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On C-Suslin spaces

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We prove a closed graph theorem for Baire locally convex spaces (for Baire linear topological spaces) in the domain and weakly C-Suslin locally convex spaces (respectively, for C-Suslin linear topological spaces) in the range which improves some classic closed graph theorems and other, more recent, related results.

1 Preliminaries

The topological spaces we use are nonempty and Hausdorff. We equip the set $\mathbb{N}$ of positive integers with the discrete topology and the universal Polish space $\mathbb{N}^{\mathbb{N}}$ with the product topology. A topological space X is called *K-Suslin* (respectively *quasi-Suslin*) [14, Chapter 1] if there is a map T from $\mathbb{N}^{\mathbb{N}}$ into the family $\mathcal{K}(X)$ of all compact subsets of X (respectively into the family $\mathcal{P}(X)$ of all subsets of X) such that $\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ covers X and if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point $x \in T(\alpha)$. A topological space X is said to admit a *relatively countably compact resolution* if X is covered by an ordered family of relatively countably compact sets $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, i.e. such that $A_\alpha \subseteq A_\beta$ whenever $\alpha(i) \leq \beta(i)$ for all $i \in \mathbb{N}$. Spaces that admit a relatively countably compact resolution have been called *strongly web-compact* in [6] (see also [8, Chapter 5] for details). A topological space X is called *C-Suslin* [13] if there is a map T from a separable metric space P into $\mathcal{P}(X)$ such that $\{T(\alpha) : \alpha \in P\}$ covers X and if $\{\alpha_n\}$ is a Cauchy sequence in P then $\bigcup_{n=1}^{\infty} T(\alpha_n)$ is a relatively countably compact set in X. Alternatively one can readily check that X is C-Suslin if and only if there is a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ and a map $T : \Sigma \to \mathcal{P}(X)$ such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$ and if $\{\alpha_n\} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point in X. A completely regular space X is K-analytic (in the sense of Choquet) if and only if it is quasi-Suslin and realcompact. Every K-analytic space is K-Suslin and every completely regular K-Suslin space is K-analytic. On the other hand, every K-Suslin space is quasi-Suslin, every quasi-Suslin space admits a relatively countably compact resolution (in other words, every quasi-Suslin space is strongly web-compact) and (as is shown later, in the proof of Corollary 3.2) every strongly web-compact space is C-Suslin. Valdivia showed [14, I.4.3 (23, 24)] that if E is a Fréchet space, its weak* bidual $(E'', \sigma(E'', E'))$ is always quasi-Suslin, but is K-analytic if and only if $(E', \mu(E', E''))$ is barrelled. It is shown in [5, Example 13] that if $\aleph_1 = \mathfrak{b}$ (the *bounding* cardinal) then the weak* dual F of the space $C_c([0, \omega_1))$ consisting of all real-valued continuous functions defined on the ordinal interval $[0, \omega_1)$ equipped with the compact-open topology is quasi-Suslin, hence C-Suslin; but since F is not Lindelöf (see for instance [6]), then F is not K-analytic. Other examples of C-Suslin spaces which are not K-analytic can be found in [13]. For a nontrivial example of a C-Suslin space take a quasi-Suslin space X with defining map $T : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(X)$ and choose an unbounded non-closed subspace Σ of $\mathbb{N}^{\mathbb{N}}$ enjoying the property that each countable bounded subset Δ of Σ has an upper bound in Σ, i.e. there exists $\delta \in \Sigma$ such that $\sup \Delta \leq \delta$, see [3, Definition 2.1]. Setting $Y := \bigcup\{T(\alpha) : \alpha \in \Sigma\}$ and $S := T|_\Sigma$, it can be shown that $S : \Sigma \to \mathcal{P}(Y)$ defines a C-Suslin structure on Y.

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The classic Schwartz's closed graph theorem for analytic spaces [12] was soon generalized by Martineau [10] and Nakamura [11, Corollary], later on by Valdivia [13, Theorem 2] and [14, I.4.2 (11)], and more recently by the first author and some collaborators in [4]. These results are stated below (the second in a slightly less general form), where all linear topological spaces are over the field of the real or complex numbers. Other extensions of Martineau's theorem for topological groups can be found in [6], [13].

Theorem 1.1 (Martineau, Nakamura) *Every linear mapping with closed graph from a Baire linear topological space into a K-Suslin linear topological space is continuous.*

Theorem 1.2 (Valdivia) *Every linear mapping with closed graph from a metrizable Baire linear topological space into a C-Suslin linear topological space is continuous.*

Theorem 1.3 (Valdivia) *Every linear mapping with closed graph from a locally convex hull of metrizable Baire locally convex spaces into a quasi-Suslin locally convex space is continuous.*

Theorem 1.4 (Ferrando, Kąkol and López Pellicer) *Every linear mapping with closed graph from a Baire linear topological space into a linear topological space which admits a relatively countably compact resolution is continuous.*

In this paper we improve all these previous theorems by developing some techniques of [4], [13], [14] which allows us to prove a closed graph theorem for Baire locally convex spaces (for Baire linear topological spaces) in the domain and weakly C-Suslin locally convex spaces (respectively, for C-Suslin linear topological spaces) in the range that, as Theorems 1.1 and 1.4 above, does not require the metrizability of the domain space. We also show that the class of C-Suslin spaces is not productive and include some applications.

2 A closed graph theorem for C-Suslin spaces

Let us recall that a subspace A of a topological space X is said to have the *Baire property* if there is an open set U in X such that both sets $U \setminus A$ and $A \setminus U$ are of first category. If $O(A)$ denotes the interior of the set of all points $x \in X$ with the property that every neighborhood of x meets A in a set of second category in X, it can be shown that A has the Baire property if and only if $O(A) \setminus A$ is of first category (see [14, I.1.1 (9)]). Let us also recall the *Banach difference theorem* [14, I.1.3 (8)], which states that if E is a linear topological space and U is a subset of E of second category in E with the Baire property then $U - U$ is a neighborhood of the origin in E. Particularly, if E is locally convex and U is an absolutely convex subset of E of second category in E with the Baire property, then U is a neighborhood of the origin in E. A locally convex space which is C-Suslin in its weak topology will be called a weakly C-Suslin locally convex space.

If Σ is a subset of $\mathbb{N}^{\mathbb{N}}$ and $T : \Sigma \to \mathcal{P}(X)$ is a map from Σ into the family of all parts of a set X, for each $(\alpha, n) \in \Sigma \times \mathbb{N}$ we shall usually write

$$T(\alpha \mid n) := \bigcup \left\{ T(\beta) : \beta \in \Sigma \beta(i) = \alpha(i), 1 \le i \le n \right\}.$$

Let us remark that $T(\alpha \mid n+1) \subseteq T(\alpha \mid n)$ for every $(\alpha, n) \in \Sigma \times \mathbb{N}$ and $T(\alpha) \subseteq T(\alpha \mid n)$ for all $n \in \mathbb{N}$.

Lemma 2.1 *Let E be a Baire locally convex space and F be a weakly C-Suslin locally convex space. If $f : E \to F$ is a linear mapping with closed graph and M is a weakly closed subset of F, then $f^{-1}(M)$ has the Baire property.*

P r o o f. Since F is a weakly C-Suslin locally convex space there exist a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a mapping $T : \Sigma \to \mathcal{P}(F)$ satisfying the conditions of the definition of C-Suslin space. Then note that the countable family

$$\mathcal{E} = \{T(\alpha \mid n) : \alpha \in \Sigma, n \in \mathbb{N}\}$$

is a web of subsets of F covering F in the sense that $F = \bigcup \{T(\alpha \mid 1) : \alpha \in \Sigma\}$ and

$$T(\alpha \mid n) = \bigcup \{T(\beta \mid n+1) : \beta \in \Sigma, \beta(i) = \alpha(i), 1 \le i \le n\}$$

for every $n \in \mathbb{N}$, where it must be pointed out that all these unions are countable.

Since E is a Baire space and $\{f^{-1}(T(\alpha \mid n)) : \alpha \in \Sigma, n \in \mathbb{N}\}$ is a web of subsets of E which covers E, setting $A[\alpha \mid n] := \overline{f^{-1}(T(\alpha \mid n))}$ for $(\alpha, n) \in \Sigma \times \mathbb{N}$ there is a sequence $\{\gamma_n\}$ in Σ with $\gamma_{n+1}(i) = \gamma_n(i)$ for $1 \leq i \leq n$ such that $A[\gamma_n \mid n]$ is of second category in E for each $n \in \mathbb{N}$. Setting $\gamma \in \mathbb{N}^{\mathbb{N}}$ with $\gamma(n) = \gamma_n(n)$ for every $n \in \mathbb{N}$ then $\gamma(i) = \gamma_n(i)$ if $1 \leq i \leq n$, which implies that $\gamma_n \to \gamma$ in $\mathbb{N}^{\mathbb{N}}$.

Let $\{U_n : n \in \mathbb{N}\}$ be a decreasing sequence of absolutely convex neighborhoods of the origin in E such that $U_n \subseteq A[\gamma_n \mid n] - A[\gamma_n \mid n]$ for every $n \in \mathbb{N}$ and let τ be a (non necessarily Hausdorff) locally convex topology on E that has as a base of neighborhoods of the origin the sequence $\{n^{-1}U_n : n \in \mathbb{N}\}$ of absolutely convex sets. Since the graph of f is closed in $E \times F$ there is a Hausdorff locally convex topology ρ on F, coarser than the weak topology of F, such that the map $f : E \to (F, \rho)$ is continuous (see [9, Lemma 3.1] or [14, I.4.1 (21)]). Let us show that $f : (E, \tau) \to (F, \rho)$ is also continuous. □

Claim 2.2 *If V is an absolutely convex neighborhood of the origin in F there exists a positive integer q such that $T(\gamma_q \mid q) \subseteq 2^{-1}qV$.*

Proof. Otherwise there exists a sequence $\{y_n\}$ in F such that $y_n \in T(\gamma_n \mid n) \setminus 2^{-1}nV$ for each $n \in \mathbb{N}$. But since $y_n \in T(\gamma_n \mid n)$ there is $\delta_n \in \Sigma$ with $\delta_n(i) = \gamma_n(i) = \gamma(i)$ for $1 \leq i \leq n$ such that $y_n \in T(\delta_n)$, which implies that $\delta_n \to \gamma$ in $\mathbb{N}^{\mathbb{N}}$. Consequently the sequence $\{\delta_n\}$ of Σ converges in $\mathbb{N}^{\mathbb{N}}$ and hence the sequence $\{y_n\}$ has a cluster point in F. This contradicts the fact that $\left\{2^{-1}n\overset{\circ}{V} : n \in \mathbb{N}\right\}$ is an open covering of F. □

If V is a closed absolutely convex neighborhood of the origin in (F, ρ), then V is a neighborhood of the origin in F and Claim 2.2 yields $\overline{T(\gamma_q \mid q)}^{\rho} \subseteq 2^{-1}qV$. Using the fact that f is continuous when considered from E into (F, ρ) then $A[\gamma_q \mid q] \subseteq f^{-1}\left(\overline{T(\gamma_q \mid q)}^{\rho}\right)$ and, consequently, the linearity of f leads to

$$U_q \subseteq f^{-1}\left(\overline{T(\gamma_q \mid q)}^{\rho}\right) - f^{-1}\left(\overline{T(\gamma_q \mid q)}^{\rho}\right) \subseteq q\, f^{-1}(V).$$

This establishes the continuity of f as linear mapping from $E(\tau)$ into $F(\rho)$.

Let us show that if M is a weakly closed subset of F then $f^{-1}(M)$ has the Baire property in E. For each $(\alpha, n) \in \Sigma \times \mathbb{N}$, put $B[\alpha \mid n] = f^{-1}(M \cap T(\alpha \mid n))$ and note that

$$B[\alpha \mid n] = \bigcup \{B[\beta \mid n+1] : \beta \in \Sigma, \beta(i) = \alpha(i), 1 \leq i \leq n\}. \tag{2.1}$$

Clearly $O(B[\beta \mid n+1]) \subseteq O(B[\alpha \mid n])$ if $\beta \in \Sigma$ satisfies that $\beta(i) = \alpha(i)$ for $1 \leq i \leq n$. Set

$$D = O\left(f^{-1}(M)\right) \setminus \bigcup \left\{O(B[\beta \mid 1]) : \beta \in \Sigma\right\},$$

$$D[\alpha \mid 1] = O(B[\alpha \mid 1]) \setminus \bigcup \left\{O(B[\beta \mid 2]) : \beta \in \Sigma, \beta(1) = \alpha(1)\right\},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D[\alpha \mid n] = O(B[\alpha \mid n]) \setminus \bigcup \left\{O(B[\beta \mid n+1]) : \beta \in \Sigma, \beta(i) = \alpha(i), 1 \leq i \leq n\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots..$$

For $f^{-1}(M) = \bigcup_{\alpha \in \Sigma} B[\alpha \mid 1]$ and (2.1) holds, from [14, I.1.1 (7)] it follows that each set of the sequence $\{D, D[\alpha \mid 1], \ldots, D[\alpha \mid n], \ldots\}$ is nowhere dense in E. Hence

$$Q = D \cup \left(\bigcup \left\{D[\alpha \mid n] : \alpha \in \Sigma, n \in \mathbb{N}\right\}\right)$$

is of first category in E.

In order to show that $f^{-1}(M)$ has the Baire property in E it is enough to prove that $O\left(f^{-1}(M)\right) \setminus f^{-1}(M) \subseteq Q$, since this implies that $O\left(f^{-1}(M)\right) \setminus f^{-1}(M)$ is a set of first category. So it suffices to show that $O\left(f^{-1}(M)\right) \setminus Q \subseteq f^{-1}(M)$.

Pick $x \in O\left(f^{-1}(M)\right) \setminus Q$ and choose a sequence $\{\beta_n\} \subseteq \Sigma$, with $\beta_{n+1}(i) = \beta_n(i)$ for $1 \leq i \leq n$, such that $x \in O(B[\beta_n \mid n])$ for every $n \in \mathbb{N}$. Since $x \in O(B[\beta_n \mid n])$ then $\left(x + n^{-1}U_n\right) \cap B[\beta_n \mid n]$ is a set of second category in E for each $n \in \mathbb{N}$, which implies that $\left(x + n^{-1}U_n\right) \cap B[\beta_n \mid n] \neq \emptyset$ for every $n \in \mathbb{N}$. Hence for each $n \in \mathbb{N}$ we can pick

$$x_n \in \left(x + n^{-1}U_n\right) \cap B[\beta_n \mid n].$$

So it is clear that $x_n \to x$ in (E,τ) and $f(x_n) \in T(\beta_n \mid n) \cap M$ for each $n \in \mathbb{N}$. Hence there is a sequence $\{\delta_n\}$ in Σ with $\delta_n(i) = \beta_n(i)$ for $1 \leq i \leq n$ such that $f(x_n) \in T(\delta_n)$ for all $n \in \mathbb{N}$. Defining $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\beta(n) = \beta_n(n)$ for every $n \in \mathbb{N}$ then $\beta(i) = \beta_n(i)$ if $1 \leq i \leq n$, which implies that $\beta_n(i) \to \beta(i)$ for each $i \in \mathbb{N}$. Hence $\delta_n \to \beta$ in $\mathbb{N}^{\mathbb{N}}$ and the sequence $\{\delta_n\} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$. Thereby there is a subnet $\{z_d : d \in D\}$ of $\{x_n\}$ such that the net $\left\{f(z_d) : d \in D\right\}$ converges weakly in F to some $z \in M$, for M weakly closed in F. Therefore $f(z_d) \to z$ in (F,ρ). But since $z_d \to x$ in (E,τ) and $f : (E,\tau) \to (F,\rho)$ is continuous, then $z = f(x)$. Thus $x \in f^{-1}(M)$, as required. □

Theorem 2.3 *Let E be a Baire locally convex space (or a locally convex hull of Baire locally convex spaces) and let F be a weakly C-Suslin locally convex space. If $f : E \to F$ is a linear mapping with closed graph, then it is continuous.*

Proof. If V is a closed absolutely convex neighborhood of the origin in F, since E is of second category in itself and $\bigcup_{n=1}^{\infty} nf^{-1}(V) = E$ we can see that $f^{-1}(V)$ is of second category in E. As in addition V is weakly closed, then Lemma 2.1 guarantees that $f^{-1}(V)$ has the Baire property. Consequently, Banach's difference theorem for locally convex spaces establishes that $f^{-1}(V)$ is a neighborhood of the origin in E.

If E is a locally convex hull of Baire spaces, there is a standard technique to get the conclusion (see [14, Chapter 1]). □

Theorem 2.4 *Let E be a Baire linear topological space and F a C-Suslin linear topological space. If $f : E \to F$ is a linear mapping with closed graph, then it is continuous.*

Proof. The proof of Lemma 2.1 can be adapted to the non locally convex setting, as well as the proof of the supplementary result [14, I.4.1 (21)]. Consequently, if M is a closed subset F then $f^{-1}(M)$ has the Baire property and an application of Banach's difference theorem completes the proof. □

3 Some applications

The following result is a version for C-Suslin spaces of an old theorem of De Wilde and Sunyach [2] which improves the main result of [7], as follows from the subsequent corollary (Corollary 3.2).

Theorem 3.1 *Let E be a Baire locally convex space. If E is C-Suslin, then E is a Fréchet space.*

Proof. First let us show that E is a metrizable locally convex space. Since E is a C-Suslin space there is $T : \Sigma \to \mathcal{P}(E)$ with $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = E$ and if $\{\alpha_n\} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$ then $\bigcup_{n=1}^{\infty} T(\alpha_n)$ is a relatively countably compact set in E. Given that E is a Baire space and $\mathcal{E} = \{T(\alpha \mid n) : \alpha \in \Sigma, n \in \mathbb{N}\}$ is a web of subsets of E covering E, there is a sequence $\{\gamma_n\}$ in Σ with $\gamma_{n+1}(i) = \gamma_n(i)$ for $1 \leq i \leq n$ such that $\overline{T(\gamma_n \mid n)}$ has an interior point in E for every $n \in \mathbb{N}$. Hence we may choose a decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of absolutely convex neighborhoods of the origin in E such that

$$U_n \subseteq \overline{T(\gamma_n \mid n)} - \overline{T(\gamma_n \mid n)}$$

for every $n \in \mathbb{N}$.

If V is an absolutely convex neighborhood of the origin in E, let W be a closed absolutely convex neighborhood of the origin in E such that $2W \subseteq V$. Defining $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\gamma(n) = \gamma_n(n)$ for every $n \in \mathbb{N}$ then $\gamma_n \to \gamma$ in $\mathbb{N}^{\mathbb{N}}$ and applying Claim 2.2 with $f = id_E$ there exists $q \in \mathbb{N}$ such that $T(\gamma_q \mid q) \subseteq qW$, which implies that

$$U_q \subseteq \overline{T(\gamma_q \mid q)} - \overline{T(\gamma_q \mid q)} \subseteq qV.$$

This shows that $q^{-1}U_q \subseteq V$, so that $\left\{n^{-1}U_n : n \in \mathbb{N}\right\}$ is a countable base of absolutely convex neighborhoods of the origin in E. Consequently, E is metrizable.

If F denotes the completion of E, let us show that E has the Baire property in F. Indeed, setting

$$D = O(E) \setminus \bigcup \left\{O(T(\alpha \mid 1)) : \alpha \in \mathbb{N}\right\},$$

$$D[\alpha \mid 1] = O(T(\alpha \mid 1)) \setminus \bigcup \left\{O(T(\beta \mid 2)) : \beta \in \Sigma, \beta(1) = \alpha(1)\right\},$$

. . . .

$$D[\alpha \mid n] = O(T(\alpha \mid n)) \setminus \bigcup \{O(T(\beta \mid n+1)) : \beta \in \Sigma, \beta(i) = \alpha(i), 1 \leq i \leq n\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

then $Q = D \cup \left(\bigcup \left\{ D\left[\alpha \mid n\right] : \alpha \in \Sigma, n \in \mathbb{N} \right\} \right)$ is a set of first category in E and we may reason as in the final part of the proof of Lemma 2.1 with E instead of $f^{-1}(M)$ to show that $O(E) \setminus Q \subseteq E$. Then it follows from [14, I.1.3 (10)] that $F = E$. □

Corollary 3.2 (Kakol and López Pellicer, [7, Theorem]) *If E is a Baire locally convex space which admits a relatively countably compact resolution, then E is a separable Fréchet space.*

P r o o f. This is a consequence of the fact that every topological space X with a relatively countably compact resolution is C-Suslin with $\Sigma = \mathbb{N}^{\mathbb{N}}$. Let us provide a proof of this fact. According to the proof of [6, Proposition 1.1], if X is covered by an ordered family $\left\{ A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$ of relatively countably compact sets, given $\gamma \in \mathbb{N}^{\mathbb{N}}$ and $\{\gamma_n\} \subseteq \mathbb{N}^{\mathbb{N}}$ with $\gamma_n(i) = \gamma(i)$ for $1 \leq i \leq n$, then each sequence $\{x_n\}$ in X such that $x_n \in A_{\gamma_n}$ for every $n \in \mathbb{N}$ has a cluster point in X. Hence, defining $T : \mathbb{N}^{\mathbb{N}} \to \mathcal{P}(X)$ by $T(\alpha) = A_\alpha$ it happens that $\bigcup \{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$, and if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ we claim that $\{x_n\}$ has a cluster point in X. Indeed, if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ we can fix a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ satisfying that $\alpha_{n_k}(i) = \alpha(i)$ for $1 \leq i \leq k$. Since $x_{n_k} \in T(\alpha_{n_k}) = A_{\alpha_{n_k}}$ for every $k \in \mathbb{N}$, we can see that $\{x_{n_k}\}$ has a cluster point in X. Hence $\{x_n\}$ has also a cluster point in X, which proves that X is C-Suslin.

The fact that E is a C-Suslin space allows us to apply Theorem 3.1 to ensure that E is a Fréchet space. Consequently $\left\{ \overline{A_\gamma} : \gamma \in \mathbb{N}^{\mathbb{N}} \right\}$ is an ordered covering of E by compact sets, which as is well known forces E to be a quasi-Suslin space. Thus E is a K-analytic metric space, hence separable. □

4 *C*-Suslin spaces are not productive

Every closed subspace of a C-Suslin topological space is C-Suslin [13, Proposition 1] and every continuous image of a C-Suslin space is C-Suslin [13, Proposition 2]. However, the finite product of C -Suslin spaces need not be a C-Suslin space, as the following example shows. Let us recall that a completely regular space X is called *Lindelöf* Σ (see [1, Chapter 0]) if it is the continuous image of a Lindelöf p-space, i.e. of a space Y that can be perfectly mapped onto a separable metric space Z. If $\mathcal{E}$ and $\mathcal{A}$ are two families of subsets of a topological space X, $\mathcal{E}$ is called a *network* modulo $\mathcal{A}$ if for each set $A \in \mathcal{A}$ and each neighborhood U of A in X there is some $E \in \mathcal{E}$ with $A \subseteq E \subseteq U$. All completely regular spaces X with a countable network modulo a cover of X by compact sets are Lindelöf Σ-spaces (see [1, Proposition IV.9.1]).

Example 4.1 There exists a C-Suslin space G such that $G \times G$ is not a C-Suslin space.

P r o o f. In [6, Example 2.1] it is shown that if X is a discrete space of cardinality $\mathfrak{c}$ there exists a countably compact topological space G such that X is homeomorphic to a closed subspace of the product $G \times G$. Since G is countably compact, G is quasi-Suslin and hence C-Suslin. If $G \times G$ is C-Suslin, then X is C-Suslin as well and there is a map $T : \Sigma \to X$ such that $\bigcup \{T(\alpha) : \alpha \in \Sigma\} = X$ and if $\{\alpha_n\} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$, then $\bigcup_{n=1}^{\infty} T(\alpha_n)$ is a relatively countably compact subset of X. Setting $A_\alpha = \bigcap_{n=1}^{\infty} T(\alpha \mid n)$ then A_α is a compact set in X. In fact, if $\{x_n\}$ is a sequence in A_α then $x_n \in T(\alpha \mid n)$ for each $n \in \mathbb{N}$. So we may choose $\{\beta_n\} \subseteq \Sigma$ with $\beta_n(i) = \alpha(i)$ for $1 \leq i \leq n$ such that $x_n \in T(\beta_n)$ for every $n \in \mathbb{N}$. Since $\beta_n \to \alpha$, then $\{x_n\}$ has a cluster point x in X. Due to the fact that $x_k \in T(\alpha \mid k) \subseteq T(\alpha \mid n)$ for every $k \geq n$ then $x \in \overline{T(\alpha \mid n)} = T(\alpha \mid n)$ for every $n \in \mathbb{N}$. So $x \in A_\alpha$ and hence A_α is countably compact. Thus A_α is compact.

Therefore $\mathcal{A} := \{A_\alpha : \alpha \in \Sigma\}$ is a covering of X by compact (actually finite) sets. We claim that the family

$$\mathcal{N} := \{T(\alpha \mid n) : (\alpha, n) \in \Sigma \times \mathbb{N}\}$$

is a countable network modulo $\mathcal{A}$. Indeed, if U is an open neighborhood of A_α in X there is $n \in \mathbb{N}$ such that $A_\alpha \subseteq T(\alpha \mid n) \subseteq U$. Otherwise for each $n \in \mathbb{N}$ there is $y_n \in T(\alpha \mid n) \setminus U$. Choosing $\{\beta_n\} \subseteq \Sigma$ with $\beta_n(i) = \alpha(i)$ for $1 \leq i \leq n$ such that $y_n \in T(\beta_n)$ for each $n \in \mathbb{N}$, then $\{y_n\}$ has a cluster point y in X. Reasoning as above we can see that $y \in A_\alpha$. But since $y_n \in X \setminus U$ for every $n \in \mathbb{N}$ then it must be $y \notin U$, a contradiction.

The fact that $\mathcal{N}$ is a countable network modulo the covering $\mathcal{A}$ of X (consisting of compact sets) assures that X is a Lindelöf Σ-space, hence Lindelöf. Since X is metrizable, it must be separable, which implies that $|X| \leq \aleph_0$. But $|X| = \mathfrak{c}$, a contradiction. □

Remark 4.2 Alternatively, we can use the Δ-Lemma to establish the previous example. Indeed, let X be as above and assume by contradiction that there is a map $T : \Sigma \to X$ such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$ and if $\{\alpha_n\}_{n=1}^{\infty} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$, then $\bigcup_{n=1}^{\infty} T(\alpha_n)$ is a relatively countably compact subset of X . Note that for each $\alpha \in \Sigma$ the set $T(\alpha)$ is finite, since it is a relatively countably compact subset of a discrete space. Since $X$is uncountable, the index set Σ must be uncountable too. So, with the help of the Δ -Lemma, we can choose an uncountable subset Δ of Σ such that $\{T(\alpha) : \alpha \in \Delta\}$ is a Δ -system, i.e. such that there is a finite subset Z of X with $T(\alpha) \cap T(\beta) = Z$for all $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. Now, if $\alpha \in \overline{\Delta}$, where the closure is in metric space $\mathbb{N}^{\mathbb{N}}$, there exists an injective sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq \Delta$ such that $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$. But then $\bigcup_{n=1}^{\infty} T(\alpha_n)$ must be an infinite relatively countably compact subset of X, a contradiction.

5 Further results

We end with two observations. In first place it has been pointed out in the proof of Corollary 3.2 that every topological space X that admits an ordered cover $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ by relatively countably compact sets is a C-Suslin space. On the opposite direction one has Proposition 5.1 below. On the other hand, every metrizable K-analytic space does contain many C-Suslin subspaces which need not be K-Suslin, as Proposition 5.2 suggests.

Proposition 5.1 *If X is a C-Suslin space, there exists a covering $\{A_\alpha : \alpha \in \Sigma\}$ of X made up of relatively countably compact sets such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$ and $\alpha, \beta \in \Sigma$.*

P r o o f . Let $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ and $T : \Sigma \to \mathcal{P}(X)$ be such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$ and if $\{\alpha_n\} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point in X. Setting $A_\alpha := \bigcup\{T(\beta) : \beta \leq \alpha, \beta \in \Sigma\}$ for every $\alpha \in \Sigma$, clearly $\bigcup\{A_\alpha : \alpha \in \Sigma\} = X$ and if $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$ then $A_\alpha \subseteq A_\beta$. On the other hand, if $\{x_n\} \subseteq A_\alpha$ and $\{\beta_n\} \subseteq \Sigma$ is such that $\beta_n \leq \alpha$ and $x_n \in T(\beta_n)$ for all $n \in \mathbb{N}$, the fact that $\{\beta_n\} \subseteq \prod_{i=1}^{\infty}\{1, \ldots, \alpha(i)\}$ guarantees the existence of a subsequence $\{\beta_{n_k}\}$ of $\{\beta_n\}$ which converges in $\mathbb{N}^{\mathbb{N}}$. This implies that $\{x_{n_k}\}$, and hence $\{x_n\}$, has a cluster point in X, which shows that A_α is a relatively countably compact set. □

Proposition 5.2 *If X is a metrizable K-analytic space (so analytic by [8, Proposition 6.3]), for every nonempty subset Σ of $\mathbb{N}^{\mathbb{N}}$ there exists a subspace Y_Σ of X which is C-Suslin.*

P r o o f . If $T : \mathbb{N}^{\mathbb{N}} \to \mathcal{K}(X)$ is a usc map such that $\bigcup\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$, define $S : \Sigma \to \mathcal{P}(X)$ by $S(\alpha) = \bigcap_{n=1}^{\infty} \overline{T(\alpha \mid n)}$ and put $Y_\Sigma := \bigcup\{S(\alpha) : \alpha \in \Sigma\}$. If $\{\gamma_n\} \subseteq \Sigma$ converges to some γ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in S(\gamma_n)$ for each $n \in \mathbb{N}$, working with a subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$, we may suppose that $\gamma_{n_k}(i) = \gamma(i)$ for $1 \leq i \leq k$.

Hence $x_{n_k} \in \overline{T(\gamma_{n_k} \mid k)} = \overline{T(\gamma \mid k)}$ for every $k \in \mathbb{N}$. If $B(x, \epsilon)$ denotes the open ball of center x and radius $\epsilon > 0$, for each $k \in \mathbb{N}$ take $y_{n_k} \in B\left(x_{n_k}, \frac{1}{k}\right) \cap T(\gamma \mid k)$ and pick $\delta_k \in \mathbb{N}^{\mathbb{N}}$ with $\delta_k(i) = \gamma(i)$ for $1 \leq i \leq k$ such that $y_{n_k} \in T(\delta_k)$. Then $\delta_k \to \gamma$ in $\mathbb{N}^{\mathbb{N}}$ and consequently $\left\{y_{n_k}\right\}$ has a cluster point y in X. Clearly y is a cluster point of the sequence $\{x_n\}$ as well, and since $T(\gamma \mid k+1) \subseteq T(\gamma \mid k)$ for every $n \in \mathbb{N}$ one can see that $x_{n_j} \in \overline{T(\gamma \mid k)}$ for each $j \geq k$, so that $y \in \overline{T(\gamma \mid k)}$ for all $k \in \mathbb{N}$. This ensures that $y \in S(\gamma) \subseteq Y_\Sigma$ and, therefore, that Y_Σ is a C-Suslin space. □

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