Topology and its Applications 192 (2015) 123–137

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On topological properties of Fréchet locally convex spaces with the weak topology [☆,☆☆]

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A R T I C L E I N F O

Article history:
Received 25 March 2014
Accepted 8 August 2014
Available online 3 June 2015

In honour of Ofelia T. Alas on the occasion of her 70th birthday

MSC:
primary 46A04, 54H11
secondary 22A05, 54C35, 54F65

Keywords:
Locally convex Fréchet space
Weak topology
$\aleph_0$-space
k-network
Banach space

A B S T R A C T

We describe the topology of any cosmic space and any $\aleph_0$-space in terms of special bases defined by partially ordered sets. Using this description we show that a Baire cosmic group is metrizable. Next, we study those locally convex spaces (lcs) E which under the weak topology $\sigma(E, E')$ are $\aleph_0$-spaces. For a metrizable and complete lcs E not containing (an isomorphic copy of) ℓ_1 and satisfying the Heinrich density condition we prove that $(E, \sigma(E, E'))$ is an $\aleph_0$-space if and only if the strong dual of E is separable. In particular, if a Banach space E does not contain ℓ_1, then $(E, \sigma(E, E'))$ is an $\aleph_0$-space if and only if E' is separable. The last part of the paper studies the question: Which spaces $(E, \sigma(E, E'))$ are $\aleph_0$-spaces? We extend, among the others, Michael's results by showing: If E is a metrizable lcs or a (DF)-space whose strong dual E' is separable, then $(E, \sigma(E, E'))$ is an $\aleph_0$-space. Supplementing an old result of Corson we show that, for a Čech-complete Lindelöf space X the following are equivalent: (a) X is Polish, (b) $C_c(X)$ is cosmic in the weak topology, (c) the weak*-dual of $C_c(X)$ is an $\aleph_0$-space.

1. Introduction

The class of $\aleph_0$-spaces in sense of Michael [30] is the most immediate extension of the class of separable metrizable spaces.

Definition 1.1. ([30]) A topological space X is called

(i) *cosmic*, if X is a regular space with a countable network (a family $\mathcal{N}$ of subsets of X is called a *network* in X if, whenever $x \in U$ with U open in X, then $x \in N \subset U$ for some $N \in \mathcal{N}$);

☆ The authors thank to Professor J.C. Ferrando, who suggested Proposition 4.11.
☆☆ The second and fourth named authors were supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2013/058.
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http://dx.doi.org/10.1016/j.topol.2015.05.075

(ii) an $\aleph_0$-*space*, if X is a regular space with a countable k-network (a family $\mathcal{N}$ of subsets of X is called a *k-network* in X if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \bigcup \mathcal{F} \subset U$ for some finite family $\mathcal{F} \subset \mathcal{N}$).

Note that both of these classes of topological spaces are closed under taking subspaces, countable Tychonoff products, countable direct sums, etc. [30] (see also [2]). The concept of network is one of a well recognized good tool, coming from the pure set-topology, which turned out to be of great importance to study successfully renorming theory in Banach spaces, we refer the reader to the recent survey of Cascales and Orihuela [10].

Michael [30] characterized cosmic and $\aleph_0$-spaces: *A regular space is a cosmic (resp. $\aleph_0$-) space if and only if X is a continuous (resp. continuous compact-covering) image of a separable metric space.* Consequently, every cosmic space (hence $\aleph_0$-space as well) is Lindelöf. Another characterization of $\aleph_0$-spaces is given by Guthrie in [23]. It is known [33] that an $\aleph_0$-space (even an $\aleph$-space) which is either first countable or locally compact is metrizable. For further properties of $\aleph_0$-spaces we refer to papers [19,22,31]. Although Michael's (above) result provides a nice characterization of cosmic and $\aleph_0$-spaces, it seems that there does not exist an appropriate description of the topology of cosmic and $\aleph_0$-spaces. For example, each countable regular space X is cosmic as a continuous image of the discrete underlying countable space X. However, this does not describe the topology of X.

In the first part of the paper we describe the topology of cosmic and $\aleph_0$-spaces in terms of special bases defined by partially ordered sets (Theorem 2.2). The second part of the paper deals with $\aleph_0$-spaces in the class of locally convex spaces (lcs) E. We examine the following two natural problems being well motivated both from topology and functional analysis.

Problem 1.2. Characterize those lcs E which are weakly $\aleph_0$-spaces, i.e. E with the weak topology $\sigma(E, E')$ is an $\aleph_0$-space.

Problem 1.3. Describe possible large class of lcs which are weakly or weakly* $\aleph_0$-spaces, i.e. the topological dual E' of E with the weak* topology $\sigma(E', E)$ is an $\aleph_0$-space.

Michael [30, §7] proved the following two facts for a Banach space E: (i) If E is separable, the normed dual E' is a weakly* $\aleph_0$-space. (ii) If E' is also separable, E is a weakly $\aleph_0$-space. Problems 1.2 and 1.3 have been also studied for Banach spaces and for E being a separable metrizable lcs with E' endowed with the compact-open topology in [3, Sections 11 and 12].

If E is a Banach space with separable normed dual E', then, by [14, Theorem (1)–(4), p. 215], the space E does not contain (an isomorphic copy of) ℓ_1, but E is a weakly $\aleph_0$-space. Let now (E, ξ) be a separable Banach space with the Schur property (i.e., every weakly null-sequence in E converges in the original topology ξ), for example ℓ_1. Then the Eberlein–Šmulian theorem implies that $\sigma(E, E')$ and ξ have the same compact sets. So, $(E, \sigma(E, E'))$ is an $\aleph_0$-space trivially because E is an $\aleph_0$-space. Hence each of the following two conditions guarantees that a separable Banach space E is a weakly $\aleph_0$-space: (1) the normed dual E' is separable, and (2) E has the Schur property. Note that the space $E := \ell_1 \times \ell_2$ is a weakly $\aleph_0$-space, but E does not have the Schur property and its normed dual is nonseparable.

For a lcs E by the *strong dual* of E we mean the dual E' endowed with the strong topology $\beta(E', E)$. By a *Fréchet lcs* space we mean a metrizable and complete lcs. Having in mind Problem 1.2 first we prove the following general fact for any lcs which is a weakly $\aleph_0$-space.

Proposition 1.4. *Let E be a lcs which is a weakly $\aleph_0$-space. Then the strong dual E' of E is trans-separable if and only if every bounded set in E is Fréchet–Urysohn in the weak topology of E.*

Next, we extend some recent results of Barroso, Kalenda and Lin [4], which with Proposition 1.4 provide the following

Theorem 1.5. *Let E be a Fréchet lcs and E' be its strong dual. Then*

(i) *If E' is separable, then E is a weakly $\aleph_0$-space.*
(ii) *If E is a weakly $\aleph_0$-space not containing ℓ_1, then E' is trans-separable.*

It is natural to ask whether the trans-separability in Theorem 1.5(ii) can be strengthened to separability. Since for metrizable spaces trans-separability and separability coincide, the space whose strong dual has bounded sets metrizable is of interest. It is known that the class of Fréchet lcs E for which the strong dual E' has bounded sets metrizable coincides with the class of Fréchet lcs which satisfy the *density condition of Heinrich*; it contains every Fréchet–Montel lcs and every quasinormable Fréchet lcs (in sense of Grothendieck). The latter class of lcs contains the most usual function spaces, all Banach spaces, as well as every (FS)-space. These spaces were studied in [6,7]. We prove the following

Theorem 1.6. *Let E be a Fréchet lcs satisfying the Heinrich density condition and not containing ℓ_1. Then E is a weakly $\aleph_0$-space if and only if the strong dual of E is separable.*

Consequently, for a Banach space E not containing ℓ_1 the normed dual E' is separable if and only if E is a weakly $\aleph_0$-space (noticed also in [3, Theorem 12.3]). The James tree space JT (see [14]) is a separable Banach space having a nonseparable normed dual and containing no isomorphic copy of ℓ_1. So JT is not a weakly $\aleph_0$-space (also mentioned in [3, §12]).

Applying recent results of Cascales, Orihuela and Tkachuk [12], we extend Michael's results [30, §7] by showing, among the others, that if E is a metrizable lcs or a (DF)-space whose *strong dual* E' is separable, then $(E, \sigma(E, E'))$ is an $\aleph_0$-space (Theorem 4.5).

By $C_c(X)$ and $C_p(X)$ we denote the space $C(X)$ of all real-valued continuous functions on a completely regular Hausdorff space X endowed with the compact-open topology and the pointwise topology, respectively. Corson proved [30, Proposition 10.8] that for an uncountable compact metrizable space X, the Banach space $C_c(X)$ (clearly the normed dual is not separable) is not a weakly $\aleph_0$-space. Nevertheless, we show that for a Čech-complete Lindelöf space X the following are equivalent (Proposition 4.7): (a) X is Polish, (b) $C_c(X)$ is cosmic in the weak topology, (c) the weak*-dual of $C_c(X)$ is an $\aleph_0$-space. As an application we prove that if there exists a continuous linear surjection from $C_c(X)$ onto $C_p(Y)$, every closed first countable subspace Z of Y is Polish provided X is Polish (Corollary 4.8); this extends a Pelant's result [2, Theorem 3.27].

2. Description of the topology of cosmic and $\aleph_0$-spaces

Let Ω be a set and I a partially ordered set with an order $\leq$. We say that a family $\{A_i\}_{i\in I}$ of subsets of Ω is *I-decreasing* (respectively, *I-increasing*) if $A_j \subset A_i$ (respectively, $A_i \subset A_j$) for every $i \leq j$ in I. One of the most important example of partially ordered sets is the product $\mathbb{N}^\mathbb{N}$ endowed with the natural partial order, i.e., $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i \in \mathbb{N}$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}}$ and $\beta = (\beta_i)_{i\in\mathbb{N}}$. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{N}^\mathbb{N}$ and each $k \in \mathbb{N}$, set

$$I_k(\alpha) := \left\{\beta \in \mathbb{N}^\mathbb{N} : \beta_i = \alpha_i \text{ for } i = 1, \ldots, k\right\}.$$

The following concept is used in our description.

Definition 2.1. A topological space (X, τ) has a *small base* if there exists an $\mathbf{M}$-decreasing base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ of τ for some $\mathbf{M} \subseteq \mathbb{N}^\mathbb{N}$.

If a topological space X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ in X, we define the countable family $\mathcal{D}_\mathcal{U}$ of subsets of X by

$$\mathcal{D}_\mathcal{U} := \{D_k(\alpha) : \ \alpha \in \mathbf{M}, \ k \in \mathbb{N}\}, \text{ where } D_k(\alpha) = \bigcap_{\beta \in I_k(\alpha) \cap \mathbf{M}} U_\beta,$$

and say that $\mathcal{U}$ satisfies the *condition* $(\mathbf{D})$ if $U_\alpha = \bigcup_{k \in \mathbb{N}} D_k(\alpha)$ for every $\alpha \in \mathbf{M}$. A similar condition naturally appears and is essentially used in [21].

Next theorem describes the topology of cosmic and $\aleph_0$-spaces.

Theorem 2.2. *Let (X, τ) be a topological space. Then:*

(i) *X has a countable network (and is cosmic) if and only if X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ satisfying the condition $(\mathbf{D})$ (and is regular). In that case the family $\mathcal{D}_\mathcal{U}$ is a countable network in X.*

(ii) *X has a countable k-network (and is an $\aleph_0$-space) if and only if X has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ satisfying the condition $(\mathbf{D})$ such that the family $\mathcal{D}_\mathcal{U}$ is a countable k-network in X (and is regular).*

In both cases we can find a small base $\mathcal{U}$ such that $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$ and $\mathcal{U} = \tau$, what means that for every $W \in \tau$ there exists $\alpha \in \mathbf{M}$ such that $W = U_\alpha$.

Proof. (i) Assume that X is cosmic with a countable network $\mathcal{D} = \{D_i : i \in \mathbb{N}\}$. Let $f : \tau \to \mathbb{N}^\mathbb{N}$ be the map defined by $f(W) = (a_n)_n$, where $a_n = 1$ if $D_n \subseteq W$, and $a_n = 2$ if $D_n \nsubseteq W$, for each $W \in \tau$. Let $\mathbf{M} := \{f(W) : W \in \tau\}$ and for each $\alpha = (a_n)_n \in \mathbf{M}$ let

$$U_\alpha := \bigcup \{D_n : n \in \mathbb{N}, a_n = 1\}.$$

Note that $W = U_{f(W)}$ for each $W \in \tau$ because $\mathcal{D}$ is a network. Now it is clear that the family $\mathcal{U} := \{U_\alpha : \alpha \in \mathbf{M}\}$ is a small base, $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$ and $\mathcal{U} = \tau$.

Let $\alpha = (a_n) \in \mathbf{M}$ and $k \in \mathbb{N}$. If $\beta \in I_k(\alpha) \cap \mathbf{M}$, then from the formula

$$\bigcup \{D_n : n \in \mathbb{N}, n \leqslant k, a_n = 1\} \subset U_\beta$$

it follows that

$$\bigcup \{D_n : n \in \mathbb{N}, n \leqslant k, a_n = 1\} \subset \bigcap \{U_\beta : \beta \in I_k(\alpha) \cap \mathbf{M}\} = D_k(\alpha) \subset U_\alpha.$$

On the other hand, from

$$U_\alpha = \bigcup_k \left[\bigcup \{D_n : n \in \mathbb{N}, n \leqslant k, a_n = 1\} \right] \subset \bigcup_k D_k(\alpha) \subset U_\alpha$$

we deduce the equality $U_\alpha = \bigcup_k D_k(\alpha)$. It proves that $\mathcal{U}$ verifies the condition $(\mathbf{D})$. Conversely, if X has a small base satisfying the condition $(\mathbf{D})$ it is clear that the family $\mathcal{D}_\mathcal{U}$ is a countable network of X.

(ii) Assume that X is an $\aleph_0$-space with a countable k-network $\mathcal{D} = \{D_i : i \in \mathbb{N}\}$, and let $\mathcal{U} := \{U_\alpha : \alpha \in \mathbf{M}\}$ be the small base constructed as in (i) satisfying the condition $(\mathbf{D})$. We show that the countable family $\mathcal{D}_\mathcal{U}$ is also a k-network in X.

Fix $U_\alpha \in \mathcal{U}$ and a compact subset K, with $K \subset U_\alpha$. As $\mathcal{D}$ is a countable k-network, there exists a finite increasing set $\{n_i : 1 \leqslant i \leqslant h\}$ such that

$$K \subset \bigcup \{D_{n_i} : 1 \leqslant i \leqslant h\} \subset U_\alpha.$$

If $\beta \in I_{n_h}(\alpha) \cap \mathbf{M}$, then

$$K \subset \bigcup \{D_{n_i} : 1 \leqslant i \leqslant h\} \subset U_\beta,$$

and therefore

$$K \subset \bigcap \{U_\beta : \beta \in I_{n_h}(\alpha) \cap \mathbf{M}\} = D_{n_h}(\alpha) \subset U_\alpha.$$

Hence the family $\mathcal{D}_{\mathcal{U}}$ is a countable k-network of X. The converse assertion is trivial. □

We were kindly informed by Prof. Tkachenko that the following Corollary 2.3 has been also proved in [38, Corollary 3.23].

Corollary 2.3. *Let G be a Baire topological group. Then G is cosmic if and only if G is metrizable and separable.*

Proof. It is enough to show that, if G is a Baire and cosmic, then G is metrizable. We prove that G has a countable base of neighborhoods at the unit e. By Theorem 2.2 there exists a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ satisfying the condition (**D**). We show that the countable family $\{\overline{D_k(\alpha)} \cdot \overline{D_k(\alpha)}^{\,-1} : \alpha \in \mathbf{M}, k \in \mathbb{N}\}$ contains a base of neighborhoods at e in G. Let W be an open neighborhood of e. Choose V, a symmetric open neighborhood of e such that $V \cdot V \subset \overline{V} \cdot \overline{V} \subset W$. There exists $\alpha \in \mathbf{M}$ with $V = U_\alpha = \bigcup_k D_k(\alpha)$. Since U_α is open in G, there exists $k \in \mathbb{N}$ such that $U_\alpha \cap \overline{D_k(\alpha)}$ has a non-empty interior in U_α, so also in G. Therefore $\overline{D_k(\alpha)} \cdot \overline{D_k(\alpha)}^{\,-1}$ is a neighborhood of e, contained in W. □

The cofinality of a partially ordered set P we denote by $\mathrm{cf}(P)$. The cofinality of $\mathbb{N}^{\mathbb{N}}$ is denoted by $\mathfrak{d}$. It is well known that $\aleph_1 \leq \mathfrak{d} \leq \mathfrak{c}$ and that the hypothesis $\mathfrak{d} < \mathfrak{c}$ is consistent with ZFC.

Example 2.4. There is a subset P of $\mathbb{N}^{\mathbb{N}}$ such that $\mathrm{cf}(P) = \mathfrak{c}$.

Proof. Let $G = (\mathbb{Z}, \tau_b)$ be the group of integers $\mathbb{Z}$ endowed with the Bohr topology τ_b. It is well-known that $\chi(G) = \mathfrak{c}$. Since G is countable it is a cosmic space. Now Theorem 2.2(i) implies that G has a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ with $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$ and $\mathcal{U} = \tau_b$. Set $P := \{\alpha \in \mathbf{M} : 0 \in U_\alpha\}$. Then P is a base at 0. Hence $\mathfrak{c} = \chi(G) \leq \mathrm{cf}(P) \leq |P| \leq \mathfrak{c}$. Thus $\mathrm{cf}(P) = \mathfrak{c}$. □

Note that the condition (**D**) is essential in Theorem 2.2, since there is a compact non-cosmic abelian group (H, τ) with a small base $\mathcal{U}$ satisfying $\mathcal{U} = \tau$, see Example 2.6. First we prove the following useful

Proposition 2.5. *If a regular topological space (X, τ) has a dense subset A with a small base $\mathcal{U} = \{U_\alpha : \alpha \in \mathbf{M}\}$ such that $U_\alpha \neq U_\beta$ for $\alpha \neq \beta$ and $\mathcal{U} = \tau|_A$, then X also has a small base $\mathcal{V}$ such that $\mathcal{V} = \tau$.*

Proof. Since A is dense, the assumption on $\mathcal{U}$ implies that, for every $V \in \tau$ there exists a unique $\alpha \in \mathbf{M}$ such that $U_\alpha = V \cap A$. Set $V_\alpha := V$. Then the family $\{V_\alpha : \alpha \in \mathbf{M}\}$ is as required. □

Example 2.6. There is a compact abelian group with a small base which is not a cosmic space.

Proof. Let $H = b\mathbb{Z}$ be the Bohr compactification of $\mathbb{Z}$ with discrete topology. So H is the completion of the group G defined in Example 2.4. Now the proof of Example 2.4 and Proposition 2.5 imply that H has a small base. Since H is not metrizable, it is not cosmic by Corollary 2.3. □

It is well known that any Baire lcs is barrelled. Next example shows that Corollary 2.3 cannot be extended to barrelled $\aleph_0$-spaces. Recall that E is *Fréchet–Montel* if E is a metrizable and complete lcs whose every closed bounded set is compact; we refer to [34, 8.5.8, p.283] for concrete examples.

Example 2.7. The strong dual E' of an infinite-dimensional Fréchet–Montel space E is a barrelled non-metrizable lcs which is an $\aleph_0$-space.

Proof. Let $(U_n)_n$ be a decreasing basis of neighbourhoods of zero in E; set $K_n := U_n^\circ$ for each $n \in \mathbb{N}$. Being a Fréchet–Montel space, E is separable and every null-sequence in $\sigma(E', E)$ is a null-sequence in $\beta(E', E)$ by [25, 11.6.2]. So $(E', \sigma(E', E))$ is submetrizable and every $\sigma(E', E)$-compact set is $\beta(E', E)$-compact and metrizable. Hence each K_n is $\beta(E', E)$-compact and metrizable, so E' is an $\aleph_0$-space (as every $\beta(E', E)$-compact set is contained in some K_n and we apply [30, Proposition 7.7]). The strong dual E' is barrelled by [25, 11.5.4]. Also E' is nonmetrizable since, otherwise, E has a fundamental sequence of bounded sets. So E is normable [25, 12.4.4] and hence finite-dimensional, a contradiction. □

3. Weakly $\aleph_0$-spaces not containing a copy of ℓ_1

Recall that a topological space X has the *property* (α_4) if for any $\{x_{m,n} : (m, n) \in \mathbb{N} \times \mathbb{N}\} \subset X$ with $\lim_n x_{m,n} = x \in X$, $m \in \mathbb{N}$, there exists a sequence $(m_k)_k$ of distinct natural numbers and a sequence $(n_k)_k$ of natural numbers such that $\lim_k x_{m_k,n_k} = x$.

Recall also that a topological space X is *strongly Fréchet–Urysohn* if for every $x \in X$ and for each decreasing family (A_n) of X with $x \in \bigcap_n \overline{A_n}$, there are $x_n \in A_n$ $(n \in \mathbb{N})$ with $\lim_n x_n = x$ (see [13]). A topological group G is Fréchet–Urysohn if and only if it is strongly Fréchet–Urysohn (see [1] or [13]).

Recall that a uniform space $(X, \mathcal{N})$ is *trans-separable* (see [26] or [34]), if for every entourage N in $\mathcal{N}$ there exists a countable subset Q of X such that $X = \bigcup_{x\in Q} U_N(x)$, where $U_N(x) := \{y \in X : (x, y) \in N\}$. Every metrizable trans-separable uniform space is separable. A lcs E is trans-separable if and only if for each neighbourhood of zero U in E there exists a countable subset N of E with $E = N + U$. Note that a lcs E does not contain ℓ_1 provided the strong dual E' is trans-separable. In order to prove Theorem 1.5 we recall the following result from [17] (see also [26, Corollary 6.8]).

Lemma 3.1. *([17]) The strong dual of a lcs E is trans-separable if and only if every bounded set in E is metrizable in the weak topology $\sigma(E, E')$ of E.*

We need the following lemma; its proof uses some technics from [13, Lemma 1.3].

Lemma 3.2. *Let E be a topological vector space (resp. topological group) such that every bounded (resp. precompact) set is Fréchet–Urysohn. Then, every bounded (resp. precompact) set has the property (α_4) and therefore it is strongly Fréchet–Urysohn.*

Proof. For the case when E is a topological group, we assume that E is not discrete; otherwise, the conclusion holds trivially. By 0 we will denote the neutral element of E.

Let B be a bounded (resp. precompact) subset and suppose that $x_{m,n} \in B$, for each $(m, n) \in \mathbb{N} \times \mathbb{N}$, and that $\lim_n x_{m,n} = x \in B$ for every $m \in \mathbb{N}$. Then $B' = B - x$ contains each $z_{m,n} := x_{m,n} - x$ and $\lim_n z_{m,n} = 0 \in B'$ for every $m \in \mathbb{N}$. To prove that B has the property (α_4) it is enough to find sequences $(p_k)_k$ and $(n_k)_k$ in $\mathbb{N}$, with $p_k < p_{k+1}$ for each $k \in \mathbb{N}$, such that $\lim_k z_{p_k,n_k} = 0$. The proof is obvious if the set $\{m \in \mathbb{N} : z_{m,n} = 0$ for some $n \in \mathbb{N}\}$ is infinite. Therefore we assume that $z_{m,n} \neq 0$ for each $(m,n) \in \mathbb{N} \times \mathbb{N}$.

Choose any sequence $(v_m)_m \subset B' \setminus \{0\}$ with $\lim_m v_m = 0$. Define

$$y_{m,n} := \begin{cases} v_m, & \text{if } z_{m,n+m} = v_m \\ z_{m,n+m} - v_m, & \text{if } z_{m,n+m} \neq v_m \end{cases} \qquad (m,n \in \mathbb{N}).$$

Clearly, $0 \neq y_{m,n} \in B' - B'$ for all $n, m \in \mathbb{N}$. It follows from $\lim_m v_m = 0$ and $\lim_m \lim_n (z_{m,n+m} - v_m) = 0$ that 0 belongs to the closure of the bounded (resp. precompact) set $\{y_{m,n} : (m,n) \in \mathbb{N} \times \mathbb{N}\}$ (note that $B' - B'$ is bounded (resp. precompact)). Therefore there exist two sequences $(p_k)_k$ and $(s_k)_k$ such that $\lim_k y_{p_k,s_k} = 0$.

Since $v_m \to 0$ if $m \to \infty$, we have to show that the sequence $(p_k)_k$ is unbounded. Suppose for a contradiction that $(p_k)_k$ is bounded. We may suppose (taking a subsequence if it was necessary) that $p_k = p$ for every $k \in \mathbb{N}$. If the sequence $(s_k)_k$ is unbounded we may assume (taking a subsequence if it was necessary) that $s_k < s_{k+1}$ for each $k \in \mathbb{N}$. Since $v_p = z_{p,s_k+p}$ or $v_p = z_{p,s_k+p} - y_{p,s_k}$, from the facts $\lim_k y_{p,s_k} = \lim_k y_{p_k,s_k} = 0$ and $\lim_k z_{p,s_k+p} = 0$ we deduce that $v_p = 0$, a contradiction.

If the sequence $(s_k)_k$ is bounded we may suppose (taking a subsequence if it was necessary) that $s_k = s$ for each $k \in \mathbb{N}$. Then $y_{p,s} = \lim_k y_{p_k,s_k} = 0$, that contradicts the choice of $y_{p,s}$. So $(p_k)_k$ is unbounded. Thus B has the property (α_4).

The set B is strongly Fréchet–Urysohn by [13, Proposition 1.4]. □

We are ready for the proof of Proposition 1.4.

Proof of Proposition 1.4. If E' is trans-separable, then every bounded set in E is metrizable in $\sigma(E, E')$ by Lemma 3.1. Conversely, if every bounded set in E is Fréchet–Urysohn in $\sigma(E, E')$, we apply Lemma 3.2 to see that every bounded set B in E is strongly Fréchet–Urysohn in $\sigma(E, E')$. As a subspace of the $\aleph_0$-space $(E, \sigma(E, E'))$, the set B is also an $\aleph_0$-space. By [31, Theorem 9.11], B is second countable, hence metrizable. Finally, again Lemma 3.1 applies to get the trans-separability of E'. □

A lcs E will be said to have *the Rosenthal property* if every bounded sequence in E either (R_1) has a subsequence which is Cauchy in the weak topology $\sigma(E, E')$, or (R_2) has a subsequence which is equivalent to the unit vector basis of ℓ_1. Recently, Ruess [36, Proposition 3.3] proved the following

Proposition 3.3. *([36]) Every sequentially complete lcs E whose every bounded set is metrizable has the Rosenthal property.*

Note that there is a quite large class of spaces E satisfying the assumptions quoted by Ruess: The strong dual of any metrizable lcs with *the Heinrich density condition* is an example of a space E of this type (see [6, Theorem 2]). In particular all quasinormable metrizable lcs satisfy the Heinrich density condition (see [6] for more details about this class).

A family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of a set E covering E is called a *resolution* if $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$. Following Cascales and Orihuela [9], a lcs E is said to be in *class* $\mathfrak{G}$ if E' admits a $\sigma(E', E)$-resolution $\{A_\alpha : \alpha \in \mathbb{N}\}$ (called a $\mathfrak{G}$-representation for E) such that every sequence in any A_α is equicontinuous, see [26] for several results about this class. The class $\mathfrak{G}$ contains "almost all" important lcs (including (LM)-spaces (hence metrizable lcs), (DF)-spaces, etc.), and it is stable under taking subspaces, Hausdorff quotients, countable direct sums and products.

A family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighborhoods of zero in E is called a $\mathfrak{G}$*-base* if $\mathcal{U}$ is an $\mathbb{N}^{\mathbb{N}}$-decreasing base of neighborhoods of zero [9,26]. Topological groups with a $\mathfrak{G}$-base were considered in [20]. A lcs E

is *quasibarrelled* (*barrelled*) if every $\beta(E', E)$-bounded ($\sigma(E', E)$-bounded) set in E' is equicontinuous [34]. Metrizable lcs are quasibarrelled. By [11], a quasibarrelled lcs E has a $\mathfrak{G}$-base if and only if E is in class $\mathfrak{G}$.

Recall that a lcs E is called a (DF)-*space* if E has a fundamental sequence of bounded absolutely convex sets and E is $\aleph_0$-quasibarrelled (see [34, 8.3]). Every normed space is a (DF)-space. The strong dual of a metrizable lcs is a complete (DF)-space [34, 8.3.9] and the strong dual of a (DF)-space is a metrizable and complete lcs [34, 8.3.7].

The following lemmas extend [27, Theorem 2.4] and [4, Lemma 3.6, Proposition 4.4], although the main ideas for the proofs are similar.

Lemma 3.4. *Let E be a lcs in class $\mathfrak{G}$ having the Rosenthal property (R_1). Then every bounded, separable set of E is Fréchet–Urysohn in the weak topology.*

Proof. Let $B \subset E$ be a bounded, separable set having the Rosenthal property (R_1). Since the linear span of B is separable, and every linear subspace of E is in class $\mathfrak{G}$, we can assume that E is separable. So, there exists a resolution $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of $(E', \sigma(E', E))$ consisting of relatively countably $\sigma(E', E)$-compact sets. Since E is separable, E' admits a coarser metrizable locally convex topology. Then Šmulian's theorem [18, 3.2 Theorem] guarantees that every V_α is relatively $\sigma(E', E)$-compact. Hence, $\{\overline{V}_\alpha^{\sigma(E',E)} : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\sigma(E', E)$-compact resolution of $(E', \sigma(E', E))$. By Talagrand's theorem (see [9, Theorem 15]) the space $(E', \sigma(E', E))$ is analytic. Thus, there exists a continuous surjection $G : \mathbb{N}^\mathbb{N} \to (E', \sigma(E', E))$.

Now, similarly as in the proof of [4, Lemma 3.6], define the map $H : E \to \mathbb{R}^{\mathbb{N}^\mathbb{N}}$ by the formula $H(x)(\alpha) = G(\alpha)(x)$, where $x \in E$, $\alpha \in \mathbb{N}^\mathbb{N}$. We can easily verify that H is a linear homeomorphism of $(E, \sigma(E, E'))$ onto $H(E)$ and elements of $H(E)$ are continuous functions defined on $\mathbb{N}^\mathbb{N}$. As every sequence of B has a $\sigma(E, E')$-Cauchy subsequence, each sequence of $H(B)$ has a Cauchy subsequence in the topology induced from $\mathbb{R}^{\mathbb{N}^\mathbb{N}}$. Hence, by [27, Corollary 2.2], the closure of $H(B)$ is Fréchet–Urysohn. Thus, B is Fréchet–Urysohn in the weak topology. □

Lemma 3.5. *Let E be a quasibarrelled lcs in class $\mathfrak{G}$. Then the following assertions are equivalent.*

(i) *Any bounded subset of E is Fréchet–Urysohn in the weak topology.*
(ii) *Any bounded sequence in E has a weakly Cauchy subsequence.*

Proof. (i) ⇒ (ii): See the proof of [4, Proposition 4.4, i)⇒ii)].

(ii) ⇒ (i): Let B be a bounded set of E. By [26, Theorem 4.8] the space $(E, \sigma(E, E'))$ has countable tightness. Hence for any $x \in \overline{B}^{\sigma(E,E')}$ we can select a countable subset $C \subset B$ such that $x \in \overline{C}^{\sigma(E,E')}$. Now Lemma 3.4 applies. □

The following corollary provides a non-metrizable counterpart of [4, Proposition 4.4].

Corollary 3.6. *Let E be the strong dual of a metrizable lcs F with the Heinrich density condition. Then the following assertions are equivalent:*

(i) *Every bounded set is Fréchet–Urysohn in the topology $\sigma(E, E')$.*
(ii) *Every bounded sequence in E contains a weakly Cauchy subsequence.*
(iii) *E does not contain ℓ_1.*

Proof. By assumptions on F, every bounded set in E is metrizable and E is barrelled, see [6, §1 and Theorem 2]. Further, E is a complete (DF)-space, so it belongs also to class $\mathfrak{G}$, see [26, 11.1]. Apply Lemma 3.5 and Proposition 3.3. □

Proof of Theorem 1.5. (i) follows from Theorem 4.5(ii) below.

(ii) Assume E does not contain ℓ_1 and E is a weakly $\aleph_0$-space. Since E is metrizable, it is quasibarrelled and has trivially a $\mathfrak{G}$-base; so E is in class $\mathfrak{G}$. Proposition 3.3 implies that E has the Rosenthal property (R_1). So every bounded set in E is Fréchet–Urysohn in $\sigma(E, E')$ by Lemma 3.5. Finally Proposition 1.4 yields that E' is trans-separable. □

Remark 3.7. The same conclusion as in Theorem 1.5 holds for E being the strong dual of a metrizable lcs with the Heinrich density condition. Indeed, Corollary 3.6 enables to apply the same argument as above.

4. Weakly and weakly* cosmic and $\aleph_0$-spaces

Let $\mathbb{P} := \mathbb{N}^{\mathbb{N}}$. Following [12] we say that a topological space X is (*strongly*) $\mathbb{P}$-*directed* (in [39] called (strongly) dominated by irrationals) if X has a compact resolution covering X (and swallowing compact sets of X), i.e., there exists a $\mathbb{P}$-increasing compact cover $\{K_\alpha : \alpha \in \mathbb{P}\}$ of X (and every compact set is contained in some K_α).

Let $\mathcal{K}(\mathbb{P})$ be the family of all compact subsets of $\mathbb{P}$. A space X is said to be *(strongly)* $\mathbb{P}$-*dominated* if there exists a family $\mathcal{F} := \{F_K : K \in \mathcal{K}(\mathbb{P})\}$ of compact sets covering X such that $F_K \subset F_L$ if $K \subset L$ (and every compact set M in X is contained in some F_K), where $K, L \in \mathcal{K}(\mathbb{P})$. If the same holds when $\mathbb{P}$ is replaced by a Polish space, or a second countable space, we say that X is *(strongly) dominated* by a Polish space, or a second countable space.

Note the following easy fact; its proof is the same as for [12, Proposition 2.2].

Lemma 4.1. *The following conditions are equivalent for a topological space X: (i) X has a compact resolution swallowing compact sets; (ii) X is strongly $\mathbb{P}$-dominated; (iii) X is strongly dominated by a Polish space.*

Second part of Theorem 4.2 follows from the first one and Lemma 4.1. Recall that a topological space X is *submetrizable* if it admits a weaker metric topology.

Theorem 4.2. *([12, Theorem 3.6]) A submetrizable space X is an $\aleph_0$-space if and only if X is strongly dominated by a second countable space. Consequently, X is an $\aleph_0$-space if X has a compact resolution swallowing compact sets.*

The submetrizabilty of X cannot be removed. Indeed, consider the locally compact space $X = [0, \omega_1)$. Under the assumption $\omega_1 = \mathfrak{b}$, the space X has a compact resolution swallowing compact sets (see [16]). Every compact set in X, being countable, is metrizable. As X is not separable, it is not an $\aleph_0$-space.

Note that each Polish space has a compact resolution swallowing compact sets (see [8]). Analogously, every metrizable topological vector space E has a bounded resolution swallowing bounded sets. Indeed, if $(U_n)_n$ is a decreasing base of neighborhoods of zero in E, then the family $\{B_\alpha : \alpha \in \mathbb{P}\}$, where $B_\alpha := \bigcap_k \alpha_k U_k$ for $\alpha = (\alpha_k) \in \mathbb{P}$, is as required.

Part (ii) of the next proposition is a substantial extension of Michael's [30, Corollary 7.10].

Proposition 4.3. *(i) Let E be a separable lcs in class $\mathfrak{G}$. Then E is a weakly $\aleph_0$-space if and only if $(E, \sigma(E, E'))$ is strongly dominated by a second countable space.*

(ii) A (barrelled) lcs E in class $\mathfrak{G}$ is separable if and only $(E', \sigma(E', E))$ is cosmic (an $\aleph_0$-space).

Proof. (i): If E is a separable lcs in class $\mathfrak{G}$, then its weak*-dual is separable [9, Theorem 14], so $(E, \sigma(E, E'))$ is submetrizable. Now we apply Theorem 4.2.

(ii): Let $\{A_\alpha : \alpha \in \mathbb{P}\}$ be a $\mathfrak{G}$-representation for E. By definition each set A_α is $\sigma(E', E)$-relatively countably compact. Assume that E is separable. Then, the space $E'_\sigma := (E', \sigma(E', E))$ admits a weaker metrizable topology. Therefore each set A_α is $\sigma(E',E)$-relatively compact. Now [9, Theorem 15] implies that E'_σ is analytic, i.e. a continuous image of $\mathbb{P}$, so E'_σ is cosmic. Conversely, if E'_σ is cosmic, $(E,\sigma(E,E'))$ is cosmic (see Theorem 4.5(i)), so E is separable.

Now assume that E is barrelled and separable in class $\mathfrak{G}$. By the remark before Lemma 3.4, E admits a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{P}\}$. For $\alpha \in \mathbb{P}$, let $U_\alpha^\circ := \{f \in E' : |f(x)| \leqslant 1, x \in U_\alpha\}$ be the polar of U_α. Then the family $\mathcal{F} := \{U_\alpha^\circ : \alpha \in \mathbb{P}\}$ is a compact resolution in E'_σ. If K is a compact subset in E'_σ, then K is equicontinuous by [37, IV.5.2], so $K \subset U_\alpha^\circ$ for some $\alpha \in \mathbb{P}$. Hence $\mathcal{F}$ is a compact resolution in E'_σ swallowing compact sets of E'_σ. Now Theorem 4.2 applies. □

Question 4.4. Does there exist a quasibarrelled separable lcs E in class $\mathfrak{G}$ whose weak*-dual is not an $\aleph_0$-space?

Recall also that E is a (*strict*) (*LF*)-*space* if E is the (strict) inductive limit of an increasing sequence $(E_n)_n$ of Fréchet spaces. We refer to [5] for concrete classes of (reflexive, strict, regular, etc.) (*LF*)-spaces which applies to Theorem 4.5 below.

Theorem 4.5. *Let E be a lcs. Then the following statements hold.*

(i) *$(E,\sigma(E,E'))$ is cosmic if and only if $(E',\sigma(E',E))$ is cosmic.*
(ii) *If E is metrizable such that the strong dual $(E',\beta(E',E))$ of E is separable, then E is a weakly $\aleph_0$-space.*
(iii) *If E is separable and metrizable, then $(E',\sigma(E',E))$ is a cosmic space. If additionally E is barrelled, then $(E',\sigma(E',E))$ is an $\aleph_0$-space.*
(iv) *If E is a (DF)-space whose strong dual is separable, then E is a weakly $\aleph_0$-space.*
(v) *If E is a separable (LF)-space, then $(E',\sigma(E',E))$ is an $\aleph_0$-space. Moreover, if E is reflexive, the same holds for $(E,\sigma(E,E'))$.*
(vi) *If E is a strict (LF)-space such that the strong dual of E is separable, then E is a weakly $\aleph_0$-space.*

Proof. (i) follows from $(E,\sigma(E,E')) \subset C_p(E',\sigma(E',E))$, $(E',\sigma(E',E)) \subset C_p(E,\sigma(E,E'))$, and [30, Proposition 10.5].

(ii): Let E be a metrizable lcs such that the strong dual space $E'_\beta := (E',\beta(E',E))$ of E is separable. Then E'_β is a complete (DF)-space by [34, 8.3.9]. As E'_β is separable, it is quasibarrelled, hence barrelled by [34, 8.3.13, 8.3.44].

Let $(U_n)_n$ be a decreasing base of absolutely convex neighborhoods of zero in E. For each $\alpha = (\alpha_k)_{k\in\mathbb{N}} \in \mathbb{P}$, set $U_\alpha := \bigcap_k \alpha_k U_k$. Then the family $\{U_\alpha : \alpha \in \mathbb{P}\}$ is a bounded resolution swallowing bounded sets in E. Therefore the polars U_α° of the sets U_α form a $\mathfrak{G}$-base in E'_β. We apply Proposition 4.3(ii) to conclude that the space $(E'',\sigma(E'',E'))$ is an $\aleph_0$-space. Finally, E is a weakly $\aleph_0$-space.

(iii) follows from Proposition 4.3(ii).

(iv): Let E be a (DF)-space. Then E'_β is a metrizable and complete lcs by [34, 8.3.7]. Hence, E'_β is barrelled [34, 8.3.13, 8.3.44] and trivially has a $\mathfrak{G}$-base. Again Proposition 4.3(ii) applies to deduce that $(E'',\sigma(E'',E'))$ is an $\aleph_0$-space. Thus E is a weakly $\aleph_0$-space.

(v): Let E be the inductive limit of a sequence (E_n) of Fréchet spaces. We claim that E has a $\mathfrak{G}$-base. Indeed, if $(U_n^k)_n$ is a decreasing basis of neighborhoods of zero in E_k for each $k \in \mathbb{N}$, then the sets of the form $U_\alpha := \bigcup_{k\in\mathbb{N}} \left(U^1_{\alpha_1} + U^2_{\alpha_2} + \cdots + U^k_{\alpha_k}\right)$, where $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \mathbb{P}$, form a base of neighbourhoods of zero in E. Finally, as E is also barrelled [37, II.7], the space E' is a weakly* $\aleph_0$-space by Proposition 4.3(ii).

Now assume that E is reflexive. Then $(E,\sigma(E,E'))$ is locally complete [25, 11.2.4] and every bounded set in E is relatively $\sigma(E,E')$-compact [37, IV.5 Corollary 2]. Since every (LF)-space is a quasi-(LB)-space in sense of Valdivia (see [26, 3.3 Example 1]), we apply Valdivia's [26, Theorem 3.5] to derive that

$E_\sigma := (E, \sigma(E, E'))$ has a compact resolution swallowing compact sets. So E_σ is separable in class $\mathfrak{G}$. Now Proposition 4.3(i) applies.

(vi): Let E be the strict inductive limit of a sequence (E_n) of Fréchet spaces and the strong dual E'_β of E be separable. For each $n \in \mathbb{N}$, the strong dual $(E'_n)_\beta$ of E_n is a (DF)-space. Since E is the strict inductive limit, the space E'_β is linearly homeomorphic with the *projective limit* of the sequence $((E'_n)_\beta)_n$ of complete (DF)-spaces, see [24]. Moreover, as E'_β can be continuously mapped onto each $(E'_n)_\beta$, each (DF)-space $(E'_n)_\beta$ is separable. Then any E_n is a weakly $\aleph_0$-space by the case (ii). By Michael's theorem [30], for each $n \in \mathbb{N}$ there exists a metrizable and separable space X_n and a continuous compact covering map from X_n onto $E_n(\sigma_n) := (E_n, \sigma(E_n, E'_n))$. Since $\sigma(E_n, E'_n) = \sigma(E, E')|E_n$, and every compact set in $(E, \sigma(E, E'))$ is contained in some E_n (note that the inductive limit is strict), the composition of the induced maps $\bigoplus_n X_n \to \bigoplus_n E_n(\sigma_n) \to \bigcup_n E_n$, where the latest space is endowed with the topology $\sigma(E, E')$, is a continuous compact covering map. This proves that E is a weakly $\aleph_0$-space. □

It is well known that, if $\Omega \subset \mathbb{R}^n$ is an open set, then the space of test functions $D(\Omega)$ is a complete separable Montel strict (LF)-space. So its strong dual, the space of distributions $D'(\Omega)$, is a complete ultrabornological (hence barrelled) non-metrizable space (see [24]). Hence, by reflexivity and Theorem 4.5(v) we note the following corollary (which completes the corresponding part of [3, Corollary 11.14] for $D'(\Omega)$).

Corollary 4.6. *The space of distributions $D'(\Omega)$ is a weakly $\aleph_0$-space.*

We know that if X is compact, $C_c(X)$ is a weakly $\aleph_0$-space if and only if X is countable, [30, Proposition 10.8]. However, $C_c(X)$ is weakly cosmic for every Polish space X.

Proposition 4.7. *Let X be a Čech-complete space. The following assertions are equivalent:*

(i) *X is Polish.*
(ii) *$C_c(X)$ is an $\aleph_0$-space.*
(iii) *$C_c(X)$ is a cosmic space.*
(iv) *$C_c(X)$ is a weakly cosmic space and X is Lindelöf.*
(v) *$C_c(X)$ is separable and X is Lindelöf.*
(vi) *The weak*-dual space of $C_c(X)$ is an $\aleph_0$-space and X is Lindelöf.*
(vii) *The weak*-dual space of $C_c(X)$ is a cosmic space and X is Lindelöf.*
(viii) *$C_c(X)$ is hereditarily separable.*

Proof. Set $E := C_c(X)$, $E_\sigma := (E, \sigma(E, E'))$ and $E'_\sigma := (E', \sigma(E', E))$. Note also that Čech-complete spaces are completely regular. The implications (i) ⇒ (ii) and (ii)⇔(iii) follow from [30, (A) and 10.3].

(ii) ⇒ (i): By [30, 10.3], the space X is an $\aleph_0$-space. Hence X is second countable by [15, 3.9.E(c)]. Thus X is a separable metrizable space. Being Čech-complete, the space X is Polish by [15, Theorem 4.3.26].

(iii) ⇒ (iv): By [30, 10.2], E_σ is cosmic. X is Lindelöf by [30, (D) and 10.3].

(iv) ⇒ (v): As any cosmic space is separable, E_σ is separable, so is E, as well.

In what follows we need the following two general facts.

Fact 1: (See [21].) Any Čech-complete Lindelöf space X has a compact resolution swallowing compact sets. Hence $E = C_c(X)$ has a $\mathfrak{G}$-base. Indeed, it is well known that X is a Čech-complete Lindelöf space if and only if it is a pre-image of a Polish space under a perfect surjective map, see [22, Corollary 3.7] and [15, Theorem 3.9.10].

A direct proof: There exists a sequence $(O^m)_m$ of open sets in βX such that $\bigcap_m O^m = X$. Since X is regular and Lindelöf, for each $m \in \mathbb{N}$ there exists an increasing covering $(O^m_n)_n$ of X such that $\bigcup_n \overline{O^m_n} \subset O^m$, where the closures are taken in βX. If K is a compact set in X, there exists $\alpha = (n_m) \in \mathbb{P}$ with $K \subset O^m_{n_m}$, consequently $K \subset K_\alpha$ and $K_\alpha := \bigcap_m \overline{O^m_{n_m}}$ is βX-compact. Hence, $\{K_\alpha : \alpha \in \mathbb{P}\}$ is as required. Finally, the space E has a $\mathfrak{G}$-base by [16, Theorem 2].

Fact 2: If X is Lindelöf it is realcompact [15, 3.11.12], and hence the space $E = C_c(X)$ is barrelled by [34, 10.1.12].

(v) ⇒ (vi): By Facts 1, 2 and Proposition 4.3(ii) E'_σ is an $\aleph_0$-space.

(vi) ⇒ (vii) is clear.

(vii) ⇒ (v) follows from Facts 1 and 2 and Proposition 4.3(ii).

(v) ⇒ (ii): Since $E = C_c(X)$ is separable, X admits a weaker separable metric topology [29, 4.4.2]. Now Fact 1 and Theorem 4.2 imply that the space X is an $\aleph_0$-space. Hence E is an $\aleph_0$-space by [30, 10.3].

(ii) ⇒ (viii): If E is an $\aleph_0$-space, then E is hereditarily separable [30].

(viii) ⇒ (v): Assume E is hereditarily separable. Then E has countable tightness, so X is Lindelöf. □

This proposition combined with Valov's [40, Corollary 4.5] extends Pelant's result, see [2, Theorem 3.27].

Corollary 4.8. *Let X be a Polish space and Y be a regular space. If there exists a continuous linear surjection from $C_c(X)$ onto $C_p(Y)$, then every closed first countable subspace Z of Y is Polish.*

Proof. Since $C_c(X)$ is cosmic, $C_p(Y)$ is cosmic as well. Hence Y is cosmic, [30, Proposition 10.5], so Y is Lindelöf. By Valov's [40, Corollary 4.5], the cosmic space Z is Čech-complete. Hence Z is second countable [15, 3.9E(c)], so metrizable. Consequently Z is Polish. □

Theorem 1.6 and Theorem 1.5 may suggest the question whether the *trans-separability* of the strong dual E' of E can be replaced by *separability* for any Fréchet lcs E. We propose only the following

Proposition 4.9. ($MA + \neg CH$) *Let E be a quasibarrelled lcs in class $\mathfrak{G}$ which is trans-separable. Then $(E, \sigma(E, E'))$ is cosmic. In particular, E is separable.*

Proof. The completion of a lcs in class $\mathfrak{G}$ is still in class $\mathfrak{G}$, and the completion of a quasibarrelled lcs is barrelled. As a subset of a cosmic space is also cosmic, so we may assume that E is a (complete) barrelled lcs in class $\mathfrak{G}$. Since every quasibarrelled lcs in class $\mathfrak{G}$ has a $\mathfrak{G}$-base by [26, Lemma 15.2], there exists a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ in E. For each $\alpha \in \mathbb{N}^\mathbb{N}$, let K_α be the polar of U_α equipped with the topology $\sigma(E', E)$. Then $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution of $(E', \sigma(E', E))$. As E is barrelled, every $\sigma(E', E)$ compact set K in E' is equicontinuous, so K is contained in K_α for some $\alpha \in \mathbb{P}$. Therefore $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows compact sets. By assumption E is trans-separable, so every K_α is $\sigma(E', E)$-metrizable by a result of Pfister, see [26, Proposition 6.8]. Consequently, every $\sigma(E', E)$-compact set is metrizable and $(E', \sigma(E', E))$ is K-analytic by [26, Theorem 12.2, Theorem 12.3]. By $MA + \neg CH$ the space $(E', \sigma(E', E))$ is analytic (Fremlin [35, Theorem 5.5.3]), hence $(E', \sigma(E', E))$ is submetrizable by Talagrand's [26, Proposition 6.3]. Finally, we apply Theorem 4.2 to derive that $(E', \sigma(E', E))$ is an $\aleph_0$-space. Hence, $(E, \sigma(E, E'))$ is cosmic by Theorem 4.5(i). □

The following example motivates also Proposition 4.9.

Example 4.10. Let $X := [0, \omega_1)$. The space $C_c(X)$ is a non-separable but trans-separable space which is not quasibarrelled. Assuming (CH) the space $C_c(X)$ is in class $\mathfrak{G}$. Under ($MA + \neg CH$) the space $C_c(X)$ is not in class $\mathfrak{G}$.

Proof. Clearly $C_c(X)$ is not separable, since $C_p(X)$ is not separable. Moreover, as Morris and Wulbert observed, $C_c(X)$ is not quasibarrelled [32]. As every compact set in X is metrizable, $C_c(X)$ is trans-separable (by Schmets [26, Lemma 6.5]). Under (CH) the space X has a compact resolution swallowing compact sets by [39, Theorem 3.6]. Hence $C_c(X)$ is in class $\mathfrak{G}$ by [16]. The space $C_c(X)$ is not in class $\mathfrak{G}$ if we assume $(MA + \neg CH)$, since by mentioned [39, Theorem 3.6] the space X even does not have a compact resolution, so by the same reason as above (use again [16]) the space $C_c(X)$ is not in class $\mathfrak{G}$. □

To prove Theorem 1.6 we need the following proposition (which provides another fact, more general then discussed in Köthe's [28, Proposition 28.5 (3)]).

Proposition 4.11. *Let (E, ν) be a lcs and $\mathcal{N}$ be the uniformity on E generated by the locally convex structure of E. Let $A \subset E$ be an absolutely convex bounded subset of E such that the set $(4A, \nu|_{4A})$ is metrizable. Then there exists a metric d on $4A$ such that*

(i) *$d(x - y, 0) = d(x, y)$ for all $x, y \in 4A$ with $x - y \in 4A$,*
(ii) *the topology generated by d on $2A$ coincides with $\nu|_{2A}$,*
(iii) *the uniformity $\mathcal{M}$ on $4A$ generated by the metric d and $\mathcal{N}$ coincide on A.*

Proof. Set $P := 4A$. Since P is metrizable, P has a decreasing basis $\{U_m\}_m$ of absolutely convex neighbourhoods of zero such that $2U_{m+1} \subset U_m$ for every $m \in \mathbb{N}$, see [25, 9.2.4] or [7, Corollary 3 (proof)]. Note that each U_m is absorbing in P. We show that, for every $x \in 2A$, the sequence $\{(x + U_m) \cap 2A\}_m$ is a basis of neighbourhoods of x. Indeed, for every U_m in P choose an absolutely convex neighbourhood $V \subset E$ of zero with $V \cap P \subset U_m$. Then $(x+V) \cap 2A \subset (x+U_m) \cap 2A$. Conversely, for an absolutely convex neighbourhood of zero W in E we have $U_p \subset W \cap P$ for some $p \in \mathbb{N}$, so $(x + U_p) \cap 2A \subset (x + W) \cap 2A$.

If p_m denotes the gauge of U_m, we define

$$d(x, y) := \sum_m 2^{-m} \min\{2p_m(2^{-1}x - 2^{-1}y), 1\}$$

So $d(x, y)$ is a metric on P satisfying the condition $d(x - y, 0) = d(x, y)$ for all $x, y \in P$ with $x - y \in P$. This proves (i).

To prove (ii), fix $x \in 2A$. If $x \in U_m$, then $d(x, 0) < 2^{-m}$. If $x \notin U_{m-2}$ $(m > 2)$, then $x/2 \notin U_{m-1}$ and $d(x, 0) \geq 2^{-(m-1)}$. Thus

$$U_m \cap 2A \subseteq H_m := \{x \in 2A : d(x, 0) < 2^{-m}\} \subseteq U_{m-2} \cap 2A.$$

Finally note that if $x, y \in 2A$, then $y \in (x + H_m)$ if and only if $d(x, y) < 2^{-m}$. This implies that d induces the relative topology on $2A$ inherited from E.

Now we prove (iii). As $\mathcal{N}$ is the uniformity on E generated by its locally convex structure, the uniform topology $\tau_{\mathcal{N}}$ generated by $\mathcal{N}$ coincides with the locally convex topology of E. If ξ is the topology on $2A$ induced by the metric d, we have $\tau_{\mathcal{N}}|_{2A} = \xi$ by (ii). Let U be an absolutely convex neighbourhood of zero in E. There exists $\epsilon > 0$ such that $M_\epsilon(0) \subset U \cap 2A$, where $M_\epsilon(0) := \{y \in 2A : d(y, 0) < \epsilon\}$. A base for the uniformity induced by the metric d on A is given by sets $M_\epsilon := \{(x, y) \in A \times A : d(x, y) < \epsilon\}$. If $(x, y) \in M_\epsilon$, then (i) implies $x - y \in M_\epsilon(0) \subset U \cap 2A$. Hence $(x, y) \in N_U \cap (A \times A)$, where

$$N_U := \{(x, y) \in E \times E : x - y \in U\}.$$

Conversely, if $\delta > 0$, there exists an absolutely convex neighbourhood of zero V in E such that $V \cap 2A \subset M_\delta(0)$. Hence, if $(x, y) \in N_V$ with $x, y \in A$, then $x - y \in V \cap 2A$, so $x - y \in M_\delta(0)$ and $d(x, y) < \delta$. This proves that $N_V \cap (A \times A) \subset M_\delta \cap (A \times A)$. □

Corollary 4.12. *Let (E, ν) be a lcs having a sequence $\{Q_n\}_{n \in \mathbb{N}}$ of absolutely convex bounded sets covering E such that $(Q_n, \nu|_{Q_n})$ is metrizable for every $n \in \mathbb{N}$. Then E is trans-separable if and only if E is separable.*

Proof. Applying Proposition 4.11 to $A = Q_n$, $n \in \mathbb{N}$, we obtain that the trans-separable uniformity on Q_n is metrizable, so Q_n is separable. Thus $E = \bigcup_n Q_n$ is separable. The converse is trivial. □

Proof of Theorem 1.6. Clearly the strong dual E' is a (DF)-space with a fundamental sequence $(Q_n)_n$ of absolutely convex bounded subsets of E'. Since E satisfies the density condition, every bounded set Q_n is metrizable by [7, Corollary 3].

Assume that E is a weakly $\aleph_0$-space. By Theorem 1.5 the strong dual E' is trans-separable. Now Corollary 4.12 implies that E is separable. Conversely, if E' is separable, we apply Theorem 4.5(ii) to complete the proof. □

We end with the following question.

Question 4.13. Is there a weakly $\aleph_0$ Fréchet lcs E not containing ℓ_1 whose strong dual E' is not separable?

References

[1] A.V. Arkhangel'skii, The frequency spectrum of a topological space and the classification of spaces, Dokl. Akad. Nauk SSSR 206 (1972) 1185–1189.
[2] A.V. Arkhangel'skii, General Topology III, Encyclopedia of Math. Sciences, vol. 51, Springer, 1991.
[3] T.O. Banakh, V.I. Bogachev, A.V. Kolesnikov, k^*-metrizable spaces and their applications, J. Math. Sci. (N.Y.) 155 (2008) 475–522.
[4] C.S. Barroso, O.F.K. Kalenda, P.K. Lin, On the approximate fixed point property in abstract spaces, Math. Z. 271 (2012) 1271–1285.
[5] K.D. Bierstedt, An introduction to locally convex inductive limits, in: Functional Analysis and Its Applications, Nice, 1986, World Sci. Publ., Singapore, 1988, pp. 33–135.
[6] K.D. Bierstedt, J. Bonet, Density conditions in Fréchet and (DF)-spaces, Rev. Mat. Complut. 2 (1989) 59–75.
[7] K.D. Bierstedt, J. Bonet, Some aspects of the modern theory of Fréchet spaces, RACSAM Rev. R. Acad. Cienc. Ser. A Mat. 97 (2003) 159–188.
[8] J.P.R. Christensen, Topology and Borel Structure, North-Holland Mathematics Studies, vol. 10, North-Holland, Amsterdam, 1974.
[9] B. Cascales, J. Orihuela, On compactness in locally convex spaces, Math. Z. 195 (1987) 365–381.
[10] B. Cascales, J. Orihuela, A biased view of topology as a tool in functional analysis, in: Recent Progress in General Topology III, Atlantis Press, 2014, pp. 93–165.
[11] B. Cascales, J. Kąkol, S.A. Saxon, Metrizability vs. Fréchet–Urysohn property, Proc. Am. Math. Soc. 131 (2003) 3623–3631.
[12] B. Cascales, J. Orihuela, V. Tkachuk, Domination by second countable spaces and Lindelöf Σ-property, Topol. Appl. 158 (2011) 204–214.
[13] M.J. Chasco, E. Martín-Peinador, V. Tarieladze, A class of angelic sequential non-Fréchet–Urysohn topological groups, Topol. Appl. 154 (2007) 741–748.
[14] J. Diestel, Sequences and Series in Banach Spaces, Springer, New York, 1984.
[15] R. Engelking, General Topology, Heldermann, Berlin, 1978.
[16] J.C. Ferrando, J. Kąkol, On precompact sets in spaces $C_c(X)$, Georgian Math. J. 20 (2013) 247–254.
[17] J.C. Ferrando, J. Kąkol, M. Lopez-Pellicer, A characterization of trans-separable spaces, Bull. Belg. Math. Soc. Simon Stevin 14 (2007) 493–498.
[18] K. Floret, Weakly Compact Sets, Lecture Notes in Math., vol. 801, Springer, Berlin, 1980.
[19] L. Foged, Characterization of ℵ-spaces, Pac. J. Math. 110 (1984) 59–63.
[20] S. Gabriyelyan, J. Kąkol, A. Leiderman, On topological groups with a small base and metrizability, Fundam. Math. 229 (2015) 129–158.
[21] S. Gabriyelyan, J. Kąkol, A. Leiderman, The strong Pytkeev property for topological groups and topological vector spaces, Monatshefte Math. 175 (2014) 519–542.
[22] G. Gruenhage, Generalized metric spaces, in: Handbook of Set-Theoretic Topology, North-Holland, New York, 1984, pp. 423–501.
[23] J.A. Guthrie, A characterization of $\aleph_0$-spaces, Appl. Gen. Topol. 1 (1971) 105–110.
[24] J. Horváth, Topological Vector Spaces and Distributions, I, Addison–Wesley, Reading, Mass., 1966.
[25] H. Jarchow, Locally Convex Spaces, B.G. Teubner, Stuttgart, 1981.
[26] J. Kąkol, W. Kubiś, M. Lopez-Pellicer, Descriptive Topology in Selected Topics of Functional Analysis, Developments in Mathematics, vol. 24, Springer, 2011.
[27] O. Kalenda, Spaces not containing l_1 have weak approximate fixed point property, J. Math. Anal. Appl. 373 (2011) 134–137.
[28] G. Köthe, Topological Vector Spaces I, Springer-Verlag, Berlin/Heidelberg, New York, 1983.
[29] R.A. McCoy, I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Notes in Math., vol. 1315, 1988.
[30] E. Michael, $\aleph_0$-spaces, J. Math. Mech. 15 (1966) 983–1002.

[31] E. Michael, A quintuple quotient quest, Gen. Topol. Appl. 2 (1972) 91–138.
[32] P.D. Morris, D.E. Wulbert, Functional representation of topological albebras, Pac. J. Math. 22 (1967) 323–337.
[33] P. O'Meara, A metrization theorem, Math. Nachr. 45 (1970) 69–72.
[34] P. Pérez Carreras, J. Bonet, Barrelled Locally Convex Spaces, North-Holland Mathematics Studies, vol. 131, North-Holland, Amsterdam, 1987.
[35] C.A. Rogers, J.E. Jayne, C. Dellacherie, F. Topsøe, J. Hoffman-Jørgensen, D.A. Martin, A.S. Kechris, A.H. Stone, Analytic Sets, Academic Press, 1980.
[36] W. Ruess, Locally convex spaces not containing ℓ_1, Funct. Approx. 50 (2014) 389–399.
[37] H.H. Schaefer, Topological Vector Spaces, Macmillan, New York, 1966.
[38] M. Tkachenko, Paratopological and semitopological groups vs topological groups, in: Recent Progress in General Topology III, Atlantis Press, 2014, pp. 825–882.
[39] V.V. Tkachuk, A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic, Acta Math. Hung. 107 (2005) 253–265.
[40] V. Valov, Function spaces, Topol. Appl. 81 (1997) 1–22.