N})$ is transseparable.*

Finally, a locally convex space (lcs) E is said to have a $\mathfrak{G}$-*base* (or a $\mathfrak{G}$-basis) [5, Chapter 1] if there exists a basis $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of (absolutely convex) neighborhoods of the origin in E such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$. As is well known, if a locally convex space E has a $\mathfrak{G}$-base, every compact set in E is metrizable.

In what follows we continue the research started in [4] on $C_b(X)$ spaces and extend the investigation of papers [10, 11] on spaces $C_c(X)$ to the $C_b(X)$ case. We begin by characterizing the spaces $C_b(X)$ whose compact sets are metrizable in terms of a particular uniformity $\mathscr{M}$ on X (cf. Theorem 5). Then we use this characterization to prove Theorem 8 below, which extends Theorem 4. Finally, we characterize those $C_b(X)$ spaces having a $\mathfrak{G}$-base.

2. Metrizable Compact Sets in Spaces $C_b(X)$

First we characterize those $C_b(X)$ spaces whose compact sets are metrizable in terms of a particular uniformity on X. This result is a translation of [10, Theorem 3] to the bounded-open topology framework.

Theorem 5. *The compact sets of $C_b(X)$ are metrizable if and only if $(X, \mathscr{M})$, where $\mathscr{M}$ is the uniformity on X generated by the pseudometrics*

$$d_A(x, y) = \sup_{f \in A} |f(x) - f(y)| \tag{1}$$

for each compact set A of $C_b(X)$, is transseparable.

Proof. Let E be the topological dual of $C_b(X)$. Let us denote by $\mathscr{K}(C_b(X))$ the family of all compact sets of $C_b(X)$ and by $\rho(E, C_b(X))$ the locally convex topology on E of uniform convergence on all compact sets of $C_b(X)$. If δ stands for the canonical homeomorphic embedding of X into $L_p(X)$, then $\delta(X) \subseteq L(X) \subseteq E$ and the topology $\rho_X = \delta^{-1}(\rho(E, C_b(X)))$ on X is stronger than the original topology of X, since $\rho(E, C_b(X))|_{\delta(X)}$ is stronger than the topology induced on $\delta(X)$ by the weak* topology $\sigma(E, C(X))$ of E. This latter fact implies that $C(X) \subseteq C(X, \rho_X)$ algebraically.

Assuming that all compact sets of $C_b(X)$ are metrizable, the dual E equipped with the locally convex topology $\rho(E, C_b(X))$ is transseparable [12, Theorem 2]. This topology $\rho(E, C_b(X))$ generates a unique admissible translation-invariant uniformity $\mathscr{N}$ on E, so that $\tau_\mathscr{N} = \rho(E, C_b(X))$. Setting $\tilde{f}$ instead of $f \in C(X)$ when f is considered as a linear functional on E, observe that $(u, v) \in N \in \mathscr{N}$ with $u, v \in E$ if and only if there are $A \in \mathscr{K}(C_b(X))$ and $\epsilon > 0$ such that $\sup_{f \in A} |\langle \tilde{f}, u - v\rangle| < \epsilon$. Particularly, the relative uniformity of $\mathscr{N}$ on $\delta(X)$ satisfies that

$$\begin{aligned} &\left(\delta_x, \delta_y\right) \in N \cap (\delta(X) \times \delta(X)) \\ &\quad \Longleftrightarrow \sup_{f \in A} \left|\left\langle \tilde{f}, \delta_x \right\rangle - \left\langle \tilde{f}, \delta_y \right\rangle\right| < \epsilon. \end{aligned} \tag{2}$$

This defines a uniformity $\mathscr{M}$ on X such that $(x, y) \in M \in \mathscr{M}$ if and only if there are $A \in \mathscr{K}(C_b(X))$ and $\epsilon > 0$ with $\sup_{f \in A} |f(x) - f(y)| < \epsilon$. Given that $(X, \mathscr{M})$ and $\delta(X)$ equipped with the relative uniformity of $\mathscr{N}$ are uniformly homeomorphic and the class of transseparable spaces is hereditary and closed under uniform continuous images, it follows that the uniform space $(X, \mathscr{M})$ is transseparable.

Assume conversely that $(X, \mathscr{M})$ is transseparable whenever $\mathscr{M}$ is the uniformity on X generated by the pseudometrics $d_A(x, y) = \sup_{f \in A} |f(x) - f(y)|$ for each $A \in \mathscr{K}(C_b(X))$ and let A be a fixed compact subset of $C_b(X)$. As observed above, A is contained in the space $C(X, \tau_\mathscr{M}) = C(X, \rho_X)$. Moreover, A is a (pointwise bounded) uniformly equicontinuous set of functions from $(X, \mathscr{M})$ to $\mathbb{R}$ since, given $\epsilon > 0$, if $d_A(x, y) < \epsilon$, obviously $|f(x) - f(y)| < \epsilon$ whenever $f \in A$. Hence, according to Theorem 2, the set A is metrizable in $C_c(X, \tau_\mathscr{M})$. Since $\tau_\mathscr{M} = \rho_X$ is stronger than the original topology on X, there are more compact sets in X than in $(X, \tau_\mathscr{M})$, which ensures that the topology on A inherited from $C_c(X)$ is stronger than that inherited from $C_c(X, \tau_\mathscr{M})$. Since A is compact in $C_b(X)$ and the bounded-open topology is stronger than the compact-open, the three topologies, that is, the bounded-open, the compact-open from X, and the compact-open from (X, τ_M), coincide on A. Thus A is metrizable. □

Next we extend [4, Theorem 2.8] by locating a class of completely regular spaces X, wider than those spaces having an ordered Σ-covering indexed by a σ-complete set of $\mathbb{N}^\mathbb{N}$, with the property that compact sets of $C_b(X)$ are metrizable.

Definition 6. A subspace Σ of $\mathbb{N}^\mathbb{N}$ will be called fair if every uniform space $(X, \mathscr{N})$ that admits an ordered Σ-covering consisting of $\mathscr{N}$-precompact sets is transseparable.

As an obvious consequence of Theorem 4, every σ-complete set of $\mathbb{N}^\mathbb{N}$, particularly the whole space $\mathbb{N}^\mathbb{N}$, is fair.

Lemma 7. *Let $(X, \mathscr{N})$ be a uniform space. If $\mathscr{M}$ stands for the uniform structure on X generated by the pseudometrics (1) when A runs over the compact sets of $C_b(X, \tau_\mathscr{N})$, then every $\tau_\mathscr{N}$-bounded set of X is $\mathscr{M}$-precompact.*

Proof. Note that the completely regular space $(X, \tau_{\mathscr{M}})$ is homeomorphic to $\delta(X)$ equipped with the relative topology of $\rho(E, C_b(X, \tau_{\mathscr{N}}))$ on E of uniform convergence on all compact sets of $C_b(X, \tau_{\mathscr{N}})$, where E stands for the topological dual of $C_b(X, \tau_{\mathscr{N}})$. If B is a $\tau_{\mathscr{N}}$-bounded set of X, then

$$B^{\diamond} := \left\{ f \in C(X, \tau_{\mathscr{N}}) : \sup_{x \in B} |f(x)| \leq 1 \right\} \tag{3}$$

is a neighborhood of the origin in $C_b(X, \tau_{\mathscr{N}})$ and, consequently, the absolutely convex set

$$B^{\diamond 0} := \left\{ u \in E : \sup_{f \in B^{\diamond}} \left| \langle \widetilde{f}, u \rangle \right| \leq 1 \right\} \tag{4}$$

is a weak* compact subset of E, that is, a $\sigma(E, C(X, \tau_{\mathscr{N}}))$-compact set. Since $\rho(E, C_b(X, \tau_{\mathscr{N}}))$ coincides with $\sigma(E, C_b(X, \tau_{\mathscr{N}}))$ on the $C_b(X, \tau_{\mathscr{N}})$-equicontinuous subsets of E, then it follows that $B^{\diamond 0}$ is a $\rho(E, C_b(X, \tau_{\mathscr{N}}))$-compact set. Given that $\delta(B)$ is contained in $B^{\diamond 0} \cap L(X)$, we conclude that $\overline{\delta(B)}^{\rho(E, C_b(X, \tau_{\mathscr{N}}))}$ is $\rho(E, C_b(X, \tau_{\mathscr{N}}))$-compact, so that $\delta(B)$ is $\rho(E, C_b(X, \tau_{\mathscr{N}}))$-precompact in the locally convex space $(E, \rho(E, C_b(X, \tau_{\mathscr{N}})))$. This is tantamount to saying that B is an $\mathscr{M}$-precompact subset of X. □

Theorem 8. *Let Y be a dense subspace of a completely regular space X which admits an ordered Σ-covering consisting of Y-bounded sets. If the index set of the covering is fair, then every compact set of $C_b(X)$ is metrizable.*

Proof. It suffices to prove the theorem assuming that $Y = X$, since the restriction map $S : C_b(X) \to C_b(Y)$ given by $S(f) = f|_Y$ is a continuous (linear) injection, so if each compact set of $C_b(Y)$ is metrizable, every compact set of $C_b(X)$ is metrizable as well. So, let us assume that there exists an ordered Σ-covering $\{A_\alpha : \alpha \in \Sigma\}$ of X consisting of bounded sets.

If $\mathscr{M}$ denotes the uniformity on X generated by the pseudometrics (1) when A runs over the compact sets of $C_b(X)$, Lemma 7 informs us that the family $\{A_\alpha : \alpha \in \Sigma\}$ is an ordering Σ-covering of X consisting of $\mathscr{M}$-precompact sets. Since Σ is fair, the uniform space $(X, \mathscr{M})$ is transseparable. Consequently, an application of Theorem 5 shows that every compact set in $C_b(X)$ is metrizable. □

Lemma 9. *If $\mathscr{M}$ is the uniformity on X generated by the pseudometrics (1) when A runs over the compact sets of $C_b(X)$, then $\tau_{\mathscr{M}}$ coincides with the original topology of X on each X-bounded set.*

Proof. Let B be a bounded set of X. If E stands for the topological dual of $C_b(X)$, we proceed as in the proof of Lemma 7 to show that $\rho(E, C_b(X))|_{\delta(B)} = \sigma(E, C(X))|_{\delta(B)}$. Moreover, if F stands for the topological dual of $C(X, \tau_{\mathscr{M}})$, then $(X, \tau_{\mathscr{M}})$ coincides with $\delta(B)$ when this latter is provided with the relative weak* topology $\sigma(F, C(X, \tau_{\mathscr{M}}))|_{\delta(B)}$ of F. So bearing in mind the definition of $\tau_{\mathscr{M}}$ we conclude that

$$\sigma(F, C(X, \tau_{\mathscr{M}}))|_{\delta(B)} = \rho(E, C_b(X))|_{\delta(B)} = \sigma(E, C(X))|_{\delta(B)} \tag{5}$$

which implies that $\tau_{\mathscr{M}}$ coincides on B with the original topology of X. □

Theorem 10. *Let X be k-space. If every admissible uniformity on X is transseparable, then compact sets of $C_b(X)$ are metrizable.*

Proof. As a consequence of the previous lemma, X and $(X, \tau_{\mathscr{M}})$ have the same compact sets. But since X is a k-space, its topology coincides with the strongest topology on X that has the same compact sets as the original one. Consequently, $\tau_{\mathscr{M}}$ coincides with the original topology of X, which shows that $\mathscr{M}$ is an admissible uniformity on X. So, if the hypothesis holds, the uniform space $(X, \mathscr{M})$ must be transseparable. In this case Theorem 5 ensures that every compact set of $C_b(X)$ is metrizable. □

The fact that $\tau_{\mathscr{M}}$ coincides with the original topology of X on each X-bounded set does not mean that X and $(X, \mathscr{M})$ have the same bounded sets. But that can be taken for granted if X is a normal space, as the following theorem shows.

Theorem 11. *If X is a normal space, then X and $(X, \tau_{\mathscr{M}})$ have the same bounded sets. Consequently $C_b(X)$ embeds into $C_b(X, \tau_{\mathscr{M}})$.*

Proof. Assume without loss of generality that B is a closed X-bounded subset of X. If $f \in C(X, \tau_{\mathscr{M}})$, Lemma 9 shows that $f|_B \in C(B)$, where B is considered as a topological subspace of X. So there is $g \in C(X)$ such that $g|_B = f|_B$. But since B is X-bounded, it holds that

$$\sup_{x \in B} |f(x)| = \sup_{x \in B} |g(x)| < \infty, \tag{6}$$

so that B is $\tau_{\mathscr{M}}$-bounded. Conversely, if B is $\tau_{\mathscr{M}}$-bounded set of X, the fact that $\tau_{\mathscr{M}}$ is finer than the original topology of X implies that B is X-bounded. On the other hand, since the identity map $\varphi : (X, \tau_{\mathscr{M}}) \to X$ is continuous, the first part of the theorem ensures that the inclusion map $f \mapsto f \circ \varphi$ from $C_b(X)$ into $C_b(X, \tau_{\mathscr{M}})$ is a linear embedding. □

3. Characterizing Spaces $C_b(X)$ with a $\mathfrak{G}$-Base

It is shown in [11, Theorem 2] that the space $C_c(X)$ has a $\mathfrak{G}$-base if and only if X has an ordered covering consisting of compact sets which swallows the compact sets of X, that is, such that each compact set of X is contained in some member of the covering. A similar strategy provides the following characterization for $C_b(X)$.

Theorem 12. *The space $C_b(X)$ has a $\mathfrak{G}$-base if and only if X has an ordered covering consisting of bounded sets which swallows the bounded sets of X.*

Proof. For each bounded set $B \subseteq X$ and $\epsilon > 0$ define

$$[B, \epsilon] = \left\{ f \in C(X) : \sup_{x \in B} |f(x)| \leq \epsilon \right\}. \tag{7}$$

If $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is an ordered covering of X consisting of bounded sets, define

$$U_\alpha = \left[B_\alpha, \alpha(1)^{-1}\right] \tag{8}$$

for $\alpha \in \mathbb{N}^{\mathbb{N}}$ and put $\mathfrak{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Clearly $\mathfrak{U}$ is a family of absolutely convex and absorbing sets in $C(X)$ such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$, composing a filter base. The reader may easily check that $\mathfrak{U}$ is a $\mathfrak{G}$-base for a locally convex topology τ on $C(X)$ with $\tau_p \leq \tau \leq \tau_b$. Now assume that $\{B_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows the bounded sets and let V be a neighborhood of the origin of $C_b(X)$. If B is a bounded set in X with $[B, \epsilon] \subseteq V$ for some $\epsilon > 0$, choosing $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $B \subseteq B_\gamma$ and $\gamma(1)^{-1} < \epsilon$, then $U_\gamma \subseteq [B, \epsilon] \subseteq V$. This shows that $\tau = \tau_b$, so $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base for $C_b(X)$.

Conversely, suppose that $C_b(X)$ has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. For every set U in $C(X)$ define a corresponding set $U^\diamond$ in X by writing

$$U^\diamond = \{x \in X : |f(x)| \leq 1 \ \forall f \in U\}. \tag{9}$$

Clearly $U^\diamond$ is closed in X and $U \subseteq V$ implies that $U^\diamond \supseteq V^\diamond$. If B is a closed bounded set in X and $\epsilon > 0$, then $[B, \epsilon]^\diamond \subseteq B$ since if $x \in X \setminus B$, there is $f \in C(X)$ with $f(x) = 2$ and $f(B) = \{0\}$, so that $f \in [B, \epsilon]$ and $x \notin [B, \epsilon]^\diamond$. If B is bounded and $0 < \epsilon \leq 1$, then $B \subseteq [B, \epsilon]^\diamond$; hence $[B, \epsilon]^\diamond = B$ for every closed bounded set B. In addition, if U is a neighborhood of the origin in $C_b(X)$, then $U^\diamond$ is a bounded subset of X. Indeed, if B is a bounded set in X such that $[B, \epsilon] \subseteq U$ for some $\epsilon > 0$, by the previous observations $U^\diamond \subseteq [B, \epsilon]^\diamond \subseteq B$ and hence $U^\diamond$ is bounded because it is subset of a bounded set.

Now for any B closed bounded in X there is some $U_\alpha \subseteq [B, 1]$, which means that $B = [B, 1]^\diamond \subseteq U_\alpha^\diamond$. This shows that the family $\mathscr{A} = \{U_\alpha^\diamond : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows all bounded sets, so particularly it covers X. As in addition $U_\alpha^\diamond \subseteq U_\beta^\diamond$ for $\alpha \leq \beta$, it follows that $\mathscr{A}$ is an ordered covering of X consisting of bounded sets. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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