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On topological spaces and topological groups with certain local countable networks

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A B S T R A C T

Being motivated by the study of the space $C_c(X)$ of all continuous real-valued functions on a Tychonoff space X with the compact–open topology, we introduced in [16] the concepts of a *cp*-network and a *cn*-network (at a point x) in X. In the present paper we describe the topology of X admitting a countable *cp*- or *cn*-network at a point $x \in X$. This description applies to provide new results about the strong Pytkeev property, already well recognized and applicable concept originally introduced by Tsaban and Zdomskyy [44]. We show that a Baire topological group G is metrizable if and only if G has the strong Pytkeev property. We prove also that a topological group G has a countable *cp*-network if and only if G is separable and has a countable *cp*-network at the unit. As an application we show, among the others, that the space $D'(\Omega)$ of distributions over open $\Omega \subseteq \mathbb{R}^n$ has a countable *cp*-network, which essentially improves the well known fact stating that $D'(\Omega)$ has countable tightness. We show that, if X is an $\mathcal{MK}_\omega$-space, then the free topological group $F(X)$ and the free locally convex space $L(X)$ have a countable *cp*-network. We prove that a topological vector space E is p-normed (for some $0 < p \leq 1$) if and only if E is Fréchet–Urysohn and admits a fundamental sequence of bounded sets.

1. Introduction

All topological spaces are assumed to be Hausdorff. Various topological properties generalizing metrizability have been studied intensively by topologists and analysts, especially like first countability, Fréchet–Urysohness, sequentiality and countable tightness (see [9,25]). It is well-known that

$$\text{metric} \Longrightarrow \begin{array}{c}\text{first}\\ \text{countable}\end{array} \Longrightarrow \begin{array}{c}\text{Fréchet–}\\ \text{Urysohn}\end{array} \Longrightarrow \text{sequential} \Longrightarrow \begin{array}{c}\text{countable}\\ \text{tight}\end{array},$$

and none of these implications can be reversed.

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One of the most immediate extensions of the class of separable metrizable spaces are the classes of cosmic and $\aleph_0$-spaces in sense of Michael [30].

Definition 1.1. ([30]) A topological space X is called

- *cosmic*, if X is a regular space with a countable network (a family $\mathcal{N}$ of subsets of X is called a *network* in X if, whenever $x \in U$ with U open in X, then $x \in N \subseteq U$ for some $N \in \mathcal{N}$);
- an $\aleph_0$-*space*, if X is a regular space with a countable k-network (a family $\mathcal{N}$ of subsets of X is called a k-*network* in X if, whenever $K \subseteq U$ with K compact and U open in X, then $K \subseteq \bigcup \mathcal{F} \subseteq U$ for some finite family $\mathcal{F} \subseteq \mathcal{N}$).

These classes of topological spaces were intensively studied in [20,23,30] and [31].

Having in mind the Nagata–Smirnov metrization theorem, Okuyama [38] and O'Meara [34] introduced the classes of σ-spaces and $\aleph$-spaces, respectively.

Definition 1.2. A topological space X is called

- ([38]) a σ-*space* if X is regular and has a σ-locally finite network;
- ([34]) an $\aleph$-*space* if X is regular and has a σ-locally finite k-network.

Any metrizable space X is an $\aleph$-space. O'Meara [33] proved that an $\aleph$-space which is either first countable or locally compact is metrizable. Every compact subset of a σ-space is metrizable [32]. Further results see [22].

Pytkeev [40] proved that every sequential space satisfies the property, known actually as the *Pytkeev property*, which is stronger than countable tightness: a topological space X has the *Pytkeev property* if for each $A \subseteq X$ and each $x \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \dots$ of A such that each neighborhood of x contains some A_n. Tsaban and Zdomskyy [44] strengthened this property as follows. A topological space X has the *strong Pytkeev property* if for each $x \in X$, there exists a countable family $\mathcal{D}$ of subsets of X, such that for each neighborhood U of x and each $A \subseteq X$ with $x \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite. Next, Banakh [1] introduced the concept of the Pytkeev network in X as follows: A family $\mathcal{N}$ of subsets of a topological space X is called a *Pytkeev network at a point* $x \in X$ if $\mathcal{N}$ is a network at x and for every open set $U \subseteq X$ and a set A accumulating at x there is a set $N \in \mathcal{N}$ such that $N \subseteq U$ and $N \cap A$ is infinite. Hence X has the strong Pytkeev property if and only if X has a countable Pytkeev network at each point $x \in X$.

In [18] we proved that the space $C_c(X)$ has the strong Pytkeev property for every Čech-complete Lindelöf space X. For the proof of this result we constructed a family $\mathcal{D}$ of sets in $C_c(X)$ such that for every neighborhood $U_{\mathbf{0}}$ of the zero function $\mathbf{0}$ the union $\bigcup\{D \in \mathcal{D} : \mathbf{0} \in D \subseteq U_{\mathbf{0}}\}$ is a neighborhood of $\mathbf{0}$ (see the condition (**D**) in [18]). Having in mind this idea for $C_c(X)$ we proposed in [16] the following types of networks which will be applied in the sequel.

Definition 1.3. ([16]) A family $\mathcal{N}$ of subsets of a topological space X is called

- a *cn-network at a point* $x \in X$ if for each neighborhood O_x of x the set $\bigcup\{N \in \mathcal{N} : x \in N \subseteq O_x\}$ is a neighborhood of x; $\mathcal{N}$ is a *cn-network* in X if $\mathcal{N}$ is a *cn*-network at each point $x \in X$;
- a *ck-network at a point* $x \in X$ if for any neighborhood O_x of x there is a neighborhood U_x of x such that for each compact subset $K \subseteq U_x$ there exists a finite subfamily $\mathcal{F} \subseteq \mathcal{N}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subseteq \bigcup \mathcal{F} \subseteq O_x$; $\mathcal{N}$ is a *ck-network* in X if $\mathcal{N}$ is a *ck*-network at each point $x \in X$;

- a *cp-network at a point* $x \in X$ if $\mathcal{N}$ is a network at x and for any subset $A \subseteq X$ with $x \in \overline{A} \setminus A$ and each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and $N \cap A$ is infinite; $\mathcal{N}$ is a *cp-network* in X if $\mathcal{N}$ is a Pytkeev network at each point $x \in X$.

Therefore the notion of the *cn*-network naturally connects the notions of the network and the base at a point $x \in X$.

Now Definitions 1.1, 1.2 and 1.3 motivated the following concepts.

Definition 1.4. A topological space X is called

- ([1]) a $\mathfrak{P}_0$-*space* if X has a countable Pytkeev network;
- ([16]) a $\mathfrak{P}$-*space* if X has a σ-locally finite *cp*-network.

It is known that: $\mathfrak{P}_0$-space $\Rightarrow$ $\aleph_0$-space $\Rightarrow$ cosmic; but the converse is false (see [1,30]). Each $\mathfrak{P}$-space X has the strong Pytkeev property and is an $\aleph$-space [16].

Definition 1.3 allows us to define the following cardinals of topological spaces.

Definition 1.5. ([16]) Let x be a point of a topological space X and $\mathfrak{n} \in \{cp, ck, cn\}$. The smallest size $|\mathcal{N}|$ of an $\mathfrak{n}$-network $\mathcal{N}$ at x is called the $\mathfrak{n}$-*character of* X *at the point* x and is denoted by $\mathfrak{n}_\chi(X, x)$. The cardinal $\mathfrak{n}_\chi(X) = \sup\{\mathfrak{n}_\chi(X, x) : x \in X\}$ is called the $\mathfrak{n}$-*character* of X. The $\mathfrak{n}$-*netweight*, $\mathfrak{n}w(X)$, of X is the least cardinality of an $\mathfrak{n}$-network for X.

We shall say that a topological space X has *countable* $\mathfrak{n}$-*character* if $\mathfrak{n}_\chi(X) \le \aleph_0$. Note that $\mathfrak{n}_\chi(X)$ is finite if and only if X is discrete, so $\mathfrak{n}_\chi(X) = 1$. It is clear that if $ck_\chi(X)$ is countable, then also $cn_\chi(X)$ is countable. We shall need the following

Proposition 1.6. *([2]) Any countable cp-network at a point x of a topological space X is a ck-network at x.*

So, for a topological space X, we have the following

$$cp_\chi(X) \le \aleph_0 \Longrightarrow ck_\chi(X) \le \aleph_0 \Longrightarrow cn_\chi(X) \le \aleph_0.$$

Our first result is a characterization of cosmic, $\aleph_0$- and $\mathfrak{P}_0$-groups G by their separability and countability of the corresponding types of networks at the unit $e \in G$. Recall ([30]) that every cosmic space is (even hereditary) separable. So it is natural to ask: For $\mathfrak{n} \in \{cp, ck, cn\}$, does any separable space X of countable $\mathfrak{n}$-character have also a countable $\mathfrak{n}$-network? In general the answer is negative. Indeed, any first countable separable non-metrizable compact space K trivially has countable *cp*-character, but K is not even a σ-space because it is not metrizable. However, for the group case we have the following

Theorem 1.7. *Let G be a topological group. Then G is a cosmic space (resp. an $\aleph_0$-space or a $\mathfrak{P}_0$-space) if and only if G is separable and has countable cn-character (resp. countable ck-character or countable cp-character).*

In [18] we proved that the space of distributions $\mathfrak{D}'(\Omega)$ over an open set $\Omega \subset \mathbb{R}^n$ has the strong Pytkeev property. Since $\mathfrak{D}'(\Omega)$ is also separable, Theorem 1.7 strengthens this result as follows.

Corollary 1.8. *The space of distributions $\mathfrak{D}'(\Omega)$ is a $\mathfrak{P}_0$-space.*

For other applications of Theorem 1.7, see Corollaries 2.1 and 2.2.

One of the most interesting and essential problems while studying a certain class of topological spaces is to describe the topology of spaces from this class. Recently we have described in [17] the topology τ of any cosmic space (or $\aleph_0$-space and $\mathfrak{P}_0$-space) (X,τ) in term of a "small base" $\mathcal{U}(\tau)$. Since the class of spaces with a countable *cn*-character (or a countable *ck*- and *cp*-character) provides the most natural *local* generalization of cosmic spaces (or $\aleph_0$-spaces or $\mathfrak{P}_0$-spaces, respectively), it is natural to ask whether such a description of the topology (as mentioned above) can be also obtained for this wider class. This question is of an independent interest for the class of topological groups G because the topology of G is defined essentially by the filter of open neighborhoods at the unit e of G. The main result of this paper is to give a positive answer to this question. For this purpose we need the following concepts.

Let Ω be a set and I be a partially ordered set with an order $\le$. We say that a family $\{A_i\}_{i\in I}$ of subsets of Ω is *I-decreasing* if $A_j \subseteq A_i$ for every $i \le j$ in I. One of the most important example of partially ordered sets is the product $\mathbb{N}^\mathbb{N}$ endowed with the natural partial order, i.e., $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i \in \mathbb{N}$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set $I_k(\alpha) := \{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1,\dots,k\}$. In fact $I_k(\alpha)$ is completely defined by the finite subset $\{\alpha_1,\dots,\alpha_k\}$ of $\mathbb{N}$. So the family $\{I_k(\alpha) : k \in \mathbb{N}, \alpha \in \mathbb{N}^\mathbb{N}\}$ is countable. Let $\mathbf{M} \subseteq \mathbb{N}^\mathbb{N}$ and $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ be an $\mathbf{M}$-decreasing family of subsets of a set Ω. Then we define the countable family $\mathcal{D}_\mathcal{U}$ of subsets of Ω by

$$\mathcal{D}_\mathcal{U} := \{D_k(\alpha) :\ \alpha \in \mathbf{M},\ k \in \mathbb{N}\},\ \text{where } D_k(\alpha) := \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}} U_\beta,$$

and say that $\mathcal{U}$ satisfies the *condition* $(\mathbf{D})$ if $U_\alpha = \bigcup_{k\in\mathbb{N}} D_k(\alpha)$ for every $\alpha \in \mathbf{M}$. A similar condition naturally appears and is essentially used in [15,17–19].

Definition 1.9. ([17]) A topological space (X,τ) has a *small base* if there exists an $\mathbf{M}$-decreasing base of τ for some $\mathbf{M} \subseteq \mathbb{N}^\mathbb{N}$.

The above notion has been used to describe the topology of cosmic spaces, $\aleph_0$-spaces and $\mathfrak{P}_0$-spaces, respectively. The item (iii) of the following theorem immediately follows from the proof of (i) and (ii) of Theorem 1.10 given in [17].

Theorem 1.10. *([17]) Let (X,τ) be a topological space. Then:*

(i) *X has a countable network (and is cosmic) if and only if X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ satisfying the condition $(\mathbf{D})$ (and is regular). In that case the family $\mathcal{D}_\mathcal{U}$ is a countable network in X.*

(ii) *X has a countable k-network (and is an $\aleph_0$-space) if and only if X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ satisfying the condition $(\mathbf{D})$ such that the family $\mathcal{D}_\mathcal{U}$ is a countable k-network in X (and is regular).*

(iii) *X has a countable cp-network (and is a $\mathfrak{P}_0$-space) if and only if X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ satisfying the condition $(\mathbf{D})$ such that the family $\mathcal{D}_\mathcal{U}$ is a countable cp-network in X (and is regular).*

For this three cases we can find a small base $\mathcal{U}$ such that $U_\alpha \ne U_\beta$ for $\alpha \ne \beta$ and $\mathcal{U} = \tau$, what means that for every $W \in \tau$ there exists $\alpha \in \mathbf{M}$ such that $W = U_\alpha$.

Note that the condition $(\mathbf{D})$ is essential in Theorem 1.10: the Bohr compactification $b\mathbb{Z}$ of the discrete group $\mathbb{Z}$ of integers has a small base [17], but the compact group $b\mathbb{Z}$ is not cosmic since it is not metrizable.

The following local version of the concept from Definition 1.9 will play an essential role in this paper.

Definition 1.11. Let x be a point in a topological space (X,τ). We say that X has a *small base at x* if there exists an $\mathbf{M}_x$-decreasing base at x for some $\mathbf{M}_x \subseteq \mathbb{N}^\mathbb{N}$.

If a topological space X has an $\mathbf{M}_x$-decreasing base at x, we shall also say that the space X has a (local) $\mathbf{M}_x$-*base* at x. Clearly, if G is a topological group with an $\mathbf{M}_e$-base at the unit e, then $g\mathbf{M}_e$ is a small base at each point $g \in G$. So we shall say simply that the group G has a (local) $\mathbf{M}$-*base* omitting the subscript e. A number of specialists in their papers dealing with locally convex spaces (*lcs* for short) used the notation of $\mathfrak{G}$-*base* for a local $\mathbb{N}^{\mathbb{N}}$-base at zero (see, for example, [6,25]). In this special case (i.e., when $\mathbf{M}_x = \mathbb{N}^{\mathbb{N}}$) we shall follow the same notation. Topological groups with a $\mathfrak{G}$-base are thoroughly studied in [19] (see also [15,18]). Note that any metrizable group G has a $\mathfrak{G}$-base.

Below we describe the topology of a topological space X at a point x in which it has countable *cn*-, *ck*- or *cp*-character.

Theorem 1.12. *Let x be a point of a topological space X. Then:*

(i) *X has a countable cn-network at x if and only if X has a small base $\mathcal{U}(x) = \{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition* (**D**)*. In that case the family $\mathcal{D}_{\mathcal{U}(x)}$ is a countable cn-network at x.*
(ii) *X has a countable ck-network at x if and only if X has a small base $\mathcal{U}(x) = \{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition* (**D**) *such that the family $\mathcal{D}_{\mathcal{U}(x)}$ is a countable ck-network at x.*
(iii) *X has a countable cp-network at x if and only if X has a small base $\mathcal{U}(x) = \{U_\alpha : \alpha \in \mathbf{M}_x\}$ at x satisfying the condition* (**D**) *such that the family $\mathcal{D}_{\mathcal{U}(x)}$ is a countable cp-network at x.*

The main application of the above theorem is the next theorem which provides a characterization of metrizability for topological groups with the Baire property; this also gives a positive answer to Question 10 of [18] and a partial positive answers to Question 4.2 of [19] and Question 9 of [18].

Theorem 1.13. *Let G be a Baire topological group. Then the following are equivalent:*

(i) *G is metrizable.*
(ii) *G has the strong Pytkeev property.*
(iii) *G has countable ck-character.*
(iv) *G has countable cn-character.*
(v) *G has a $\mathfrak{G}$-base satisfying the condition* (**D**).

Since a regular topological space is cosmic if and only if it has a countable *cn*-network, we have the following

Corollary 1.14. *([17]) A Baire separable topological group G is metrizable if and only if G is cosmic.*

It turns out that even Fréchet–Urysohn lcs which are Baire need not to have a countable *cn*-character, see Remark 3.9 below.

Section 3 contains some applications of presented results for the class of topological vector spaces and for the free (abelian) topological groups $F(X)$ ($A(X)$, respectively), as well as for the free locally convex space $L(X)$ over a Tychonoff space X. This section deals with a natural question: For which topological spaces X the free groups $A(X)$, $F(X)$ and $L(X)$ have a $\mathfrak{G}$-base, countable $\mathfrak{n}$-character or are $\aleph_0$-spaces? As usual, $\chi(X)$ denotes the character of a topological space X. Recall also that a topological space X is called an $\mathcal{MK}_\omega$-*space* if the topology of X is defined by an increasing sequence of compact metrizable subsets. Denote by $\mathfrak{d}$ the cofinality of the partially ordered set $\mathbb{N}^{\mathbb{N}}$. The next theorem gives a partial answer to the aforementioned question and provides an alternative and simple proof of the equality $\chi(A(X)) = \chi(F(X)) = \mathfrak{d}$ for a non-discrete $\mathcal{MK}_\omega$-space X (which is one of the principal results of [35]). Also this theorem generalizes Theorem 4.16 of [19] and gives an affirmative answer to Question 4.17 of [19].

Theorem 1.15. *Let X be an $\mathcal{MK}_\omega$-space. Then:*

(i) *$A(X)$, $F(X)$ and $L(X)$ have a $\mathfrak{G}$-base satisfying the condition* (**D**) *and are $\mathfrak{P}_0$-spaces.*
(ii) (a) *If X is not discrete, then $\chi(A(X)) = \chi(F(X)) = \chi(L(X)) = \mathfrak{d}$.*
(b) *If X is discrete, then $\chi(A(X)) = \chi(F(X)) = 1$, and $\chi(L(X)) = \aleph_0$ if X is finite and $\chi(L(X)) = \mathfrak{d}$ if X is infinite.*

At the end of Section 3 we mostly deal with topological vector spaces (*tvs* for short) having a fundamental sequence of bounded sets. In particular, we extend Theorem 5.1 in [26] from the class of locally convex spaces to the class of tvs.

Theorem 1.16. *A tvs E is p-normed for some $0 < p \le 1$ if and only if E is Fréchet–Urysohn and admits a fundamental sequence of bounded sets.*

2. Proofs of Theorems 1.7, 1.12 and 1.13

We start this section with the proof of Theorem 1.7.

Proof of Theorem 1.7. We assume that G is not discrete since, otherwise, the theorem is trivial.

Noting that any cosmic space is (hereditary) separable, the necessity follows from the fact that a regular topological space X is cosmic (respectively, an $\aleph_0$-space or a $\mathfrak{P}_0$-space) if and only if X has a countable *cn*-network (respectively, a countable *ck*-network or a countable *cp*-network).

Let us prove sufficiency. Let $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ be a countable $\mathfrak{n}$-network at the unit e of G and let $\{g_n\}_{n\in\mathbb{N}}$ be a dense subset of G. Without loss of generality we can assume that $\mathcal{D}$ is closed under taking finite products. We show that the countable family

$$\mathcal{N} := \{g_n D_m : \ n,m \in \mathbb{N}\}$$

is an $\mathfrak{n}$-network in G.

Fix $g \in G$ and let gU be an open neighborhood of g. Take an open symmetric neighborhood W of e such that $W^3 \subseteq U$. In all three cases $\mathcal{D}$ is also a *cn*-network at e. Hence the set

$$W_0 := \bigcup \{D \in \mathcal{D} : e \in D \subseteq W\}$$

is a neighborhood of e. As $G = \cup_n g_n W_0$, we can find $r,t \in \mathbb{N}$ such that $g = g_r \cdot h$ and $h \in D_t \subseteq W_0$.

(1) Assume that $\mathcal{D}$ is a *cn*-network at e. Clearly,

$$\bigcup \{g_r D_t \cdot D_m : \ D_m \in \mathcal{D}, D_m \subseteq W\} = g_r D_t \cdot W_0 \subseteq g_r W_0^2 \subseteq g W^3 \subseteq gU,$$

and $g = g_r h \in \bigcap \{g_r D_t \cdot D_m : \ D_m \in \mathcal{D}, D_m \subseteq W\}$. So $\mathcal{N}$ is a *cn*-network at g.

(2) Assume that $\mathcal{D}$ is a *ck*-network at e. Take an open neighborhood $W_1 \subseteq W$ of e such that for every compact subset K of W_1 there exists a finite subfamily $\mathcal{F}$ of $\mathcal{D}$ satisfying $e \in \bigcap \mathcal{F}$ and $K \subseteq \bigcup \mathcal{F} \subseteq W$. As $G = \cup_n g_n W_1$, we can take $a,b \in \mathbb{N}$ such that $g = g_a \cdot h$ and $h \in D_b \subseteq W$. Now, for each compact subset gK of gW_1 we have

$$g_a D_b \cdot \mathcal{F} \subseteq \mathcal{N}, \ \ g \in \bigcap g_a D_b \cdot \mathcal{F}$$

and

$$gK = g_a h \cdot K \subseteq \bigcup g_a D_b \cdot \mathcal{F}.$$

Thus $\mathcal{N}$ is a *ck*-network at g.

(3) Assume that $\mathcal{D}$ is a *cp*-network at e. Let $A \subseteq G$ be such that $g \in \overline{A} \setminus A$. Since $e \in \overline{g^{-1}A} \setminus g^{-1}A$, there is $D_s \in \mathcal{D}$ such that $e \in D_s \subseteq W_0$ and $D_s \cap g^{-1}A$ is infinite. So

$$g_r h D_s \cap A \subseteq g_r(D_t \cdot D_s) \cap A$$

is infinite. As $g \in g_r(D_t \cdot D_s) \in \mathcal{N}$ and

$$g_r(D_t \cdot D_s) = g(h^{-1} \cdot D_t \cdot D_s) \subseteq g \cdot W_0^{-1} \cdot W_0^2 \subseteq g \cdot W^3 \subseteq g \cdot U,$$

we obtain that $\mathcal{N}$ is a *cp*-network at g. □

As a corollary of this theorem we obtain the following extensions of some results from [18] in the class of *separable* locally convex spaces.

Corollary 2.1. *Let E be a separable lcs satisfying one of the following conditions:*

(i) *E is a (DF)-space with countable tightness;*
(ii) *E is a sequential dual metric space;*
(iii) *E is a strict (LM)-space;*
(iv) *E is a quasibarrelled lcs with a $\mathfrak{G}$-base.*

Then E is a $\mathfrak{P}_0$-space.

Proof. By Theorems 3 and 9 of [18], the space E has the strong Pytkeev property, so Theorem 1.7 applies. □

In Corollary 9 of [18] we proved that the strong dual F' of a Fréchet space F has countable tightness if and only if F' has the strong Pytkeev property. If additionally F' is separable, Theorem 1.7 implies

Corollary 2.2. *Let F be a Fréchet space whose strong dual F' is separable. Then the strong dual F' has countable tightness if and only if F' is a $\mathfrak{P}_0$-space.*

In what follows we need the following result due to Banakh [1] (its independent proof is given in [16, Corollary 6.4]).

Theorem 2.3. *([1]) If X is an $\aleph_0$-space and Y is a $\mathfrak{P}_0$-space, then $C_c(X,Y)$ is a $\mathfrak{P}_0$-space.*

Theorems 1.7 and 1.10 with the comments after it, may suggest also the following

Example 2.4. There exists a large class of lcs E for which the existence of a $\mathfrak{G}$-base (the condition (**D**) is not required) in E implies that E is a $\mathfrak{P}_0$-space.

Proof. We prove the following two facts. (i) If $C_c(X)$ is a separable space admitting a $\mathfrak{G}$-base, then $C_c(X)$ is a $\mathfrak{P}_0$-space. (ii) There exists a $\mathfrak{P}_0$-spaces $C_c(X)$ (hence having countable *cp*-character) which do not admit a $\mathfrak{G}$-base. We need the following main result of [10]: $C_c(X)$ has a $\mathfrak{G}$-base if and only if X admits a compact resolution swallowing compact sets, i.e. a family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets covering X such that $K_\alpha \subseteq K_\beta$ for all $\alpha \le \beta$ in $\mathbb{N}^{\mathbb{N}}$ and each compact set of X is contained in some K_α. Since

$C_c(X)$ is separable, it admits a weaker metrizable and separable topology; hence X admits such a weaker topology. By the above remark we conclude that X has a compact resolution swallowing compact sets. Then applying [7, Theorem 3.6] the space X is an $\aleph_0$-space. Hence $C_c(X)$ is a $\mathfrak{P}_0$-space by Theorem 2.3. To complete the claim (ii) let $X = \mathbb{Q}$ be the space of rational numbers. Theorem 2.3 implies that $C_c(\mathbb{Q})$ is a $\mathfrak{P}_0$-space but (as $\mathbb{Q}$ does not have a compact resolution swallowing compact sets) it does not have a $\mathfrak{G}$-base. □

Below we prove Theorem 1.12.

Proof of Theorem 1.12. If x is an isolated point, we set $\mathbf{M}_x := \mathbb{N}^\mathbb{N}$ and $U_\alpha := \{x\}$ for each $\alpha \in \mathbf{M}_x$. Clearly, the family $\{U_\alpha : \alpha \in \mathbf{M}_x\}$ is as desired in all three cases of the theorem. So we shall assume that x is not isolated.

Since the sufficiency in all cases (i)–(iii) is obvious, we need to show only the necessity. Fix a countable *cn*-network (respectively, a countable *cp*- or *ck*-network) $\mathcal{D} = \{D_i\}_{i\in\mathbb{N}}$ at x which is closed under taking finite unions. Recall that D_i contains x for every $i \in \mathbb{N}$. Further, as a countable *cp*- or *ck*-network is also a *cn*-network, for all cases (i)–(iii) we first prove (in four steps below) that there is a small base at x satisfying condition (**D**). This proves also item (i) of the theorem.

Step 1. For every $k, i \in \mathbb{N}$, set

$$D_k^i := \bigcap_{l=1}^{k} D_{i-1+l}.$$

So, for each $i \in \mathbb{N}$, the sequence $\{D_k^i\}_{k\in\mathbb{N}}$ is decreasing. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$, set

$$A_\alpha := \bigcup_{i\in\mathbb{N}} D_{\alpha_i}^i = \bigcup_{i\in\mathbb{N}} \bigcap_{l=1}^{\alpha_i} D_{i-1+l}.$$

Clearly, $x \in A_\alpha$ and $A_\alpha \subseteq A_\beta$ for each $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ with $\beta \le \alpha$.

Step 2. Let V be a neighborhood of x. Set $J(V) := \{j \in \mathbb{N} : D_j \subseteq V\}$. Since x is not isolated, the family $J(V)$ is infinite. Now we prove that the following condition holds.

(**A**) If W is a neighborhood of x and $J(W) := \{j \in \mathbb{N} : D_j \subseteq W\} = \{n_k\}_{k\in\mathbb{N}}$ with $n_1 < n_2 < \ldots$, then there is $\alpha = \alpha(W) \in \mathbb{N}^\mathbb{N}$ such that
- (A_1) $\alpha_{n_k} = 1$ for every $k \in \mathbb{N}$;
- (A_2) $A_\alpha = \bigcup_{k\in\mathbb{N}} D_{n_k} (\subseteq W)$ is a neighborhood of x.

We construct $\alpha = \alpha(W)$ as follows. If $i = n_k$ for some $k \in \mathbb{N}$ we set $\alpha_i = 1$. So $D_{\alpha_i}^i = D_{n_k}$. Set $n_0 := 0$. Now, if $n_{k-1} < i < n_k$ for some $k \in \mathbb{N}$, we set $\alpha_i := n_k - i + 1$. Then

$$D_{\alpha_i}^i = \bigcap_{l=1}^{\alpha_i} D_{i-1+l} \subseteq D_{i-1+\alpha_i} = D_{n_k}.$$

Hence $A_\alpha = \bigcup_{k\in\mathbb{N}} D_{n_k}$. Since $\mathcal{D}$ is a *cn*-network at x, A_α is a neighborhood of x. Thus (A_1) and (A_2) are satisfied.

Step 3. Denote by $\mathbf{M}_x$ the set of all $\alpha \in \mathbb{N}^\mathbb{N}$ of the form $\alpha = \alpha(W)$ for some neighborhood W of x. For each $\alpha \in \mathbf{M}_x$ set $U_\alpha := A_\alpha$. Now, by (**A**), the family $\{U_\alpha : \alpha \in \mathbf{M}_x\}$ is a small base at x.

Step 4. Now we check that the condition (**D**) holds. It is clear that $\bigcup_{k\in\mathbb{N}} D_k(\alpha) \subseteq U_\alpha$. We prove the converse inclusion as follows

$$\begin{aligned}\bigcup_{k\in\mathbb{N}} D_k(\alpha) &= \bigcup_{k\in\mathbb{N}} \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}_x} U_\beta \\ &= \bigcup_{k\in\mathbb{N}} \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}_x} \left(\bigcup_{i\in\mathbb{N}} \bigcap_{l=1}^{\beta_i} D_{i-1+l}\right) \text{ (take only } i=k) \\ &\supseteq \bigcup_{k\in\mathbb{N}} \bigcap_{\beta\in I_k(\alpha)\cap\mathbf{M}_x} \left(\bigcap_{l=1}^{\beta_k} D_{k-1+l}\right) \text{ (since } \beta_k=\alpha_k) \\ &= \bigcup_{k\in\mathbb{N}} \bigcap_{l=1}^{\alpha_k} D_{k-1+l} = U_\alpha.\end{aligned}$$

(ii) Assume that $\mathcal{D}$ is a countable *ck*-network at x. It remains to show that the countable family $\mathcal{D}_{\mathcal{U}(x)}$ is also a *ck*-network at x.

Let O_x be a neighborhood of x. Set $W := \bigcup_{j\in J(O_x)} D_j$. Since $\mathcal{D}$ is a *ck*-network at x, there is a neighborhood $U_x \subseteq O_x$ of x such that for each compact subset $K \subseteq U_x$ there is $j \in J(O_x)$ such that $K \subseteq D_j \subseteq W \subseteq O_x$. So $U_x \subseteq W$ and hence W is a neighborhood of x.

Let K be a compact subset of U_x. Set $\alpha = \alpha(O_x)$. By the construction of W, there exists $i \in J(O_x)$ such that $K \subseteq D_i \subseteq W$. By the definition of $J(O_x)$, we have $i = n_k$ for some $k \in \mathbb{N}$. So $x \in D_i = D_{n_k}$ and $\alpha_{n_k} = 1$ by (A_1). As

$$\begin{aligned}D_{n_k}(\alpha) = \bigcap_{\beta\in I_{n_k}(\alpha)\cap\mathbf{M}_x} A_\beta &= \bigcap_{\beta\in I_{n_k}(\alpha)\cap\mathbf{M}_x} \left(\bigcup_{i\in\mathbb{N}} \bigcap_{l=1}^{\beta_i} D_{i-1+l}\right) \text{ (take } i=n_k) \\ &\supseteq \bigcap_{\beta\in I_{n_k}(\alpha)\cap\mathbf{M}_x} \left(\bigcap_{l=1}^{\beta_{n_k}} D_{n_k-1+l}\right) \text{ (since } \beta_{n_k}=\alpha_{n_k}=1) = D_{n_k},\end{aligned}$$

we obtain that $K \subseteq D_i \subseteq D_{n_k}(\alpha) \subseteq W$. Thus $\mathcal{D}_{\mathcal{U}(x)}$ is a countable *ck*-network at x.

(iii) Assume that $\mathcal{D}$ is a countable *cp*-network at x. We show that the countable family $\mathcal{D}_{\mathcal{U}(x)}$ is also a *cp*-network at x.

Let $A \subseteq X$ with $x \in \overline{A} \setminus A$ and let O_x be a neighborhood of x. Set $\alpha = \alpha(O_x)$ and $W := \bigcup_{j\in J(O_x)} D_j$. Since $\mathcal{D}$ is also a *cn*-network, W is a neighborhood of x. By the construction of W and the definition of *cp*-network, there exists $i \in J(O_x)$ such that $x \in D_i \subseteq W$ and $D_i \cap A$ is infinite. By the definition of $J(O_x)$, we have $i = n_k \in J(O_x)$ for some $k \in \mathbb{N}$. So $D_i = D_{n_k}$. In (ii) we proved that $D_i \subseteq D_{n_k}(\alpha)$. So $A \cap D_{n_k}(\alpha)$ is infinite. Thus $\mathcal{D}_{\mathcal{U}(x)}$ is a countable *cp*-network at x. □

Remark 2.5. In Proposition 2 of [18] it is shown that there is a topological group G with a $\mathfrak{G}$-base which has uncountable tightness (see also Example 3.12 below). So $cn_\chi(G) > \aleph_0$ (see [16]). Also the compact group $b\mathbb{Z}$ has a small base by [17], and $\chi(b\mathbb{Z}) = 2^{\aleph_0} = \mathfrak{c}$.

It is somewhat surprising that the validity of the condition (**D**) essentially depends on the chosen family $\mathbf{M}_x$ as the following example shows.

Example 2.6. We consider the Banach separable space ℓ^1 and build a small base at $\mathbf{0}$ in ℓ^1 as follows. Set $\mathbf{M}_0 := \mathbb{N}^{\mathbb{N}} \cap \ell^\infty$. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbf{M}_0$, set

$$U_\alpha = \left\{ (x_i)_{i\in\mathbb{N}} \in \ell^1 : \sum_i \alpha_i |x_i| < 1 \right\}.$$

Clearly, $\{U_\alpha : \alpha \in \mathbf{M}_0\}$ is a small base in ℓ^1. For each $\alpha = (\alpha_i) \in \mathbf{M}_0$ and every $k \in \mathbb{N}$ we have

$$D_k(\alpha) = \{(x_i) \in \ell^1 : \alpha_1|x_1| + \cdots + \alpha_k|x_k| < 1 \text{ and } 0 = x_{k+1} = x_{k+2} = \ldots\},$$

and $\bigcup_{k\in\mathbb{N}} D_k(\alpha) \neq U_\alpha$. So condition $(\mathbf{D})$ does not hold.

Now we are ready to prove Theorem 1.13.

Proof of Theorem 1.13. The implications (i)⇒(ii) and (iii)⇒(iv) are clear, (ii)⇒(iii) follows from Proposition 1.6, and (v)⇒(iv) follows from Theorem 1.12(i).

(i)⇒(v) If $\{V_n\}_{n\in\mathbb{N}}$ is a decreasing base of neighborhoods at the unit e of G, then the family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $U_\alpha := V_{\alpha_1}$ for $\alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}}$, is a $\mathfrak{G}$-base satisfying the condition $(\mathbf{D})$.

(iv)⇒(i) Let G have a countable cn-character. We have to show that G is metrizable. We prove that G has a countable base of neighborhoods at the unit e. By Theorem 1.12(i) there exists a small local base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ at e satisfying the condition $(\mathbf{D})$. We show that the countable family $\{\overline{D_k(\alpha)} \cdot \overline{D_k(\alpha)}^{\,-1} : \alpha \in \mathbf{M}, k \in \mathbb{N}\}$ contains a base of neighborhoods of e in G. Indeed, let W be an open neighborhood of e. Choose a symmetric open neighborhood V of e such that $V \cdot V \subseteq \overline{V} \cdot \overline{V} \subseteq W$. There exists $\alpha \in \mathbf{M}$ with $U_\alpha = \bigcup_k D_k(\alpha) \subseteq V$. Since $\mathrm{Int}(U_\alpha)$ is open in G and G is Baire, there exists $k \in \mathbb{N}$ such that $\mathrm{Int}(U_\alpha) \cap \overline{D_k(\alpha)}$ has a non-empty interior in U_α, so also in G. Therefore $\overline{D_k(\alpha)} \cdot \overline{D_k(\alpha)}^{\,-1}$ is a neighborhood of e which is contained in W. □

We do not know whether the assumption on a $\mathfrak{G}$-base to satisfy the condition $(\mathbf{D})$ can be omitted in Theorem 1.13(v). However, Example 2.6 shows that the condition $(\mathbf{D})$ essentially depends on the chosen family $\mathbf{M}_x$. This suggests the next question:

Problem 2.7. Let G be a Baire topological group with a $\mathfrak{G}$-base $\mathcal{U}$. Does $\mathcal{U}$ necessarily satisfy the condition $(\mathbf{D})$?

3. Applications to free (abelian) topological groups and topological vector spaces

Now we apply the obtained results to the important classes of free lcs and free (abelian) topological groups. The following concept is due to Markov [28], see also Graev [21].

Definition 3.1. Let X be a Tychonoff space. A topological group $F(X)$ (respectively, $A(X)$) is called *the (Markov) free (*respectively*, abelian) topological group* over X if $F(X)$ (respectively, $A(X)$) satisfies the following conditions:

(i) There is a continuous mapping $i : X \to F(X)$ (respectively, $i : X \to A(X)$) such that $i(X)$ algebraically generates $F(X)$ (respectively, $A(X)$).

(ii) If $f : X \to G$ is a continuous mapping to a (respectively, abelian) topological group G, then there exists a continuous homomorphism $\bar{f} : F(X) \to G$ (respectively, $\bar{f} : A(X) \to G$) such that $f = \bar{f} \circ i$.

The topological groups $F(X)$ and $A(X)$ always exist and are essentially unique. Note that the mapping i is a topological embedding [21,28]. If X is a discrete space, it is clear that $F(X)$ and $A(X)$ are also discrete. It is known (see [27]) that for each $\mathcal{MK}_\omega$-space X, the groups $F(X)$ and $A(X)$ are also $\mathcal{MK}_\omega$-spaces and hence sequential.

Analogously we can define free lcs (see [28,41]):

Definition 3.2. Let X be a Tychonoff space. The *free lcs* $L(X)$ on X is a pair consisting of a lcs $L(X)$ and a continuous mapping $i : X \to L(X)$ such that every continuous mapping f from X to a lcs E gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$.

Also the free lcs $L(X)$ always exists and is unique. The set X forms a Hamel basis for $L(X)$, and the mapping i is a topological embedding [41,12,13,45]. The identity map $id_X : X \to X$ extends to a canonical homomorphism $id_{A(X)} : A(X) \to L(X)$. It is known that $id_{A(X)}$ is an embedding of topological groups [43,46]. For example, if X is a finite space of cardinality n, then $L(X) \cong \mathbb{R}^n$; and if X is a countably infinite discrete space, then $L(X) \cong \phi$, where ϕ is the countable inductive limit of the increasing sequence $(\mathbb{R}^k)_{k\in\mathbb{N}}$.

It is well-known that the space $L(X)$ admits a canonical continuous monomorphism $L(X) \to C_c(C_c(X))$. If X is a k-space, this monomorphism is an embedding of lcs [12,13,45]. So, for k-spaces, we obtain the next chain of topological embeddings:

$$A(X) \hookrightarrow L(X) \hookrightarrow C_c(C_c(X)). \tag{3.1}$$

Denote by $Q = [0,1]^{\mathbb{N}}$ the Hilbert cube. Since Q is a subspace of $\mathbb{R}^{\mathbb{N}}$, Q has a $\mathfrak{G}$-base at each its point satisfying the condition (**D**). Below we prove Theorem 1.15.

Proof of Theorem 1.15. (i) Since X is an $\mathcal{MK}_\omega$-space, the space $C_c(X)$ is a Polish space by [29, 4.2.2 and 5.8.1]. Thus $C_c(C_c(X))$ has a $\mathfrak{G}$-base satisfying the condition (**D**) by Theorems 2 and 9 of [18] and is a $\mathfrak{P}_0$-space by Theorem 2.3. Now (3.1) implies that $L(X)$ and $\phi = L(\mathbb{N})$ also have a $\mathfrak{G}$-base satisfying the condition (**D**) and are $\mathfrak{P}_0$-spaces. As the groups $A(X)$ and $F(X)$ are $\mathcal{MK}_\omega$-spaces, they embed into $\phi \times Q$ by [42]. Thus $A(X)$ and $F(X)$ also have a $\mathfrak{G}$-base satisfying the condition (**D**) and are $\mathfrak{P}_0$-spaces (see [1,19]).

(ii) It is well-known that, if X is not discrete, then $A(X)$ and $F(X)$ are not even Fréchet–Urysohn. Now Proposition 2.4 and Corollary 3.14 of [19] and (i) imply

$$\begin{aligned}\mathfrak{d} &\le \min\{\chi(A(X)), \chi(F(X)), \chi(L(X))\} \\ &\le \max\{\chi(A(X)), \chi(F(X)), \chi(L(X))\} \le \mathfrak{d},\end{aligned}$$

that proves (a). Now let X be discrete, clearly $\chi(A(X)) = \chi(F(X)) = 1$. If X is finite, then $L(X) = \mathbb{R}^{|X|}$ is metrizable, and hence $\chi(L(X)) = \aleph_0$. If X is infinite, then X is countably infinite as an $\mathcal{MK}_\omega$-space. So $L(X) = \phi$. Now Proposition 2.4 and Corollary 3.14 of [19] imply that $\chi(L(X)) = \mathfrak{d}$. □

Note (see [14]) that for a metrizable space X, the space $L(X)$ is a $\mathfrak{P}_0$-space if and only if $L(X)$ has countable tightness if and only if X is separable.

At the end of this section we consider some applications to topological vector spaces.

Recall that a topological space X has the *property* (α_4) *at a point* $x \in X$ if for any $\{x_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$, $m \in \mathbb{N}$, there exists a sequence $(m_k)_k$ of distinct natural numbers and a sequence $(n_k)_k$ of natural numbers such that $\lim_k x_{m_k,n_k} = x$; X has the *property* (α_4) or is an (α_4)-*space* if it has the property (α_4) at each point $x \in X$. Nyikos proved in [37, Theorem 4] that any Fréchet–Urysohn topological group satisfies (α_4). However there are Fréchet–Urysohn topological spaces which do not have (α_4). Further, in [8, Lemma 1.3] it was shown that for a Fréchet–Urysohn topological group G the property (α_4) can be strengthened by the double sequence property (AS):

(AS) For any family $\{x_{n,k} : (n,k) \in \mathbb{N} \times \mathbb{N}\} \subseteq G$, with $\lim_n x_{n,k} = x \in G, k = 1, 2, \ldots$, it is possible to choose strictly increasing sequences of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$, such that $\lim_i x_{n_i,k_i} = x$.

For a group G, $g \in G$ and $n \in \mathbb{N}$, we set $g^n := g \cdots g$ (n times) and, if G is abelian, $ng := g + \cdots + g$. Let G be a topological group with (AS) and a sequence (g_n) in G converge to the unit e. Clearly, $\lim_n g_n^m = e$ for every $m \in \mathbb{N}$. Applying (AS) to the family $\{(g_n^m) : m \in \mathbb{N}\}$ of the powers of the sequence (g_n) we propose the following property (PS) which is weaker than (AS).

Definition 3.3. We say that a topological group G has *the property* (PS) if for every sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ converging to the unit e there are strictly increasing sequences (m_k) and (n_k) of natural numbers such that $g_{n_k}^{m_k} \to e$.

Lemma 1.3 of [8] immediately implies

Proposition 3.4. *Any Fréchet–Urysohn topological group G has (PS).*

As usual we denote by $\sigma(E, E')$ the weak topology of a locally convex space E. Recall that an abelian topological group G is *maximally almost periodic* (MAP) if its continuous characters separate the points of G. A MAP abelian group G endowed with the Bohr topology we denote by G^+. Recall also that a MAP abelian group G (respectively, a lcs E) has the *Schur property* if G^+ and G (respectively, $(E, \sigma(E, E'))$ and E) have the same set of convergent sequences. The next corollary shows that the class of topological groups having (PS) is much wider than the class of Fréchet–Urysohn topological group. Proposition 3.4 applies to get the following

Corollary 3.5. *Let (G, τ) be a Fréchet–Urysohn topological group (respectively, lcs) with the Schur property. Then G^+ (respectively, $(G, \sigma(G, G'))$) has (PS).*

Recall (see [20]) that a family $\mathcal{N}$ of subsets of a topological space X is called a *cs^*-network at a point* $x \in X$ if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to x and for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subseteq O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite; the smallest size $|\mathcal{N}|$ of a cs^*-network at x is called the *cs^*-character of X at the point x*. The cardinal $cs_\chi^*(X) = \sup\{cs_\chi^*(X, x) : x \in X\}$ is called the *cs^*-character* of X. It is easy to see that if a topological space X has the strong Pytkeev property, then X has countable cs^*-character.

Now we apply (PS) for the important class of topological vector spaces having a fundamental sequence of bounded sets.

Proposition 3.6. *Let E be a tvs with a fundamental sequence of bounded sets. If E has (PS), then E has countable cs^*-character.*

Proof. Let $\{D_n\}_{n\in\mathbb{N}}$ be a fundamental sequence of closed absolutely convex bounded sets in E. We claim that the family $\mathcal{N} := \{\frac{1}{k} D_n : k, n \in \mathbb{N}\}$ is a countable cs^*-network at zero. Indeed, let $x_n \to 0$. Choose strictly increasing sequences (m_k) and (n_k) of natural numbers such that $m_k x_{n_k} \to 0$. Fix an open neighborhood U of zero. Take $a \in \mathbb{N}$ such that $(m_k x_{n_k}) \subseteq D_a$, and choose $b \in \mathbb{N}$ such that $D_a \subseteq bU$. Then, by $\frac{1}{b} D_a \subseteq U$, we have

$$x_{n_k} = \frac{b}{m_k} \cdot \left(\frac{m_k}{b} x_{n_k} \right) \subseteq \frac{b}{m_k} \left(\frac{1}{b} D_a \right) \subseteq \frac{1}{b} D_a$$

for every $k \geq k_0$, where $b/m_{k_0} < 1$. Thus $(x_n) \cap \frac{1}{b} D_a$ is infinite. $\square$

For a lcs E with a fundamental sequence of bounded sets which is also *quasibarrelled* (this implies that E is a (DF)-space, see [39]) it is known essentially more than in Proposition 3.6.

Remark 3.7. Any (DF)-space E, by definition, admits a fundamental sequence of bounded sets and must be $\aleph_0$-quasibarrelled, see [25]. On the other hand, by Cascales–Kąkol–Saxon result, see Lemma 15.2 from [25], it follows that any quasibarrelled (DF)-space L admits a $\mathfrak{G}$-base. So L has the strong Pytkeev property by Theorem 9 of [18], and hence L has countable cs^*-character. We recall also that the strong dual F' of any Fréchet space F is a (DF)-space, see [25], and F' is quasibarrelled if and only if F is a distinguished space, see [4].

Recall that a tvs E is *locally bounded* if E has a bounded neighborhood of zero U. Then E is metrizable and $(n^{-1}U)_n$ forms a countable base of neighborhoods of zero for E. Recall also that for every locally bounded topological vector space E there exists a p-norm for some $0 < p \leq 1$ generating the original vector topology of E, see [24, Theorem 6.8.3].

Proof of Theorem 1.16. We need to show only sufficiency. Let E be a Fréchet–Urysohn tvs and let $(B_n)_n$ be a fundamental (increasing) sequence of balanced bounded sets of E. Applying Propositions 3.4 and 3.6 and [3, Theorem 3] (stating that every topological groups which is Fréchet–Urysohn and has countable cs^*-character is metrizable) we conclude that E is metrizable. We claim that E is locally bounded. Indeed, let (U_n) be a decreasing base of balanced neighborhoods of zero in E. We show that some B_n is a (bounded) neighborhood of zero. If this is not the case we have $U_n \nsubseteq nB_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ choose $n^{-1}x_n \in U_n \setminus B_n$. But then the set $B := \{n^{-1}x_n : n \in \mathbb{N}\}$ is bounded and not included in any B_n, a contradiction. The proof is completed if we apply the notice above concerning p-normed spaces. □

Note that a *metrizable* lcs E is normable if and only if the strong dual E' of E is a Fréchet–Urysohn lcs by [8, Theorem 2.8] (this result can be derived also from [5]).

Corollary 3.8. *Let E be the topological product of a family $(E_j)_{j\in J}$ of metrizable topological vector spaces. Then E has a fundamental sequence of bounded sets if and only if J is finite.*

Proof. Let E_0 be the Σ-product of E, i.e.

$$E_0 := \{(x_i) \in E : |j \in J : x_j \neq 0| \leq \aleph_0\}.$$

It is known (see [36]) that E_0 is Fréchet–Urysohn. Assume that E has a fundamental sequence $(B_n)_n$ of bounded sets, so E_0 admits such a sequence, too. Theorem 1.16 applies to deduce that E_0 is metrizable, hence E is metrizable (since E_0 is dense in E). Then some B_m is a bounded neighborhood of zero in E, see the proof of Theorem 1.16. This implies that J is finite, otherwise E would contain $\mathbb{K}^{\mathbb{N}}$ ($\mathbb{K}$ the field of either real or complex numbers) having a bounded neighborhood of zero, which is impossible. □

Remark 3.9. If in Corollary 3.8 each E_j is additionally complete and J is uncountable, E_0 is a Baire non-metrizable subspace of E. By Theorem 1.13 the space E_0 does not have a countable cn-character.

The following lemma supplements Remark 3.7.

Lemma 3.10. *Let (E, τ) be a lcs with a fundamental sequence $(B_n)_n$ of bounded sets. Then E endowed with the finest locally convex topology ξ having the same bounded sets as τ (which exists) has a $\mathfrak{G}$-base and has the strong Pytkeev property. In particular, if (E, τ) is bornological, then $\tau = \xi$.*

Proof. We may assume that all sets B_n are absolutely convex. For each $n \in \mathbb{N}$ let E_n be the linear span of B_n endowed with the Minkowski functional norm topology. Let (E, ξ) be the strict inductive limit space of the sequence $(E_n)_n$ of normed spaces. Then (E, ξ) is bornological, i.e. every absolutely convex bornivorous set in (E, ξ) is a ξ-neighborhood of zero, and the topologies ξ and τ have the same bounded sets. Then clearly the topology ξ is the finest one as we claimed. By the proof of Theorem 3 of [18] the space (E, ξ) has a $\mathfrak{G}$-base and the strong Pytkeev property. Finally, if (E, τ) is bornological, then (by definition) we have $\tau = \xi$. □

Proposition 3.11. *Let (E, τ) be a lcs such that:*

(i) *E admits a fundamental sequence $(B_n)_n$ of absolutely convex bounded sets.*
(ii) *Every linear functional f over E is continuous if and only if any restriction $f|_{B_n}$ is continuous.*

Then E with the weak topology $\sigma(E, E')$ is angelic and a $\sigma(E, E')$-compact set K is $\sigma(E, E')$-metrizable if and only if K is contained in a $\sigma(E, E')$-separable subset of E.

Proof. Let ξ be the locally convex topology as in Lemma 3.10. Observe that τ and ξ have the same continuous linear functionals; hence the both spaces (E, τ) and (E, ξ) have the same weak topology. Since (E, ξ) has a $\mathfrak{G}$-base by Lemma 3.10, we apply Cascales–Orihuela's results, see [25, Proposition 11.3] and [11, Corollary 9] to complete the proof. □

Example 3.12. Let $X = [0, \mathfrak{b})$, where $\mathfrak{b}$ is the small uncountable cardinal equal to the smallest cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which cannot be covered by a σ-compact subset of $\mathbb{N}^{\mathbb{N}}$. It is well-known that the cofinality $\mathrm{cf}(\mathfrak{b})$ of $\mathfrak{b}$ is uncountable, and hence X is not Lindelöf. The space $C_c(X)$ has a $\mathfrak{G}$-base by Proposition 16.14 of [25]; in particular, $C_c(X)$ has countable cs^*-character by Theorem 3.12 of [19]. On the other hand, since X is not Lindelöf, $C_c(X)$ has uncountable tightness by a result of McCoy (see [25, Lemma 16.4]). Taras Banakh noted that the pseudocharacter $\psi(C_c(X))$ of $C_c(X)$ is uncountable (this follows from the inequality $\mathrm{cf}(\mathfrak{b}) > \aleph_0$ and the Ferrando–Kąkol duality theorem, see [10]), and hence $C_c(X)$ is not submetrizable that gives a negative answer to Question 2.15 of [19]. Finally, we note that the space $C_c(X)$ is not a σ-space because $\psi(C_c(X)) > \aleph_0$ (see [22, 4.3]).

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