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On $\mathfrak{P}$-spaces and related concepts

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A B S T R A C T

The concept of the strong Pytkeev property, recently introduced by Tsaban and Zdomskyy in [32], was successfully applied to the study of the space $C_c(X)$ of all continuous real-valued functions with the compact-open topology on some classes of topological spaces X including Čech-complete Lindelöf spaces. Being motivated also by several results providing various concepts of networks we introduce the class of $\mathfrak{P}$-spaces strictly included in the class of $\aleph$-spaces. This class of generalized metric spaces is closed under taking subspaces, topological sums and countable products and any space from this class has countable tightness. Every $\mathfrak{P}$-space X has the strong Pytkeev property. The main result of the present paper states that if X is an $\aleph_0$-space and Y is a $\mathfrak{P}$-space, then the function space $C_c(X,Y)$ has the strong Pytkeev property. This implies that for a separable metrizable space X and a metrizable topological group G the space $C_c(X,G)$ is metrizable if and only if it is Fréchet–Urysohn. We show that a locally precompact group G is a $\mathfrak{P}$-space if and only if G is metrizable.

1. Introduction

All topological spaces are assumed to be Hausdorff. Various topological properties generalizing metrizability have been studied intensively by topologists and analysts, especially like first countability, the Fréchet–Urysohn property, sequentiality and countable tightness (see [8,18]). Pytkeev [30] proved that every sequential space satisfies the property, known actually as the *Pytkeev property*, which is stronger than countable tightness: a topological space X has the *Pytkeev property* if for each $A \subseteq X$ and each $x \in \overline{A} \setminus A$, there are infinite subsets $A_1, A_2, \dots$ of A such that each neighborhood of x contains some A_n. Tsaban and Zdomskyy [32] strengthened this property as follows. A topological space X has the *strong Pytkeev property* if for each $x \in X$, there exists a countable family $\mathcal{D}$ of subsets of X, such that for each neighborhood U of x and each $A \subseteq X$ with $x \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite. Generalizing the

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property of the family $\mathcal{D}$, Banakh in [4] introduced the notion of the Pytkeev network in X. A family $\mathcal{N}$ of subsets of a topological space X is called a *Pytkeev network at a point* $x \in X$ if $\mathcal{N}$ is a network at x and for every open set $U \subset X$ and a set A accumulating at x there is a set $N \in \mathcal{N}$ such that $N \subset U$ and $N \cap A$ is infinite; $\mathcal{N}$ is a *Pytkeev network* in X if $\mathcal{N}$ is a Pytkeev network at each point $x \in X$. Hence X has the strong Pytkeev property if and only if X has a countable Pytkeev network at each point $x \in X$.

Now the main result of [32] states that the space $C_c(X)$ of all continuous real-valued functions on a Polish space X (more generally, a separable metrizable space X, see [4] or Corollary 6.6 below) endowed with the compact-open topology has the strong Pytkeev property. This result was essentially strengthened in [13]: The space $C_c(X)$ has the strong Pytkeev property for every Čech-complete Lindelöf space X. Being inspired by the idea used to prove the last assertion for $C_c(X)$, we propose the following types of networks which will be applied in the sequel.

Definition 1.1. A family $\mathcal{N}$ of subsets of a topological space X is called

- a *cn-network* at a point $x \in X$ if for each neighborhood O_x of x the set $\bigcup\{N \in \mathcal{N} : x \in N \subseteq O_x\}$ is a neighborhood of x; $\mathcal{N}$ is a *cn-network* in X if $\mathcal{N}$ is a *cn*-network at each point $x \in X$.
- a *ck-network* at a point $x \in X$ if for any neighborhood O_x of x there is a neighborhood U_x of x such that for each compact subset $K \subset U_x$ there exists a finite subfamily $\mathcal{F} \subset \mathcal{N}$ satisfying $x \in \bigcap \mathcal{F}$ and $K \subset \bigcup \mathcal{F} \subset O_x$; $\mathcal{N}$ is a *ck-network* in X if $\mathcal{N}$ is a *ck*-network at each point $x \in X$.
- a *cp-network* at a point $x \in X$ if either x is an isolated point of X and $\{x\} \in \mathcal{N}$, or for each subset $A \subset X$ with $x \in \overline{A} \setminus A$ and each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and $N \cap A$ is infinite; $\mathcal{N}$ is a *cp-network* in X if $\mathcal{N}$ is a *cp*-network at each point $x \in X$.

These notions relate as follows:

$$\text{base (at } x) \Longrightarrow ck\text{-network (at } x) \Longrightarrow cn\text{-network (at } x) \Longrightarrow \text{network (at } x).$$

The following fact (see Proposition 2.3) additionally explains our interest to the study of spaces X with countable *cn*-network at each point $x \in X$: If X has a countable *cn*-network at a point x then X has a countable tightness at x. In Section 2 we recall other important types of networks and related results used in the article.

Let us recall the following classes of topological spaces admitting certain countable networks of various types.

Definition 1.2. A topological space X is said to be

- ([23]) a *cosmic* space if X is regular and has a countable network;
- ([23]) an $\aleph_0$*-space* if X is regular and has a countable k-network;
- ([4]) a $\mathfrak{P}_0$*-space* if X is regular and has a countable Pytkeev network.

It is known also that: $\mathfrak{P}_0$-space $\Rightarrow$ $\aleph_0$-space $\Rightarrow$ cosmic, but the converse is false (see [4, Example 1.11] and [23, Example 12.4]).

It is easy to see that for each network (resp. each k-network or a Pytkeev network) $\mathcal{N}$ in a topological space X the family $\mathcal{N} \vee \mathcal{N} := \{A \cup B : A, B \in \mathcal{N}\}$ is a *cn*-network (resp. a *ck*-network or a *cp*-network) in X. Hence, a regular space X is cosmic (resp. an $\aleph_0$-space or a $\mathfrak{P}_0$-space) if and only if X has a countable *cn*-network (resp. a countable *ck*-network or a countable *cp*-network). On the other hand, these versions of network and *cn*-network differ at the σ-locally finite level.

Okuyama [25] and O'Meara [27], having in mind the Nagata–Smirnov metrization theorem, introduced the classes of σ-spaces and $\aleph$-spaces, respectively, which contain all metrizable spaces.

Definition 1.3. A topological space X is called

- ([25]) a *σ-space* if X is regular and has a σ-locally finite network.
- ([27]) an *$\aleph$-space* if X is regular and has a σ-locally finite k-network.

This motivates us to propose the following concept.

Definition 1.4. A topological space X is called a *$\mathfrak{P}$-space* if X has a σ-locally finite cp-network.

Each $\mathfrak{P}$-space X has the strong Pytkeev property (see Corollary 3.7).

As one can expect, any $\mathfrak{P}$-space is an $\aleph$-space. Moreover, it turns out that $\mathfrak{P}$-spaces satisfy even a stronger condition. In Section 3 we study also the following *strict* versions of σ-spaces and $\aleph$-spaces.

Definition 1.5. A topological space X is called

- a *strict σ-space* if X has a σ-locally finite cn-network;
- a *strict $\aleph$-space* if X has a σ-locally finite ck-network.

The following diagram describes the relation between new, as well as, known classes of generalized metric spaces and justifies the study of strict σ-spaces and strict $\aleph$-spaces.

$$\begin{tikzcd}
\text{separable metrizable space} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathfrak{P}_0\text{-space} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \aleph_0\text{-space} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{cosmic space} \arrow[d, Rightarrow] \\
\text{metrizable space} \arrow[r, Rightarrow] & \mathfrak{P}\text{-space} \arrow[r, Rightarrow] & \text{strict } \aleph\text{-space} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{strict } \sigma\text{-space} \arrow[d, Rightarrow] \\
 & & \aleph\text{-space} \arrow[r, Rightarrow] & \sigma\text{-space.}
\end{tikzcd}$$

None of the implications in this diagram can be reversed (see Theorem 1.6 and Examples 3.9 and 4.7).

In Section 4 we propose some criterions for metrizability of topological spaces (see Theorem 4.5). For the locally precompact topological groups, i.e. which can be embedded into locally compact ones, we prove the following.

Theorem 1.6. *For a locally precompact topological group G the following conditions are equivalent:*

(i) *G is metrizable.*
(ii) *G is a $\mathfrak{P}$-space.*
(iii) *G has the strong Pytkeev property.*

It is well-known that the classes of $\aleph$-spaces and σ-spaces are closed under taking subspaces, topological sums and countable products (see [15]). The same holds also for the new three classes of generalized metric spaces introduced in Definitions 1.4 and 1.5: they are closed under taking subspaces, topological sums and countable products (see Section 5). It is well-known that the class of $\aleph$-spaces is closed also under taking function spaces with the Lindelöf domain in this class. Given topological spaces X and Y, let $C(X,Y)$ be the family of all continuous functions from X into Y, and denote by $C_c(X,Y)$ the family $C(X,Y)$ endowed with the compact-open topology. Recall that the compact-open topology on the space $C(X,Y)$ is defined by a subbase consisting of the sets $[K;U]$ with $K \subseteq X$ compact and $U \subseteq Y$ open, where

$$[A; B] := \{f \in C(X,Y) : \ f(A) \subseteq B\},$$

for subsets $A \subseteq X$ and $B \subseteq Y$. Foged [9] (and O'Meara [27]) proved that $C_c(X,Y)$ is an $\aleph$-space for each $\aleph_0$-space X and any $\aleph$-space Y. So it is natural to ask whether an analogous result holds also for $\mathfrak{P}$-spaces Y. In the last Section 6 we prove the following partial result which is the main result of the article.

Theorem 1.7. *Let X be an $\aleph_0$-space. Then:*

(1) *If Y is a $\mathfrak{P}$-space, then the function space $C_c(X,Y)$ has the strong Pytkeev property.*
(2) *If Y is a strict $\aleph$-space, then the function space $C_c(X,Y)$ has a countable ck-network at each function $f \in C_c(X,Y)$.*

This implies that for a separable metrizable space X and a metrizable topological group G the space $C_c(X,G)$ is metrizable if and only if it is Fréchet–Urysohn, see Corollary 6.2. Note that the class of strict $\aleph$-spaces is a proper subclass of the class of $\aleph$-spaces, see Example 3.9.

Notice that the next our paper [11] describes the topology of a topological space X admitting a countable *cp*-, *ck*- or *cn*-network at a point $x \in X$, and also provides some applications for topological groups and topological vector spaces.

2. Networks in topological spaces and relations between them

Recall the most important types of networks which are used in the paper.

Definition 2.1. Let $\mathcal{N}$ be a family of subsets of a topological space X. Then:

- ([1]) $\mathcal{N}$ is a *network at a point* $x \in X$ if for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$; $\mathcal{N}$ is a *network* in X if $\mathcal{N}$ is a network at each point $x \in X$.
- ([16]) $\mathcal{N}$ is a *cs-network at a point* $x \in X$ if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X convergent to x and for each neighborhood O_x of x there are $N \in \mathcal{N}$ and $k \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \ge k\} \subset N \subset O_x$; $\mathcal{N}$ is a *cs-network* in X if $\mathcal{N}$ is a *cs*-network at each point $x \in X$.
- ([14]) $\mathcal{N}$ is a *cs*-network at a point* $x \in X$ if for each sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to x and for each neighborhood O_x of x there is a set $N \in \mathcal{N}$ such that $x \in N \subset O_x$ and the set $\{n \in \mathbb{N} : x_n \in N\}$ is infinite; $\mathcal{N}$ is a *cs*-network* in X if $\mathcal{N}$ is a *cs**-network at each point $x \in X$.
- ([5]) $\mathcal{N}$ is a *local k-network at a point* $x \in X$ if for each neighborhood O_x of x there is a neighborhood U_x of x such that for each compact subset $K \subset U_x$ there is a finite subfamily $\mathcal{F} \subset \mathcal{N}$ such that $K \subset \bigcup \mathcal{F} \subset O_x$; $\mathcal{N}$ is a *local k-network* in X if $\mathcal{N}$ is a local k-network at each point $x \in X$.
- ([23]) $\mathcal{N}$ is a *k-network* in X if whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{N}$.

For regular spaces X the notions of local k-network and k-network coincide (see Remark 2.2 below). Note that a regular space X is an $\aleph_0$-space if and only if X has a countable *cs*-network [16] if and only if X has a countable *cs**-network [14].

Below we discuss some simple relations between various types of networks.

Remark 2.2. Let X be a topological space.

(i) Each *ck*-network (at a point $x \in X$) is a local k-network (at x). On the other hand, if $\mathcal{N}$ is a local k-network at x, then the family $\mathcal{N}_x := \{N \cup \{x\} : N \in \mathcal{N}\}$ is a *ck*-network at x. Also if $\mathcal{N}$ is a k-network in X, then $\mathcal{N}$ is a local k-network and the family $\mathcal{N} \vee \mathcal{N}$ is a *ck*-network for X.

(ii) If X is a *regular* space and $\mathcal{N}$ is a local k-network for X, then $\mathcal{N}$ is a k-network. Indeed, let $K \subset U$ with K compact and U open in X. For every $x \in K$, take open neighborhoods W_x, V_x and O_x of x such that $\overline{W_x} \subset V_x \subset O_x \subset U$ and V_x satisfies the definition of k-network for O_x. So there is a finite family $\mathcal{F}_x \subset \mathcal{N}$ such that $\overline{W_x} \cap K \subset \bigcup \mathcal{F}_x \subset U_x$. Since K is compact, $K \subset \bigcup_{j=1}^m W_{x_j}$ for some $x_1, \dots, x_m \in K$. Clearly,

$$K \subset \bigcup_{j=1}^{m} \bigcup \mathcal{F}_{x_j} \subset \bigcup_{j=1}^{m} O_{x_j} \subset U.$$

Thus $\mathcal{N}$ is a k-network.

(iii) It is clear that any *cp*-network $\mathcal{N}$ at a point $x \in X$ is a cs^*-network at x. Observe that $\mathcal{N}$ is also a *cn*-network at x. Indeed, let U be a neighborhood of x and $W := \bigcup\{N \in \mathcal{N} : N \subset U\}$. We have to prove that W is a neighborhood of x. If this is not the case, then $x \in \overline{X \setminus W}$. By definition, there is $N \in \mathcal{N}$ such that $x \in N \subset U$ and $N \cap (X \setminus W)$ is infinite. Since $N \subset W$ we obtain $W \cap (X \setminus W) \neq \emptyset$, a contradiction.

(iv) If $\mathcal{N}$ is a *ck*-network (at a point $x \in X$) in X, then $\mathcal{N}$ is a *cn*-network and a cs^*-network (at x).

(v) For any point x of a topological space X the family $\{\{x\}\}$ is trivially a network at the point x. So to avoid such a trivial and unpleasant situation in which a network at a point actually has nothing common with the *topology* of X at x, we do hope that the notion of a *cn*-network is of interest.

(vi) If $\mathcal{N}$ is a Pytkeev network at a point x of X, then the family $\mathcal{N}_x = \{N \cup \{x\} : N \in \mathcal{N}\}$ is a *cp*-network at x. So the difference between these notions is not essential when they are considered only at a *fixed* point (as in the definition of the strong Pytkeev property). But these notions essentially differ on σ-locally finite level, see Example 3.9 below.

Recall that a topological space X has *countable tightness at a point* $x \in X$ if whenever $x \in \overline{A}$ and $A \subseteq X$, then $x \in \overline{B}$ for some countable $B \subseteq A$; X has *countable tightness* if it has countable tightness at each point $x \in X$.

Proposition 2.3. *Let a topological space X have a countable cn-network at a point x. Then X has countable tightness at x.*

Proof. Let $\{D_n\}_{n \in \mathbb{N}}$ be a countable *cn*-network at x and $A \subset X$ be such that $x \in \overline{A} \setminus A$. Set $J := \{n \in \mathbb{N} : D_n \cap A \neq \emptyset\}$. For every $n \in J$ take arbitrarily $a_n \in D_n \cap A$ and set $B := \{a_n\}_{n \in J} \subset A$. We show that $x \in \overline{B}$. For every neighborhood U of x set $I(U) := \{n \in \mathbb{N} : x \in D_n \subseteq U\}$. Then, by definition, the set $\bigcup_{n \in I(U)} D_n$ contains a neighborhood V of x. Since $A \cap V \neq \emptyset$, we can find $n \in I(U) \cap J$. Thus $a_n \in B \cap U$. □

It is natural to ask for which topological spaces some of the types of networks coincide. Partial answers to this question are given in three propositions below proved by Banakh [5].

Proposition 2.4. *([5]) Any countable Pytkeev network at a point x of a topological space X is a local k-network at x. Consequently, any countable cp-network at a point $x \in X$ is a ck-network at x.*

Recall that a topological space X is called a *k-space* if for each non-closed subset $A \subset X$ there is a compact subset $K \subset X$ such that $K \cap A$ is not closed in K.

Proposition 2.5. *([5]) If a topological space X is a regular k-space and $\mathcal{N}$ is a k-network at a point $x \in X$, then $\mathcal{N} \vee \mathcal{N} := \{C \cup D : C, D \in \mathcal{N}\}$ is a Pytkeev network at x.*

Corollary 2.6. *A k-space X is a $\mathfrak{P}_0$-space if and only if X is an $\aleph_0$-space.*

Since any topological group is regular, Propositions 2.4 and 2.5 yield

Corollary 2.7. *Let G be a topological group which is a k-space. Then G has countable cp-network at the unit e if and only if G has a countable ck-network at e.*

Note that the condition to be a k-space in Corollary 2.7 is essential, see Theorem 1.6 and Example 4.7.

Recall that a topological space X is called *Fréchet–Urysohn at a point* $x \in X$ if for every subset $A \subset X$ such that $x \in \overline{A}$ there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ in A converging to x; X is called *Fréchet–Urysohn* if it is Fréchet–Urysohn at each point $x \in X$.

Proposition 2.8. *([5]) Let $\mathcal{N}$ be a cs^*-network at a point x of a topological space X. If X is Fréchet–Urysohn at x, then $\mathcal{N}$ is a Pytkeev network (actually a cp-network) at x.*

Corollary 2.9. *In the class of Fréchet–Urysohn spaces the concepts of cp-network and cs^*-network are equivalent.*

Notation 2.10. If $\mathcal{N}$ is either a *ck*-, *cs*-, cs^*-, *cp*-, *cn*-network or network (at a point x) in a topological space X, we will say that $\mathcal{N}$ is an $\mathfrak{n}$-*network* (at x). Set $\mathfrak{N} = \{ck, cs, cs^*, cp, cn, 0\}$.

Definitions 2.1 and 1.1 allow us to define the following cardinals of topological spaces.

Definition 2.11. Let x be a point of a topological space X and $\mathfrak{n} \in \mathfrak{N}$. The smallest size $|\mathcal{N}|$ of an $\mathfrak{n}$-network $\mathcal{N}$ at x is called the $\mathfrak{n}$-*character of X at the point* x and is denoted by $\mathfrak{n}_\chi(X, x)$. The cardinal $\mathfrak{n}_\chi(X) = \sup\{\mathfrak{n}_\chi(X, x) : x \in X\}$ is called the $\mathfrak{n}$-*character* of X. The $\mathfrak{n}$-*netweight*, $\mathfrak{n}w(X)$, of X is the least cardinality of an $\mathfrak{n}$-network for X.

Analogously we define the local k-character (at a point x) of a topological space X.

In the paper we study topological spaces X with countable $\mathfrak{n}$-character (at a point $x \in X$), i.e. spaces X with $\mathfrak{n}_\chi(X) \le \aleph_0$ (respectively, $\mathfrak{n}_\chi(X, x) \le \aleph_0$). Recall again (see [32]) that a topological space X has the *strong Pytkeev property* if and only if X has a countable Pytkeev network at each point $x \in X$, i.e. if $cp_\chi(X) \le \aleph_0$. If a space X is first countable at a point $x \in X$, then any countable base at x is a *cp*-network at x. So, every first countable space X has the strong Pytkeev property.

As usual we denote by $\chi(X, x)$ the character of a topological space X at a point x, and the character of X is denoted by $\chi(X)$. Applying the definition of the *cn*-network we have the following

Proposition 2.12. *If x is a point of a topological space X, then $\chi(X, x) \le 2^{cn_\chi(X,x)}$.*

Example 2.13. Let G be a discrete abelian group of cardinality 2^κ, where the cardinal κ is uncountable. It is well known (see [3, 9.9.57]) that $\chi(G, \tau_b) = 2^{|G|}$, where τ_b is the Bohr topology of G. By Proposition 2.12 we have $\aleph_0 < \kappa \le cn_\chi(G, \tau_b)$. On the other hand, the group (G, τ_b) has countable tightness by [3, Problem 9.9.H]. So there are precompact abelian topological groups of countable tightness with arbitrary large *cn*-character. Since any convergent sequence in (G, τ_b) is essentially constant, we have $cs^*_\chi(G, \tau_b) = 1 < cn_\chi(G, \tau_b)$.

Proposition 2.4 and Remark 2.2 imply

Corollary 2.14. *Let x be a point of a topological space X. Then*

$$cp_\chi(X, x) \le \aleph_0 \Longrightarrow ck_\chi(X, x) \le \aleph_0 \Longrightarrow \max\{cs^*_\chi(X, x), cn_\chi(X, x)\} \le \aleph_0;$$

$$cn_\chi(X, x) \le \min\{cp_\chi(X, x), ck_\chi(X, x)\}, \quad cn_\chi(X) \le \min\{cp_\chi(X), ck_\chi(X)\};$$

$$cs^*_\chi(X, x) \le \min\{cp_\chi(X, x), ck_\chi(X, x), cs_\chi(X, x)\} \le \max\{cp_\chi(X, x), ck_\chi(X, x), cs_\chi(X, x)\} \le \chi(X, x);$$

$$cs^*_\chi(X) \le \min\{cp_\chi(X), ck_\chi(X), cs_\chi(X)\} \le \max\{cp_\chi(X), ck_\chi(X), cs_\chi(X)\} \le \chi(X).$$

3. Three new types of generalized metric spaces

Recall [17] that a topological space X is called a *cs–σ-space* if it is regular and has a σ-locally finite cs-network. Analogously we define

Definition 3.1. For $\mathfrak{n} \in \mathfrak{N}$, a topological space X is called an $\mathfrak{n}$*–σ-space* if it is regular and has a σ-locally finite $\mathfrak{n}$-network.

So cp–σ-spaces are $\mathfrak{P}$-spaces, ck–σ-spaces are strict $\aleph$-spaces, cn–σ-spaces are strict σ-spaces and 0–σ-spaces are σ-spaces, respectively.

Remark 3.2. Each ck–σ-space is both a cn–σ-space and a cs^*–σ-space.

Nagata–Smirnov metrization theorem implies

Proposition 3.3. *Any metrizable space X is a $\mathfrak{P}$-space. Each separable metrizable space is a $\mathfrak{P}_0$-space.*

Proof. By [8, 4.4.4], X has a σ-locally finite open base $\mathcal{D}$. Clearly, $\mathcal{D}$ is also a cp-network for X. If additionally X is separable, it is clear that any countable open base of X is a countable cp-network for X. □

It is known (see [14]) that each countable cs^*-network in a regular space X is a k-network. Next proposition generalizes this fact.

Proposition 3.4. *Any σ-locally finite cs^*-network in a regular topological space X is a k-network for X. Consequently, each countable cs^*-network in X is a k-network.*

Proof. Let $\mathcal{D} = \bigcup_n \mathcal{D}_n$ be an increasing σ-locally finite cs^*-network for X and let K be a compact subset of an open set $U \subset X$. We have to find an $n \in \mathbb{N}$ and a finite subfamily $\mathcal{F}$ of $\mathcal{D}_n$ such that $K \subset \bigcup \mathcal{F} \subset U$.

For each $x \in K$ and every $n \in \mathbb{N}$ take an open neighborhood $U_n(x)$ of x such that $U_n(x) \subset U$ and $U_n(x)$ intersects only members of a finite subfamily $T_n(x)$ of $\mathcal{D}_n$. Set

$$R_n(x) := \{D \in T_n(x) : \ D \subset U \text{ and } D \cap K \neq \emptyset\}.$$

Since K is compact there are $x_1^n, \dots, x_{s_n}^n \in K$ such that $K \subset \cup_{i=1}^{s_n} U_n(x_i^n)$. Set

$$A_n := \bigcup_{j=1}^{n} \bigcup_{i=1}^{s_j} \left\{D : \ D \in R_j(x_i^j)\right\}.$$

Clearly, $A_1 \subset A_2 \subset \cdots \subset U$. Since $\mathcal{D}$ is a network for X, it is enough to show that $K \setminus A_n$ is finite for some $n \in \mathbb{N}$.

Suppose for a contradiction that $K \setminus A_n$ is infinite for every $n \in \mathbb{N}$. Then we can choose a sequence $\{x_n\}$ of distinct elements of K such that $x_n \notin A_n$ for every $n \in \mathbb{N}$. By [15, Corollary 4.7], K is metrizable. So without loss of generality we may assume that $\{x_n\}$ converges to a point $z \in K$. As $\mathcal{D}$ is a cs^*-network, there is $q \in \mathbb{N}$ and $D \in \mathcal{D}_q$ such that $z \in D \subset U$ and $D \cap \{x_n\}$ is infinite. If an index i is such that $z \in U_q(x_i^q)$, then $D \in R_q(x_i^q)$, so $D \subset A_q$. By the construction of $\{x_n\}$, we have $x_n \notin D$ for every $n \ge q$. Thus $D \cap \{x_n\}$ is finite, a contradiction. □

Since any *cp*-network is trivially a cs^*-network, Proposition 3.4 implies that any $\mathfrak{P}_0$-space is an $\aleph_0$-space (see [4]).

Next theorem shows that some types of σ-locally finite networks coincide.

Theorem 3.5. *For a regular topological space X the following assertions are equivalent*

(i) *X is an $\aleph$-space;*
(ii) *([9]) X is a cs–σ-space;*
(iii) *(see [28]) X is a cs^*–σ-space.*

Proof. (i) $\Leftrightarrow$ (ii) was proved by Foged [9]. (ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (i) follows from Proposition 3.4 (this also follows from Lemma 1.17 and Theorem 1.4 of [28]). □

The following theorem is essentially used in the proof of Theorem 1.7.

Theorem 3.6. *For $\mathfrak{n} \in \{0, cn, cs^*, ck, cp\}$, let X be an $\mathfrak{n}$–σ-space and $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$ be a closed increasing σ-locally finite $\mathfrak{n}$-network for X. Then for every Lindelöf subset S of X there exists a sequence $\mathcal{N} = \{D_k\}_{k\in\mathbb{N}} \subset \mathcal{D}$ which is an $\mathfrak{n}$-network at each point of S.*

In the case $\mathfrak{n} \in \{ck, cp\}$, the family $\mathcal{N}$ satisfies additionally the following property: if $K \subset S \cap U$ with K compact and U open, then there is an open subset W of X such that

(a) *$K \subset W \subset \bigcup_{k\in I(U)} D_k \subset U$, where $I(U) = \{k \in \mathbb{N} : D_k \subset U\}$, and*
(b) *for each compact subset C of W there is a finite subfamily α of $I(U)$ for which $C \subset \bigcup_{k\in\alpha} D_k$.*

Proof. Since $\mathcal{D}$ contains closed sets, for every $x \in S$ and every $n \in \mathbb{N}$ choose a neighborhood $U_n(x)$ of x such that $U_n(x)$ intersects only members of a finite subfamily $T_n(x)$ of $\mathcal{D}_n$ such that

$$T_n(x) = \{D \in \mathcal{D}_n : D \cap U_n(x) \neq \emptyset\} = \{D \in \mathcal{D}_n : x \in D\}. \tag{1}$$

So, for every $x \in S$ the set

$$T(x) := \{D \in \mathcal{D} :\ x \in D\} = \bigcup_{n\in\mathbb{N}} T_n(x)$$

is a countable $\mathfrak{n}$-network at x.

As S is Lindelöf, for every $n \in \mathbb{N}$ there is a sequence $\{x_j^n\}_{j\in\mathbb{N}} \in S$ such that the set $\bigcup_{j\in\mathbb{N}} U_n(x_j^n)$ is an open neighborhood of S. Let $\mathcal{N} = \{D_k\}_{k\in\mathbb{N}}$ be an enumeration of the family $\bigcup_{n\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} T(x_j^n) \subset \mathcal{D}$. We show that $\mathcal{N}$ is as desired.

(1) Assume that $\mathcal{D}$ is a network. Fix arbitrarily $x \in S$ and an open neighborhood O_x of x. Then there is $n \in \mathbb{N}$ and $D \in \mathcal{D}_n$ such that $x \in D \subset O_x$. Choose $j_0 \in \mathbb{N}$ such that $x \in U_n(x_{j_0}^n)$. Then $D \cap U_n(x_{j_0}^n) \neq \emptyset$; so, by (1), $D \in T_n(x_{j_0}^n)$. Thus $D \in \mathcal{N}$.

Assume in addition that $\mathcal{D}$ is a *cn*-network. We showed above that for every $D \in \mathcal{D}$ with $x \in D \subset O_x$ there are $n \in \mathbb{N}$ and $j_0 \in \mathbb{N}$ such that $D \in T_n(x_{j_0}^n) \subset \mathcal{N}$. Then

$$\bigcup\{D_k \in \mathcal{N} : x \in D_k \subset O_x\} = \bigcup\{D \in \mathcal{D} : x \in D \subset O_x\}$$

is a neighborhood of x by the definition of *cn*-network.

(2) Assume that $\mathcal{D}$ is a cs^*-network. Fix arbitrarily $x \in S$, an open neighborhood O_x of x and a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x. Since $\mathcal{D}$ is a cs^*-network in X, there is $n \in \mathbb{N}$ and $D \in \mathcal{D}_n$ such that $x \in D \subset O_x$

and the set $\{n \in \mathbb{N} : x_n \in D\}$ is infinite. Choose $j_0 \in \mathbb{N}$ such that $x \in U_n(x^n_{j_0})$. Then $D \cap U_n(x^n_{j_0}) \neq \emptyset$; so, by (1), $D \in T_n(x^n_{j_0})$. Thus $D \in \mathcal{N}$.

(3) Assume that $\mathcal{D}$ is a *ck*-network. Fix arbitrarily $x \in S$ and an open neighborhood O_x of x. Since $\mathcal{D}$ is a *ck*-network at x, there exists a neighborhood V of x such that $V \subset O_x$ and for each compact subset C of V there are $n \in \mathbb{N}$ and a finite subset $\mathcal{F}$ of $\mathcal{D}_n$ such that $x \in \bigcap \mathcal{F}$ and $C \subset \bigcup \mathcal{F} \subset V$. Choose $j_0 \in \mathbb{N}$ such that $x \in U_n(x^n_{j_0})$. Then $D \cap U_n(x^n_{j_0}) \neq \emptyset$ for every $D \in \mathcal{F}$; so, by (1), $D \in T_n(x^n_{j_0})$. Thus $D \in \mathcal{N}$ and $\mathcal{F} \subset \mathcal{N}$. Therefore $\mathcal{N}$ is a *ck*-network at x.

(4) Assume that $\mathcal{D}$ is a *cp*-network. We show that $\mathcal{N}$ is a *cp*-network at each point $x \in S$. Fix arbitrary $x \in S$. If x is isolated in X, then $\{x\} \in \mathcal{D}$ by definition. If $n \in \mathbb{N}$ and $j \in \mathbb{N}$ are such that $x \in U_n(x^n_j)$, then $\{x\} \cap U_n(x^n_j) \neq \emptyset$. Then (1) implies that $x = x^n_j$, and hence $\{x\} \in \mathcal{N}$. Assume that x is non-isolated. Fix an open neighborhood O_x of x and a subset $A \subset X$ with $x \in \overline{A} \setminus A$. Since $\mathcal{D}$ is a *cp*-network in X, there is $n \in \mathbb{N}$ and $D \in \mathcal{D}_n$ such that $x \in D \subset O_x$ and $D \cap A$ is infinite. Choose $j_0 \in \mathbb{N}$ such that $x \in U_n(x^n_{j_0})$. Then $D \cap U_n(x^n_{j_0}) \neq \emptyset$; so, by (1), $D \in T_n(x^n_{j_0})$. Thus $D \in \mathcal{N}$.

Now assume $\mathfrak{n} \in \{ck, cp\}$. Since any countable *cp*-network at a point x is also a *ck*-network at x by Proposition 2.4, we can assume that $\mathfrak{n} = ck$. Let $K \subset S \cap U$ with K compact and $U \subset X$ open. For every $x \in K$, set

$$I(x) := \{k \in \mathbb{N} : \ x \in D_k \subset U\}.$$

Since $\mathcal{N}$ is a *ck*-network at x and hence a *cn*-network at x, the set $O(x) := \bigcup\{D_k : k \in I(x)\}$ is a neighborhood of x and $O(x) \subset U$. By the definition of *ck*-network at x and since X is regular, there are open neighborhoods $W(x)$ and $V(x)$ of x such that $\overline{W(x)} \subset V(x) \subset O(x)$ and for each compact subset C of $V(x)$ there is a finite subset $\mathcal{F}$ of $\{D_k : k \in I(x)\}$ with $C \subset \bigcup \mathcal{F} \subset O(x)$. Since K is compact there are $z_1, \dots, z_s \in K$ such that the set

$$W := \bigcup_{j=1}^{s} W(z_j)$$

is an open neighborhood of K. Clearly, (a) holds by construction. Let us check (b). Let C be an arbitrary compact subset of W. Then for every $1 \leq j \leq s$, there is a finite subfamily $\alpha_j \subset I(z_j)$ such that

$$\overline{W(z_j)} \cap C \subset \bigcup_{k \in \alpha_j} D_k \subset O(z_j).$$

Set $\alpha := \cup_{j=1}^{s} \alpha_j$. Then

$$C = \bigcup_{j=1}^{s} (\overline{W(z_j)} \cap C) \subset \bigcup_{j=1}^{s} \bigcup_{k \in \alpha_j} D_k = \bigcup_{k \in \alpha} D_k \subset \bigcup_{j=1}^{s} O(z_j) \subset U,$$

and hence (b) holds true. $\square$

Corollary 3.7. *For $\mathfrak{n} \in \mathfrak{N}$, if X is an $\mathfrak{n}$–σ-space, then $\mathfrak{n}_\chi(X) \leq \aleph_0$.*

Theorem 3.5 and Corollary 3.7 immediately imply the following result which might be also extracted from the proof of [31, Corollary 2.18] and is proved in [12].

Corollary 3.8. *Each $\aleph$-space X has countable cs^*-character.*

Below we join and extend two examples which were kindly proposed us by Taras Banakh. This example shows that the classes of *cn*–σ-spaces and *ck*–σ-space are much smaller than the classes of σ-spaces and $\aleph$-spaces respectively.

Example 3.9 *(Banakh).* For any uncountable cardinal κ there is a paracompact $\aleph$-space Ω of tightness κ with a unique non-isolated point ∞ at which $cn_\chi(\Omega,\infty)=\kappa$. In particular, the space Ω is not a strict σ-space.

Proof. Consider the space $\Omega=\{\infty\}\cup(\kappa\times\mathbb{N}\times\mathbb{N})$ in which all points $x\in\kappa\times\mathbb{N}\times\mathbb{N}$ are isolated, while a neighborhood base at ∞ is formed by the sets

$$U_{C,n,\varphi}=\{\infty\}\cup\{(\alpha,k,m)\in\kappa\times\mathbb{N}\times\mathbb{N}:\alpha\in\kappa\setminus C,\ k\ge n,\ m\ge\varphi(\alpha,k)\},$$

where $C\subset\kappa$ is a subset of cardinality $|C|<\kappa$, $n\in\mathbb{N}$ and $\varphi:\kappa\times\mathbb{N}\to\mathbb{N}$ is a function. It is easy to see that the space Ω does not contain infinite compact subsets. Consequently, each network in Ω is a k-network. Since Ω has a unique non-isolated point, it is paracompact.

For every $n,m\in\mathbb{N}$ consider the following discrete families in Ω

$$\mathcal{N}_n=\big\{\{(\alpha,k,m)\}:\alpha\in\Omega,\ k\le n,\ m\in\mathbb{N}\big\}$$

and

$$\mathcal{N}_{n,m}=\big\{\{(\alpha,k)\}\times[m,\mathbb{N}):\alpha\in\Omega,\ k\le n\big\}.$$

Then the family

$$\mathcal{N}=\big\{\{\infty\}\big\}\cup\bigcup_{n\in\mathbb{N}}\mathcal{N}_n\cup\bigcup_{n,m\in\mathbb{N}}\mathcal{N}_{n,m}$$

is a σ-discrete network in Ω; so Ω is an $\aleph$-space.

Claim 1. The family $\mathcal{N}$ is a Pytkeev network for Ω, i.e., for every open set $U\subset\Omega$ and a set A accumulating at a point $x\in U$ there is a set $N\in\mathcal{N}$ such that $N\subset U$ and $N\cap A$ is infinite.

Since all points of Ω except for ∞ are isolated, it suffices to check that $\mathcal{N}$ is a Pytkeev network at ∞. Fix a neighborhood $U\subset\Omega$ of ∞ and a set $A\subset\Omega$ accumulating at ∞. Without loss of generality we can assume that the neighborhood U is of basic form $U=U_{C,n,\varphi}$ for some set $C\subset\kappa$ with $|C|<\kappa$, $n\in\mathbb{N}$ and $\varphi:\kappa\times\mathbb{N}\to\mathbb{N}$. We claim that for some pair $(\alpha,k)\in(\kappa\setminus C)\times[n,\mathbb{N})$ the intersection $A\cap(\{(\alpha,k)\}\times\mathbb{N})$ is infinite. Indeed, otherwise, we could find a function $\psi:\kappa\times\mathbb{N}\to\mathbb{N}$ such that $\psi\ge\varphi$ and $A\cap(\{(\alpha,k)\}\times\mathbb{N})\subset\{(\alpha,k)\}\times\big[0,\psi(\alpha,k)\big)$ for every $(\alpha,k)\in(\kappa\setminus C)\times[n,\mathbb{N})$. Then $O_{C,n,\psi}$ is a neighborhood of ∞, disjoint with the set $A\setminus\{\infty\}$, which means that A does not accumulate at ∞. This contradiction shows that for some $(\alpha,k)\in(\kappa\setminus C)\times[n,\mathbb{N})$ the set A has infinite intersection with the set $\{(\alpha,k)\}\times\mathbb{N}$ and hence with the set $N=\{(\alpha,k)\}\cup[\varphi(\alpha,k),\mathbb{N})$. Taking into account that $N\in\mathcal{N}$ and $N\subset U_{C,n,\varphi}$, we conclude that $\mathcal{N}$ is a Pytkeev network at ∞.

Claim 2. $cn_\chi(\Omega,\infty)=\kappa$. The inequality $cn_\chi(\Omega,\infty)\le\kappa$ follows from the fact that the family $\{\{\infty\}\cup\{x\}: x\in\Omega\}$ is a *cn*-network of cardinality κ at ∞. Assuming that $cn_\chi(\Omega,\infty)<\kappa$, fix a *cn*-network $\mathcal{C}$ at ∞ of cardinality $|\mathcal{C}|=cn_\chi(\Omega,\infty)<\kappa$. We can suppose that $|C|>1$ and $\infty\in C$ for every $C\in\mathcal{C}$. In each set $C\in\mathcal{C}$ fix a point $(\alpha_C,y_C,z_C)\in C\cap(\kappa\times\mathbb{N}\times\mathbb{N})$ and consider the set $F=\{\alpha_C:C\in\mathcal{C}\}$. Then the set

$$U_\infty=\{\infty\}\cup\big((\kappa\setminus F)\times\mathbb{N}\times\mathbb{N}\big)$$

is a neighborhood of ∞ which does not contain any set $C\in\mathcal{C}$. So

$$\bigcup\{C \in \mathcal{C} : \infty \in C \subset U_\infty\} = \emptyset$$

is not a neighborhood of ∞ and hence $\mathcal{C}$ fails to be a *cn*-network at ∞.

Claim 3. The tightness $t(\Omega, \infty)$ of the space Ω at ∞ is κ; hence the tightness $t(\Omega)$ of Ω is κ.

Clearly, $t(\Omega, \infty) \le t(\Omega) \le |\Omega| \le \kappa$. Let us show that $t(\Omega, \infty) \ge \kappa$. Consider the set $\mathcal{A} := U_{C,n,\varphi} \setminus \{\infty\}$ for some $C \subset \kappa$ with $|C| < \kappa$, $n \in \mathbb{N}$ and a function $\varphi : \kappa \times \mathbb{N} \to \mathbb{N}$. It is clear that $\infty \in \overline{\mathcal{A}}$. Fix arbitrarily a subset A of $\mathcal{A}$ with $|A| < \kappa$. Denote by B the projection of A to κ; so $|B| < \kappa$. Then, by construction, the open neighborhood $U_{B,n,\varphi}$ of ∞ does not intersect with A. Thus $t(\Omega, \infty) \ge \kappa$.

Finally, Ω fails to be a strict σ-space by Corollary 3.7 and Proposition 2.3. □

Remark 3.10. We have two natural types of networks in a space X: *global* (for the whole space X) and *local* (at each point $x \in X$). If X is an $\aleph_0$-space, then X has a *countable* family $\mathcal{N}$ of subsets which is simultaneously a (global) k-network for X and a (local) k-network at each point $x \in X$. If X is a *strict* $\aleph_0$-space, then X has a σ-locally finite family $\mathcal{N}$ of subsets such that $\mathcal{N}$ is not only a (global) k-network for X, but also $\mathcal{N}$ defines a (local) countable k-network at each point $x \in X$ by Corollary 3.7. However, if X is only an $\aleph$-space, then X has a (global) k-network $\mathcal{N}$ for X which defines a (local) countable cs^*-network at each point $x \in X$ by Theorem 3.5, but X may not have countable k-network at some points of X, see Example 3.9. This incoordination of global and local concepts may suggest that the class of strict $\aleph$-spaces is actually a more appropriate generalization of $\aleph_0$-spaces than the class of $\aleph$-spaces.

Relations between various types of $\aleph$-spaces are given below.

Proposition 3.11. *Each $\mathfrak{P}$-space is a strict $\aleph$-space, and each strict $\aleph$-space is an $\aleph$-space.*

Proof. Let X be a $\mathfrak{P}$-space with a σ-locally finite *cp*-network $\mathcal{D}$. For every $x \in X$, the proof of Theorem 3.6 shows that the family $\mathcal{N}(x) := \{N \in \mathcal{D} : x \in N\}$ is a countable *cp*-network at x. Proposition 2.4 implies that $\mathcal{N}(x)$ is also a *ck*-network at x. Thus $\mathcal{D}$ is a σ-locally finite *ck*-network for X, i.e. X is a strict $\aleph$-space.

Let X be a strict $\aleph$-space with a σ-locally finite *ck*-network $\mathcal{D}$. As X is regular, Remark 2.2 shows that $\mathcal{D}$ is also a k-network for X. Thus X is an $\aleph$-space. □

In [12] it is shown that a topological space X is cosmic (resp. an $\aleph_0$-space) if and only if X is a Lindelöf σ-space (resp. a Lindelöf $\aleph$-space). Analogously we prove the following

Proposition 3.12. *Let X be a topological space. Then*

(i) *X is cosmic if and only if X is a Lindelöf strict σ-space;*
(ii) *X is an $\aleph_0$-space if and only if X is a Lindelöf strict $\aleph$-space;*
(iii) *X is a $\mathfrak{P}_0$-space if and only if X is a Lindelöf $\mathfrak{P}$-space.*

Proof. We prove only (iii). Assume that X is a Lindelöf $\mathfrak{P}$-space with a σ-locally finite *cp*-network $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$. It is enough to prove that every $\mathcal{D}_n$ is countable. For every $x \in X$ choose an open neighborhood U_x of x such that U_x intersects with a finite subfamily $T(x)$ of $\mathcal{D}_n$. Since X is a Lindelöf space, we can find a countable subset $\{x_k\}_{k\in\mathbb{N}}$ of X such that $X = \bigcup_{k\in\mathbb{N}} U_{x_k}$. Hence any $D \in \mathcal{D}_n$ intersects with some U_{x_k} and therefore $D \in T(x_k)$. Thus $\mathcal{D}_n = \bigcup_{k\in\mathbb{N}} T(x_k)$ is countable. Conversely, if X is a $\mathfrak{P}_0$-space, then X is Lindelöf (see [23]) and it is trivially a $\mathfrak{P}$-space. □

Remark 3.13. If X is a cosmic non-$\aleph_0$-space, then X is not a strict $\aleph$-space by Proposition 3.12.

4. Metrizability conditions for topological spaces

In this section we present some criterions for metrizability of topological spaces.

Following [2, II.2] we say that a topological space X has *countable fan tightness at a point* $x \in X$ if for each sets $A_n \subset X$, $n \in \mathbb{N}$, with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\cup_{n\in\mathbb{N}} F_n}$. A space X has *countable fan tightness* if it has countable fan tightness at each point $x \in X$.

Recall that a topological space X has the *property* (α_4) *at a point* $x \in X$ if for any $\{x_{m,n} : (m,n) \in \mathbb{N}\times\mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$, $m \in \mathbb{N}$, there exists a sequence $(m_k)_k$ of distinct natural numbers and a sequence $(n_k)_k$ of natural numbers such that $\lim_k x_{m_k,n_k} = x$; X has the *property* (α_4) or is an (α_4)*-space* if it has the property (α_4) at each point $x \in X$. Nyikos proved in [29, Theorem 4] that any Fréchet–Urysohn topological group satisfies (α_4). However there are Fréchet–Urysohn topological spaces which do not have (α_4).

Next proposition recalls some criterions for a topological space to be first countable at a point. Note that (i) $\Leftrightarrow$ (ii) is proved in [5] and (i) $\Leftrightarrow$ (iii) follows from Proposition 6 and Lemma 7 of [7].

Proposition 4.1. *Let x be a point of a topological space X. Then the following assertions are equivalent:*

(i) *X is first countable at x.*
(ii) *X has a countable Pytkeev network at x and countable fan tightness at x.*
(iii) *X has a countable cs^*-network at x and is a Fréchet–Urysohn (α_4)-space.*

Recall also (see [23]) that a point x in a topological space X is called an *r-point* if there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of x such that if $x_n \in U_n$, then $\{x_n\}_{n\in\mathbb{N}}$ has compact closure; we call X to be an *r-space* if all of its points are r-points.

Remark 4.2. The first countable spaces and the locally compact spaces are trivially r-spaces. So the Bohr compactification $b\mathbb{Z}$ of $\mathbb{Z}$ is an r-space, but since $b\mathbb{Z}$ has uncountable tightness it does not have countable fan tightness. On the other hand, there are spaces with countable fan tightness which are not r-spaces. Indeed, let $X = C_p[0,1]$. Then X has countable fan tightness by [2, II.2.12]. As X has a neighborhood base at zero determined by finite families of points in $[0,1]$, for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of zero we can find $z \in [0,1]$ and $f_n \in U_n$ such that $f_n(z) \to \infty$. Hence the closure of $\{f_n\}_{n\in\mathbb{N}}$ is non-compact. Thus X is not an r-space. We do not know non-metrizable r-spaces which have countable fan tightness.

Following Morita [24], a topological space X is called an *M-space* if there exists a sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X such that: (i) if $x_n \in \bigcup\{U \in \mathcal{U}_n : x \in U\}$ for each n, then $\{x_n\}_{n\in\mathbb{N}}$ has a cluster point; (ii) for each n, $\mathcal{U}_{n+1}$ star refines $\mathcal{U}_n$. Any countably compact space X is an M-space; just set $\mathcal{U}_n = \{X\}$ for every $n \in \mathbb{N}$. Countably compact subsets of an $\mathfrak{n}$–σ-space are metrizable as the next theorem shows.

Theorem 4.3. *For a topological space X the following assertions are equivalent:*

(i) *X is metrizable.*
(ii) *([26]) X is an $\aleph$-space and an r-space.*
(iii) *([24]) X is a σ-space and an M-space.*

Consequently, each compact subset of a σ-space (in particular, a *cn*–σ-space) X is metrizable. For locally compact spaces we have the following

Proposition 4.4. *A locally compact space X is metrizable if and only if X is a paracompact σ-space.*

Proof. If X is metrizable, then X is paracompact by Stone's theorem [8, 5.1.3] and is a σ-space by Proposition 3.3. Assume that X is a paracompact σ-space. Then Theorem 5.1.27 of [8] implies that $X = \oplus_{i\in I} X_i$, where X_i is a clopen Lindelöf subset of X for every $i \in I$. By [12], any X_i is a locally compact cosmic space. Hence X_i is a separable metrizable space by [8, 3.3.5]. Thus X is also metrizable. □

Let us note that in the following theorem the implication (i) $\Leftrightarrow$ (ii) generalizes [4, Theorem 1.9], the implication (i) $\Leftrightarrow$ (iii) is proved in [12] and (i) $\Leftrightarrow$ (v) follows from [24].

Theorem 4.5. *For a topological space X the following assertions are equivalent:*

(i) *X is metrizable.*
(ii) *X is a $\mathfrak{P}$-space and has countable fan tightness.*
(iii) *X is an $\aleph$-space and is a Fréchet–Urysohn (α_4)-space.*
(iv) *X is a strict $\aleph$-space and is a Fréchet–Urysohn (α_4)-space.*
(v) *X is a strict σ-space and is an M-space.*

Proof. (i) $\Rightarrow$ (iv) Let X be metrizable. Then clearly, X is a Fréchet–Urysohn (α_4)-space, and Proposition 3.3 implies that X is a strict $\aleph$-space. (iv) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (ii) By Corollary 3.8, X has countable cs^*-character. Now the hypothesis and Proposition 4.1 imply that X is a $\mathfrak{P}$-space and has countable fan tightness.

(ii) $\Rightarrow$ (i) By Corollary 3.7, we have $cp_\chi(X) \le \aleph_0$. So X is first countable by Proposition 4.1. Now Proposition 3.11 and Theorem 4.3(ii) imply metrizability of X.

Since any metrizable space is a strict σ-space and each strict σ-space is a σ-space, the equivalence (i) $\Leftrightarrow$ (v) immediately follows from Theorem 4.3(iii) and Proposition 3.3. □

Since any cosmic space is separable, the next corollary contains [4, Theorem 1.9]:

Corollary 4.6. *A topological space X is second countable if and only if X is a $\mathfrak{P}_0$-space and has countable fan tightness if and only if X is an $\aleph_0$-space and is a Fréchet–Urysohn (α_4)-space.*

Example 4.7. Let $\mathbb{Z}^+$ be the (discrete) group of integers $\mathbb{Z}$ endowed with the Bohr topology and $b\mathbb{Z}$ be the completion of $\mathbb{Z}^+$. It is easy to see that $\mathbb{Z}^+$ is an $\aleph_0$-space. So

$$1 = cs^*_\chi(\mathbb{Z}^+) < ck_\chi(\mathbb{Z}^+) = cn_\chi(\mathbb{Z}^+) = \aleph_0 < \mathfrak{c} = \chi(\mathbb{Z}^+).$$

Since every convergent sequence in $\mathbb{Z}^+$ is trivial, the family $\{\{0\}\}$ is a cs^*-network at 0 which is not a cn-network at zero. As each compact subset of $\mathbb{Z}^+$ is finite, the precompact group $\mathbb{Z}^+$ is not a k-space. The compact non-metrizable group $b\mathbb{Z}$ is a dyadic compactum by Ivanovskij–Kuz'minov's theorem. Thus $b\mathbb{Z}$ has uncountable cs^*-character by [7, Proposition 7] and has uncountable cn-character by [11].

It is natural to ask whether $\mathbb{Z}^+$ is a $\mathfrak{P}_0$-space. Answering this question Banakh [4] proved the following: A precompact group has the strong Pytkeev property if and only if it is metrizable. So Theorem 1.6 generalizes the above-mentioned theorem to locally precompact groups. Our proof is similar to that of Banakh. Recall that, a subset E of a topological group G is called *left-precompact* (respectively, *right-precompact, precompact*) if, for every open neighborhood U of the unit $e \in G$, there exists a finite subset F of G such that $E \subseteq F \cdot U$ (respectively, $E \subseteq U \cdot F$, $E \subseteq F \cdot U$ and $E \subseteq U \cdot F$). If E is symmetric the three different definitions coincide. A topological group G is called *locally precompact* if it has a base at the unit consisting of symmetric precompact sets (or in other words, G embeds into a locally compact group).

Proof of Theorem 1.6. The implication (i) ⇒ (ii) follows from Proposition 3.3, and (ii) ⇒ (iii) follows from Corollary 3.7. Let us prove (iii) ⇒ (i). Assume that G has the strong Pytkeev property with a closed countable *cp*-network $\mathcal{N}$ at the unit e of G. Suppose for a contradiction that G is not first countable. Then there exists a symmetric precompact neighborhood U of e such that if $N \in \mathcal{N}$ and $N \subset U$ then N is nowhere dense in G. Indeed, otherwise we can take a neighborhood $V \subset G$ of e with $V^{-1}V \subset U$ and find $N \subset V$ such that $N^{-1}N \subset U$ contains a neighborhood of e, that means that the countable family $\{N^{-1}N : N \in \mathcal{N}\}$ is a base at e. Let $\{N'_k\}_{k\in\mathbb{N}}$ be an enumeration of the family

$$\mathcal{N}' = \{N \in \mathcal{N} : N \subset U\},$$

which is a closed *cp*-network at e. Fix arbitrarily $a_1 \notin N'_1$. Using induction and the nowhere density of the sets N'_k one can construct a sequence $(a_k)_{k\in\mathbb{N}} \subset U$ of distinct points of G such that $a_n \notin \bigcup_{k<n} \bigcup_{m\le n} a_k N'_m$. Consider the set $A = \{a_k^{-1}a_n : k < n\}$. We claim that this set contains e in its closure. Indeed, for every neighborhood V of e, we can find a neighborhood W of e such that $W^{-1}W \subset V \cap U$, and using the precompactness of U one can find a finite subset $F \subset G$ such that $U \subset FW$. By the Pigeonhole Principle, there are two numbers $k < n$ such that $a_k, a_n \in xW$ for some $x \in F$. Then $a_k^{-1}a_n \in W^{-1}W \subset V$, and hence $A \cap V \neq \emptyset$. Since $\mathcal{N}'$ is a *cp*-network at e, there is a number $q \in \mathbb{N}$ such that the set $B := N'_q \cap A$ is infinite. But this is not possible because $B \subset \{a_i^{-1}a_j : i < j < q\}$ and hence B is finite. □

Consequently, this shows that the group $\mathbb{Z}^+$ does not have the strong Pytkeev property.

5. $\mathfrak{n}$-Networks and operations over topological spaces

In this section we consider some standard operations in the class of spaces with countable $\mathfrak{n}$-character. For σ-, $\aleph$-, $\aleph_0$- and $\mathfrak{P}_0$-spaces as well as for spaces with countable cs^*-character all the following results are well-known (see [4,7,15,23]).

Next two obvious propositions show that the classes of topological spaces with countable character of various types are closed under taking subspaces and topological sums.

Proposition 5.1. *For $\mathfrak{n} \in \mathfrak{N}$, if $\mathcal{N}$ is a [σ-locally finite] $\mathfrak{n}$-network (at a point x) in a topological space X, then for every subspace $A \subset X$ (such that $x \in A$) the family $\mathcal{N}|_A := \{N \cap A : n \in \mathcal{N}\}$ is an [σ-locally finite] $\mathfrak{n}$-network (at the point x) in the space A.*

Proposition 5.2. *For $\mathfrak{n} \in \mathfrak{N}$, if $\mathcal{N}_i$ is an [σ-locally finite] $\mathfrak{n}$-network in a topological space X_i, $i \in I$, then the family $\mathcal{N} = \bigcup_{i\in I} \mathcal{N}_i$ is an [σ-locally finite] $\mathfrak{n}$-network in the topological sum $\oplus_{i\in I} X_i$.*

If $X = \prod_{n\in\mathbb{N}} X_n$ and $n \in \mathbb{N}$, we denote by p_n and π_n the projections of X onto X_n and $X_1 \times \cdots \times X_n$ respectively. For countable (Tychonoff) product we have the following.

Proposition 5.3. *For $\mathfrak{n} \in \mathfrak{N}$, if $\mathcal{N}_i = \{N_n^i\}_{n\in\mathbb{N}}$ is a countable $\mathfrak{n}$-network at a point x_i of a topological space X_i, $i \in \mathbb{N}$, then the countable family*

$$\mathcal{N} := \left\{ N_{m_1}^1 \times \cdots \times N_{m_n}^n \times \prod_{k>n} X_k : n \in \mathbb{N}, x_i \in N_{m_i}^i \in \mathcal{N}_i \right\}$$

is an $\mathfrak{n}$-network at the point $x = (x_i)$ of $X = \prod_{i\in\mathbb{N}} X_i$.

Proof. Let $U = \prod_{i=1}^n U_i \times \prod_{i>n} X_i$ be a neighborhood of x, where U_i is a neighborhood of x_i for all $1 \le i \le n$.

(1) Assume that $\mathcal{N}_i$ are *cn*-networks. By definition, for every $1 \le i \le n$, the set $W_i := \bigcup\{N_k^i \in \mathcal{N}_i : x_i \in N_k^i \subseteq U_i\}$ is a neighborhood of x_i. Clearly,

$$\prod_{i=1}^n W_i \times \prod_{i>n} X_i \subseteq \bigcup\left\{\prod_{i=1}^n N_{l_i}^i \times \prod_{i>n} X_i \in \mathcal{N} :\ x_i \in N_{l_i}^i \subseteq U_i, 1 \le i \le n\right\}.$$

Thus $\mathcal{N}$ is a countable *cn*-network in X at x.

The case when $\mathcal{N}_i$ are networks is considered analogously.

(2) Assume that $\mathcal{N}_i$ are *ck*-networks. By definition, for every $1 \le i \le n$, there exists a neighborhood $W_i \subset U_i$ of x_i such that for each compact subset K_i of W_i there is a finite subfamily $\mathcal{F}_i \subset \mathcal{N}_i$ such that $x_i \in \bigcap \mathcal{F}_i$ and $K_i \subset \bigcup \mathcal{F}_i \subset U_i$. Set $W := \prod_{i=1}^n W_i \times \prod_{i>n} X_i$. Then each compact subset K of W is contained in a set of the form $\prod_{i=1}^n K_i \times \prod_{i>n} X_i$, where K_i is a compact subset of W_i. Clearly,

$$K \subset \bigcup\left\{\prod_{i=1}^n N_{l_i}^i \times \prod_{i>n} X_i \in \mathcal{N} :\ x_i \in N_{l_i}^i \in \mathcal{F}_i, 1 \le i \le n\right\} \subset U.$$

Thus $\mathcal{N}$ is a countable *ck*-network at x.

The cases $\mathcal{N}_i$ are cs^*-networks or *cs*-networks are considered analogously.

(3) Assume that $\mathcal{N}_i$ are *cp*-networks. First we prove the following claim.

Claim. *The product $X_1 \times X_2$ of two spaces X_1 and X_2 with countable cp-networks at x_1 and x_2 has a countable cp-network at $x = (x_1, x_2)$.*

Proof. Let A be a subset of $X_1 \times X_2$ such that $x \in \overline{A} \setminus A$ and $U_1 \times U_2$ be a neighborhood of x.

For $i \in \{1, 2\}$, consider the countable family $\mathcal{P}_i = \{N_n^i \in \mathcal{N}_i : x_i \in N_n^i \subset U_i\}$ and let $\mathcal{P}_i = \{N_{n_k}^i\}_{k\in\mathbb{N}}$ be its enumeration. For every $k \in \mathbb{N}$, set $P_{k,i} = \bigcup_{l\le k} N_{n_l}^i$ and $P_k = P_{k,1} \times P_{k,2}$. Then $x \in \bigcap_{k\in\mathbb{N}} P_k \subset \bigcup_{k\in\mathbb{N}} P_k \subset U_1 \times U_2$. By (1) the family $\mathcal{N}$ is a *cn*-network at x, so $V := \bigcup_{k\in\mathbb{N}} P_k$ is a neighborhood of x. We show that $A_k := P_k \cap A$ is infinite for some $k \in \mathbb{N}$.

Suppose by a contradiction that A_k is finite for all $k \in \mathbb{N}$. Then for every $a = (a_1, a_2) \in (V \cap A) \setminus A_0$ we can find a unique number $k_a \in \mathbb{N}$ such that $a \in A_{k_a+1} \setminus A_{k_a}$. Since $a \notin P_{k_a}$, we fix arbitrarily $n_a \in \{1, 2\}$ such that $a_{n_a} \notin P_{k_a,n_a}$.

For $n \in \{1, 2\}$, set $A(n) := \{a = (a_1, a_2) \in (V \cap A) \setminus A_0 : n_a = n\}$ and $B_n := p_n(A(n)) \subset X_n$. We claim that $x_n \notin \overline{B_n}$. We show first that $x_n \notin B_n$. Indeed, assuming that $x_n \in B_n$ we can find $a = (a_1, a_2) \in A(n)$ such that $a_n = x_n$. However, by the definition of $A(n)$, we have $a_n \notin P_{k_a,n}$ and hence $a_n \neq x_n$. This contradiction shows that $x_n \notin B_n$. Now we suppose for a contradiction that $x_n \in \overline{B_n}$. Since $\mathcal{N}_n$ is a *cp*-network at x_n, we can find $P_{k,n} \in \mathcal{N}_n$ such that $P_{k,n} \cap B_n$ is infinite. On the other hand, for every $a = (a_1, a_2) \in A(n) \setminus A_k$ we have $a_n \notin P_{k,n}$. So the intersection $P_{k,n} \cap B_n$ is contained in A_k and hence it is finite. This contradiction shows that $x_n \notin \overline{B_n}$.

For $n \in \{1, 2\}$, choose an open neighborhood W_n of x_n such that $W_n \cap \overline{B_n} = \emptyset$. Then

$$[(V \cap A) \setminus A_0] \cap (W_1 \times W_2) = (A(1) \cup A(2)) \cap (W_1 \times W_2) = \emptyset,$$

and hence $x \notin \overline{A}$, a contradiction. Thus $A_k := P_k \cap A$ is infinite for some $k \in \mathbb{N}$.

Since P_k is a finite union of elements from $\mathcal{N}$, we obtain that $\mathcal{N}$ is a *cp*-network at x. The claim is proved. □

Now let A be a subset of X such that $x \in \overline{A} \setminus A \subset \overline{A} \subset U$. Set $C := \pi_n(A) \setminus \{\pi_n(x)\}$ and $D := \pi_n^{-1}(\pi_n(x)) \cap A$. If D is *infinite*, then

$$D \subset \left(N^1_{m_1} \times \cdots \times N^n_{m_n} \times \prod_{i>n} X_i \right) \cap A,$$

where $x_i \in N^i_{m_i} \subseteq U_i$ and $N^i_{m_i} \in \mathcal{N}_i$ for all $1 \le i \le n$, and hence the last intersection is infinite as desired. If D is *finite*, then $\pi_n(x) \in \overline{C} \setminus C$. By Claim, there is

$$N := N^1_{m_1} \times \cdots \times N^n_{m_n}, \text{ where } x_i \in N^i_{m_i} \subseteq U_i \text{ and } N^i_{m_i} \in \mathcal{N}_i, \quad 1 \le i \le n,$$

such that $N \cap C$ is infinite. Then $(N \times \prod_{i>n} X_i) \cap A$ is infinite as well. Thus $\mathcal{N}$ is a countable *cp*-network at x. $\square$

Proposition 5.4. *For $\mathfrak{n} \in \mathfrak{N}$, the countable product of $\mathfrak{n}$–σ-spaces is an $\mathfrak{n}$–σ-space.*

Proof. Let $\mathcal{D}_k = \bigcup_{s\in\mathbb{N}} \mathcal{D}_{s,k}$ be a closed σ-locally finite $\mathfrak{n}$-network in an $\mathfrak{n}$–σ-space X_k, $k \in \mathbb{N}$. As all X_k are regular, the space $X := \prod_{k\in\mathbb{N}} X_k$ is also regular. For each $s, n \in \mathbb{N}$ set

$$\mathcal{N}_{s,n} := \left\{ N^1_{s,m_1} \times \cdots \times N^n_{s,m_n} \times \prod_{i>n} X_i : \ N^k_{s,m_k} \in \mathcal{D}_{s,k}, \ k \in \{1, \dots, n\} \right\}.$$

Clearly, $\mathcal{N}_{s,n}$ is locally finite for every $s, n \in \mathbb{N}$. Hence the family $\mathcal{N} := \bigcup_{s,n\in\mathbb{N}} \mathcal{N}_{s,n}$ is σ-locally finite. Fix $x = (x_k) \in X$. For every $k \in \mathbb{N}$, the family $\{D \in \mathcal{D}_k : x_k \in D\}$ is a countable $\mathfrak{n}$-network at x_k, see the proof of Theorem 3.6. Now Proposition 5.3 shows that $\mathcal{N}$ is an $\mathfrak{n}$-network for X. $\square$

Propositions 5.1–5.4 imply

Corollary 5.5. *For $\mathfrak{n} \in \mathfrak{N}$, the class of topological space with countable $\mathfrak{n}$-character is closed under taking subspaces, topological sums and countable products.*

Corollary 5.6. *For $\mathfrak{n} \in \mathfrak{N}$, the class of $\mathfrak{n}$–σ-spaces is closed under taking subspaces, topological sums and countable products.*

Corollary 5.7. *([4,23]) The classes of cosmic, $\aleph_0$-spaces and $\mathfrak{P}_0$-spaces are closed under taking subspaces, countable topological sums and countable products.*

6. Function spaces

The following theorem is one of the most important and interesting applications of $\aleph$-spaces.

Theorem 6.1. *Let X be an $\aleph_0$-space. Then:*

(i) *([23]) If Y is an $\aleph_0$-space, then $C_c(X,Y)$ is also an $\aleph_0$-space.*
(ii) *([9,26]) If Y is a (paracompact) $\aleph$-space, then $C_c(X,Y)$ is a (paracompact) $\aleph$-space.*

Recall that for a first countable space X the space $C_c(X)$ is metrizable if and only if it is Fréchet–Urysohn, see [22]. The following proposition says something similar for the case $C_c(X,G)$, where G is an arbitrary topological group.

Proposition 6.2. *Let X be an $\aleph_0$-space and G a topological group. If G is an $\aleph$-space, then $C_c(X,G)$ has countable cs^*-character. Consequently, $C_c(X,G)$ is metrizable if and only if it is Fréchet–Urysohn.*

Proof. By Theorem 6.1 the space $C_c(X,G)$ is an $\aleph$-space. By Corollary 3.8 the space $C_c(X,G)$ has countable cs^*-character. Assume that $C_c(X,G)$ is Fréchet–Urysohn. Since every Fréchet–Urysohn group which has countable cs^*-character is metrizable by [7, Theorem 3], the topological group $C_c(X,G)$ is metrizable. The converse is trivial. □

Theorem 6.1 inspires the following question.

Question 6.3. *Let X be an $\aleph_0$-space and Y be a (paracompact) strict $\aleph$-space or a (paracompact) $\mathfrak{P}$-space. Is $C_c(X,Y)$ a (paracompact) strict $\aleph$-space or a (paracompact) $\mathfrak{P}$-space, respectively?*

Note that $C_c(X,Y)$ is a paracompact $\mathfrak{P}$-space for any $\aleph_0$-space X and each *metrizable* space Y (see [6]). We know (see Corollary 3.7) that any $\mathfrak{P}$-space has the strong Pytkeev property. So these results and Theorem 1.7 speak in favor of an affirmative answer to Question 6.3 might be positive.

In order to prove Theorem 1.7 we need the following lemma proved in [19, Theorem 5, p. 223].

Lemma 6.4. *Let C be a compact subspace of a Hausdorff space X. Then the map $(x,f) \mapsto f(x)$, from $C \times C_c(X,Y)$ to Y, is continuous.*

We are at the position to prove Theorem 1.7.

Proof of Theorem 1.7. For the $\aleph_0$-space X fix a countable k-network $\mathcal{K}$, which is closed under taking finite unions and finite intersections. For fixed $\mathfrak{n} \in \{ck, cp\}$, fix a closed (increasing) σ-locally finite $\mathfrak{n}$-network $\mathcal{D} = \bigcup_{j\in\mathbb{N}} \mathcal{D}_j$ in the $\mathfrak{n}$–σ-space Y. To prove two cases (1) and (2) of Theorem 1.7 we need to show that for every function $f \in C_c(X,Y)$ there exists a countable $\mathfrak{n}$-network at f.

Fix $f \in C_c(X,Y)$. Since $f(X)$ is a Lindelöf subspace of Y, Theorem 3.6 implies that there exists a sequence $\mathcal{D}_f = \{D_k\}_{k\in\mathbb{N}} \subset \mathcal{D}$ which is a countable $\mathfrak{n}$-network at each point of $f(X)$ and satisfies the condition: if $\mathrm{K} \subset f(X) \cap U$ with K compact and U open, then there is an open subset W of Y such that

(a) $\mathrm{K} \subset W \subset \bigcup_{k\in I(U)} D_k \subset U$, where $I(U) = \{k \in \mathbb{N} : D_k \subset U\}$, and
(b) for each compact subset C of W there is a finite subfamily α of $I(U)$ for which $C \subset \bigcup_{k\in\alpha} D_k$.

Let $\mathcal{N}_f$ be the countable family consisting of $\mathcal{D}_f$ and closed under finite unions and intersections of its elements. We claim that the countable family

$$[\![\mathcal{K}; \mathcal{N}_f]\!] = \big\{[K_1; N_1] \cap \cdots \cap [K_n; N_n] : K_1, \dots, K_n \in \mathcal{K},\ N_1, \dots, N_n \in \mathcal{N}_f\big\}$$

is an $\mathfrak{n}$-network at f in $C_c(X,Y)$.

Fix an open neighborhood $O_f \subset C_c(X,Y)$ of f. Without loss of generality we can assume that the neighborhood O_f is of basic form

$$O_f = [C_1; U_1] \cap \cdots \cap [C_n; U_n]$$

for some compact sets $C_1, \dots, C_n$ in X and some open sets $U_1, \dots, U_n$ in Y.

For every $i \in \{1, \dots, n\}$, consider the countable family

$$\mathcal{K}_i := \{K \in \mathcal{K} : C_i \subseteq K \subseteq f^{-1}(U_i)\},$$

and let $\mathcal{K}_i = \{K'_{i,j}\}_{j\in\mathbb{N}}$ be its enumeration. For every $j \in \mathbb{N}$ we set $K_{i,j} := \bigcap_{k\le j} K'_{i,k}$. It follows that the decreasing sequence $\{K_{i,j}\}_{j\in\mathbb{N}}$ converges to C_i in the sense that each open neighborhood of C_i contains all but finitely many sets $K_{i,j}$.

For every $i \in \{1, \dots, n\}$, consider the countable family (which is non-empty by (b))

$$\mathcal{N}_i := \{N \in \mathcal{N}_f : f(C_i) \subset N \subset U_i\},$$

and let W_i be an open neighborhood of $f(C_i)$ satisfying (a) and (b). Let $\{N'_{i,j}\}_{j\in\mathbb{N}}$ be an enumeration of $\mathcal{N}_i$. For every $j \in \mathbb{N}$ we set $N_{i,j} := \bigcup_{k\le j} N'_{i,k} \in \mathcal{N}_i$. It follows from (a) that $\{N_{i,j}\}_{j\in\mathbb{N}}$ is an increasing sequence of sets in Y with

$$f(C_i) \subset W_i \subset \bigcup_{j\in\mathbb{N}} N_{i,j} \subset U_i.$$

Then the sets

$$\mathcal{F}_j := \bigcap_{i=1}^{n} [K_{i,j}; N_{i,j}] \in [\![\mathcal{K}; \mathcal{N}_f]\!], \quad j \in \mathbb{N},$$

form an increasing sequence of sets in the function space $C_c(X, Y)$. Set $W_f := \bigcap_{i=1}^{n}[C_i; W_i]$.

Claim 6.5. $f \in W_f = \bigcap_{i=1}^{n}[C_i; W_i] \subset \bigcup_{j\in\mathbb{N}} \mathcal{F}_j \subset O_f = \bigcap_{i=1}^{n}[C_i; U_i]$.

Proof. We need to prove only the first inclusion. Suppose for a contradiction that there exists a function $g \in \bigcap_{i=1}^{n}[C_i; W_i]$ which does not belong to $\bigcup_{j\in\mathbb{N}} \mathcal{F}_j$. Then for every $j \in \mathbb{N}$ we can find an index $i_j \in \{1, \dots, n\}$ such that $g \notin [K_{i_j,j}; N_{i_j,j}]$. This means that $g(x_j) \notin N_{i_j,j}$ for some point $x_j \in K_{i_j,j}$. By the Pigeonhole Principle, there is $m \in \{1, \dots, n\}$ such that the set $J_m := \{j \in \mathbb{N} : i_j = m\}$ is infinite. As the decreasing sequence $\{K_{m,j}\}_{j\in J_m}$ converges to the compact set C_m, the set $C_m \cup \{x_j\}_{j\in J_m}$ is compact.

Since each compact subset of the $\aleph_0$-space X is metrizable (see Theorem 4.3), we can find an infinite subset J' of J_m such that the sequence $\{x_j\}_{j\in J'}$ converges to some point $x' \in C_m$. As g is continuous, the sequence $\{g(x_j)\}_{j\in J'}$ converges to the point $g(x') \in g(C_m) \subset W_m$, and hence we can assume also that $g(x_j) \in W_m$ for every $j \in J'$. Then we can apply (b) to the compact set $C' = g(C_m) \cup \{g(x_j)\}_{j\in J'}$ to find a finite subfamily $\mathcal{F}$ of $\mathcal{N}_f$ such that $C' \subset \bigcup \mathcal{F}$. Consequently, by construction, there is N_{m,j_0} containing C'. But this contradicts the choice of the points x_j and hence proves the inclusion $\bigcap_{i=1}^{n}[C_i; W_i] \subset \bigcup_{j\in\mathbb{N}} \mathcal{F}_j$. $\square$

By Claim 6.5, without loss of generality we shall assume that $f \in \mathcal{F}_j$ for every $j \in \mathbb{N}$.

We continue the proof by distinguishing two cases which cover the proof of the theorem.

(1): *Assume that $\mathcal{D}$ is a cp-network.* Given a subset $A \subset C_c(X, Y)$ with $f \in \overline{A}$ we need to find a set $\mathcal{F} \in [\![\mathcal{K}; \mathcal{N}_f]\!]$ such that $f \in \mathcal{F} \subset O_f$ and moreover $A \cap \mathcal{F}$ is infinite if f is an accumulation point of the set A. We can suppose that $A \subset W_f$.

If f is an isolated point of $C_c(X, Y)$, then $f \in \mathcal{F}_j \subset O_f$ for all $j \in \mathbb{N}$. Moreover, if $O_f = \{f\}$, then $\{f\} \in [\![\mathcal{K}; \mathcal{N}_f]\!]$, and we are done.

Assume now that f is an accumulation point of A in $C_c(X, Y)$. We show that $A \cap \mathcal{F}_j$ is infinite for some $j \in \mathbb{N}$. Suppose for a contradiction that for every $j \in \mathbb{N}$ the intersection $A_j := \mathcal{F}_j \cap A$ is finite. Then, by Claim 6.5, $A = A \cap W_f = \bigcup_{j\in\mathbb{N}} A_j$ is the countable union of the increasing sequence $\{A_j\}_{j\in\mathbb{N}}$ of finite subsets of $C_c(X, Y)$. Below we follow the proof of Theorem 2.1 of [4].

For every function $\alpha \in A \setminus A_0$ we denote by j_α the unique natural number such that $\alpha \in A_{j_\alpha+1} \setminus A_{j_\alpha} = A_{j_\alpha+1} \setminus \mathcal{F}_{j_\alpha}$. Since $\alpha \notin \mathcal{F}_{j_\alpha} = \bigcap_{i=1}^{n}[K_{i,j_\alpha}; N_{i,j_\alpha}]$, fix $i_\alpha \in \{1, \dots, n\}$ such that $\alpha \notin [K_{i_\alpha,j_\alpha}; N_{i_\alpha,j_\alpha}]$ and a point $x_\alpha \in K_{i_\alpha,j_\alpha}$ such that $\alpha(x_\alpha) \notin N_{i_\alpha,j_\alpha}$.

For every $i \in \{1, \dots, n\}$ consider the subsequence

$$A(i) := \{\alpha \in A \setminus A_0 : \ i_\alpha = i\}$$

and observe that $A \setminus A_0 = \bigcup_{i=1}^{n} A(i)$.

For every $i \in \{1, \dots, n\}$, set $B_i := \{\alpha(x_\alpha) : \alpha \in A(i)\} \subset Y$. We claim that the set $f(C_i)$ does not have accumulation points of B_i. Indeed, suppose for a contradiction that there is a point $y \in f(C_i)$ which is an accumulation point of B_i. As $\mathcal{N}_f$ is a *cp*-network at y, there is $N \in \mathcal{N}_f$ such that $y \in N \subset W_i$ and $N \cap B_i$ is infinite. Since $\mathcal{N}_f$ is closed under taking finite unions, there is $N_{i,j}$ such that $f(C_i) \cup N \subset N_{i,j} \subset W_i$. But the choice of the points x_α guarantees that $\alpha(x_\alpha) \notin N_{i,j}$ for all $\alpha \in A(i) \setminus A_j$, which yields that the intersection $B_i \cap N_{i,j} \subset \{\alpha(x_\alpha) : \alpha \in A(i) \cap A_j\}$ is finite. This contradicts the choice of the set $N \subset N_{i,j}$. Thus every point $y \in f(C_i)$ has an open neighborhood $O_y \subset W_i$ with finite $O_y \cap B_i$. Since $f(C_i)$ is compact we can find a finite family $Z_i \subset f(C_i)$ such that $V_i := \bigcup_{y \in Z_i} O_y \subset W_i$ is an open neighborhood of $f(C_i)$ having a finite intersection with the set B_i.

Since the decreasing sequence $\{K_{i,j}\}_{j\in\mathbb{N}}$ converges to C_i, there is a number $j_i \in \mathbb{N}$ such that $K_{i,j_i} \subset f^{-1}(V_i)$. Take a sufficiently large j_i such that $V_i \cap \{\alpha(x_\alpha) : \alpha \in A(i) \setminus A_{j_i}\} = \emptyset$. Then the set

$$C_i' := C_i \cup \{x_\alpha : \alpha \in A(i) \setminus A_{j_i}\} \subset K_{i,j_i}$$

is a compact subset of $f^{-1}(V_i)$, and hence the set $\widetilde{O}_f := \bigcap_{i=1}^n [C_i'; V_i]$ is an open neighborhood of f in $C_c(X, Y)$. By construction, for every $1 \le i \le n$ and each $\alpha \in A(i) \setminus A_{j_i}$, we have $x_\alpha \in C_i'$ and $\alpha(x_\alpha) \notin V_i$; so $\alpha \notin \widetilde{O}_f$. Hence

$$A \cap \widetilde{O}_f = \left(\widetilde{O}_f \cap \bigcup_{i=1}^n A_{j_i}\right) \cup \left(\widetilde{O}_f \cap \bigcup_{i=1}^n A(i) \setminus A_{j_i}\right) \subset \widetilde{O}_f \cap \bigcup_{i=1}^n A_{j_i}$$

is finite and f is not an accumulation point of A, a contradiction. Thus $A \cap \mathcal{F}_j$ is infinite for some $j \in \mathbb{N}$. Therefore $[\![\mathcal{K}; \mathcal{N}_f]\!]$ is a countable *cp*-network at f.

(2): *Assume that $\mathcal{D}$ is a ck-network.* We have to show that the countable family $[\![\mathcal{K}; \mathcal{N}_f]\!]$ is a *ck*-network at f. For this purpose it is enough to prove that the open neighborhood W_f of f witnesses O_f in the definition of *ck*-network at f, i.e. for each compact subset $A \subset W_f$ there is $j \in \mathbb{N}$ such that $A \subset \mathcal{F}_j$.

Fix a compact subset A of W_f. We show that $A \subset \mathcal{F}_j$ for some $j \in \mathbb{N}$. Suppose for a contradiction that $A \setminus \mathcal{F}_j \neq \emptyset$ for every $j > 0$. Observe that $C_c(X, Y)$ is an $\aleph$-space (see Theorem 6.1(ii)), and hence A is metrizable by Theorem 4.3. As A is compact, we can find a function $g_j \in A \setminus \mathcal{F}_j$ such that the sequence $\{g_j\}_{j\in\mathbb{N}}$ converges to some function $g_0 \in A$ in $C_c(X, Y)$. So we can assume that $A = \{g_j\}_{j\in\mathbb{N}}$.

For every $j > 0$ we can find an index $i_j \in \{1, \dots, n\}$ such that $g_j \notin [K_{i_j,j}; N_{i_j,j}]$. This means that $g_j(x_j) \notin N_{i_j,j}$ for some point $x_j \in K_{i_j,j}$. By the Pigeonhole Principle, there is $m \in \{1, \dots, n\}$ such that the set $J_m := \{j \in \mathbb{N} : i_j = m\}$ is infinite. As the decreasing sequence $\{K_{m,j}\}_{j\in J_m}$ converges to the compact set C_m, the set $C_m \cup \{x_j\}_{j\in J_m}$ is compact. Since each compact subset of the $\aleph_0$-space X is metrizable, we can find an infinite subset J' of J_m such that the sequence $\{x_j\}_{j\in J'}$ converges to some point $x_0 \in C_m$.

Observe that $g_0(x_0)$ belongs to the open set $W_m \subset Y$. Applying Lemma 6.4 with $C = \{x_0\} \cup \{x_j\}_{j\in J'}$ we can find an infinite subset J'' of J' such that $g_k(x_j) \in W_m$ for every $j, k \in J'' \cup \{0\}$. Now we apply once again Lemma 6.4 to the compact set $C' = C_m \cup \{x_j\}_{j\in J''}$ to obtain that the set

$$T := \{g_j(x) : \ x \in C', j \in J'' \cup \{0\}\}$$

is a compact subset of Y contained in W_m. Then we apply (b) for the compact set T to find a finite subfamily $\mathcal{F}$ of $\mathcal{N}_f$ such that $T \subset \bigcup \mathcal{F}$. Consequently, by construction, there is N_{m,j_0} containing T. But this contradicts the choice of the points x_j and hence proves that $A \subset \mathcal{F}_j$ for some $j \in \mathbb{N}$, and hence $[\![\mathcal{K}; \mathcal{N}_f]\!]$ is a *ck*-network at f. $\square$

If Y is a $\mathfrak{P}_0$-space, then the family $\mathcal{D}_f$ in the proof of Theorem 1.7 can be chosen to be common for all $f \in C_c(X, Y)$, so we obtain the following remarkable results.

Corollary 6.6. *([4]) If X is an $\aleph_0$-space and Y is a $\mathfrak{P}_0$-space, then $C_c(X,Y)$ is a $\mathfrak{P}_0$-space.*

Let $X = \mathbb{Q}$ be the set of rational numbers. Then $C_c(\mathbb{Q})$ is a $\mathfrak{P}_0$-space by Corollary 6.6; so the answer to Question 4 from [13] is negative that was noticed by Banakh in [4]. The locally convex space $C_c(\mathbb{Q})$ gives also a negative answer to Questions 1 and 7 from [13], see Remarks 3 and 6 in [13].

Below we pose a few natural questions which are inspired by the corresponding results for $\aleph_0$-spaces and $\aleph$-spaces.

Recall that a map $f : Y \to X$ of topological spaces is *compact-covering* if each compact subset of X is the image of a compact subset of Y. Michael [23] obtained the following characterizations of cosmic and $\aleph_0$-spaces.

Theorem 6.7. *([23]) Let X be a regular space. Then:*

(i) *X is cosmic if and only if X is a continuous image of a separable metric space.*
(ii) *X is an $\aleph_0$-space if and only if X is a compact-covering image of a separable metric space.*

Question 6.8. *Find a characterization of $\mathfrak{P}_0$-spaces analogous to the characterization of $\aleph_0$-spaces given in Theorem 6.7.*

Recall that a mapping $f : X \to Y$ is *sequence-covering* if each convergent sequence with the limit point of Y is the image of some convergent sequence with the limit point of X. Following Lin [21], a mapping $f : X \to Y$ is a *mssc-mapping* (i.e., metrizably stratified strong compact mapping) if X is a subspace of the product space $\prod_{n\in\mathbb{N}} X_n$ of a family $\{X_n\}_{n\in\mathbb{N}}$ of metric spaces satisfying the following condition: for each $y \in Y$, there exists an open neighborhood sequence $\{V_i\}$ of y such that each $\mathrm{cl}\left(p_i(f^{-1}(V_i)\right)$ is compact in X_i, where $p_i : \prod_{n\in\mathbb{N}} X_n \to X_i$ is the projection. Li [20] characterized $\aleph$-spaces as follows:

Theorem 6.9. *([20]) A regular space X is an $\aleph$-space if and only if X is a sequence-covering mssc-image of a metric space.*

Question 6.10. *Characterize analogous strict $\aleph$-spaces and $\mathfrak{P}$-spaces.*

Denote by $C_p(X)$ the space $C(X) := C(X,\mathbb{R})$ endowed with the pointwise topology. Sakai [31] proved that $C_p(X)$ has countable cs^*-character if and only if X is countable. It is well known that $\chi(C_p(X)) = |X|$ and hence $cs^*(C_p(X)) \le |X|$ for every infinite X.

Question 6.11. *Is $cs^*(C_p(X)) = |X|$ for every infinite Tychonoff space X?*

If this question has a positive answer then also $cp(C_p(X)) = ck(C_p(X)) = |X|$ by Corollary 2.14. It is well known that $C_p(X)$ is b-Baire-like for every Tychonoff space X. In [10] we proved that a b-Baire-like locally convex space E is metrizable if and only if E has countable cs^*-character.

Question 6.12. *Let E be a b-Baire-like locally convex space. Is $cs^*(E) = \chi(E)$?*

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