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On uniform spaces with a small base and K-analytic $C_c(X)$ spaces ☆

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ABSTRACT

A base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of a uniformity is a $\mathfrak{G}$-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. If X is a completely regular space we show that there exists an admissible uniformity on X with a $\mathfrak{G}$-base that contains the Nachbin uniformity if and only if there exists a resolution of the space $C_c(X)$ of real-valued continuous functions on X equipped with the compact-open topology consisting of equicontinuous sets. This result is applied to show, among other things, that if G is a k_R-space topological group such that $C_c(G)$ is K-analytic then G has a $\mathfrak{G}$-base. In the opposite direction, if G is a topological group with a $\mathfrak{G}$-base and enjoys the so-called property U, then $C_c(G)$ has a resolution consisting of equicontinuous sets.

1. Preliminaries

In what follows, unless otherwise stated, X will be a Hausdorff completely regular space and $C_p(X)$ and $C_c(X)$ will denote the space $C(X)$ of all real-valued continuous functions defined on X provided with the pointwise convergence topology and with the compact-open topology, respectively.

Let us recall that a family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of a set X is called a *resolution* of X if $\bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} = X$ and $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$, [10, Chapter 3]. A topological group G is said to have a $\mathfrak{G}$-base if there is a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods of the identity $\imath$ in G such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. Clearly, every metrizable topological group has a $\mathfrak{G}$-base. Conversely, every Fréchet–Urysohn topological group with a $\mathfrak{G}$-base is metrizable, [9, Theorem 1.2]. A $C_c(X)$ space has a $\mathfrak{G}$-base of (absolutely convex)

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neighborhoods of the origin if and only if X has a compact resolution (a resolution made up of compact sets) that swallows the compact sets, [4, Theorem 2]. Since, by Christensen's theorem [2, Theorem 3.3] (see also [6, Theorem 6.4]), every Polish space has a compact resolution which swallows the compact sets whereas, according to the classic Arens theorem, $C_c(X)$ is metrizable if and only if X is hemicompact, it turns out that if P is a non-hemicompact Polish space then $(C_c(P), +)$ is a non-metrizable Abelian topological group with a $\mathfrak{G}$-base. A topological group G has *property* U provided that each continuous function mapping G into the real line is uniformly continuous, [3].

A space X is called K-*analytic* if there is an upper semi-continuous compact-valued map T from the product space $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}$ is equipped with the discrete topology, into X such that $\bigcup\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$. A space X is *analytic* if X is the continuous image of a Polish space.

A family $\mathcal{F}$ of functions from a uniform space $(X, \mathcal{N})$ into a uniform space $(Y, \mathcal{M})$ is called *uniformly equicontinuous* [11, Chapter 7, Problem G] if for each $V \in \mathcal{M}$ there is $U \in \mathcal{N}$ such that $(f(x), f(y)) \in V$ whenever $f \in \mathcal{F}$ and $(x, y) \in U$. If $\{\mathcal{U}_\lambda : \lambda \in \Lambda\}$ is the family of admissible uniformities for a completely regular space (X, τ), the smallest uniformity $\mathcal{U}_{\lambda_0}$ that makes all τ-continuous functions $f : X \to \mathbb{R}$ uniformly continuous, is called the *Nachbin uniform structure* of X, [13]. Explicitly, the Nachbin uniformity is the admissible uniform structure for X generated by the pseudometrics $\{d_f : f \in C(X)\}$ with

$$d_f(x, y) = |f(x) - f(y)|$$

for every $(x, y) \in X \times X$.

In this paper a general result concerning uniformities (see Theorem 1 below) is applied to topological groups with a small base. This generalize some of the research of [9] to uniformities by showing the interplay between the existence of certain topological groups G with a $\mathfrak{G}$-base and the K-analyticity, or at least the existence of a resolution consisting of equicontinuous sets, of the locally convex space $C_c(G)$. We also show that under CH there exists a topological group with a $\mathfrak{G}$-base and enjoying property U that has such a resolution but is not K-analytic.

2. Main theorem

Let $\mathcal{N}$ be a uniformity on a (nonempty) set X and denote by $\tau_{\mathcal{N}}$ the uniform topology defined by $\mathcal{N}$. We shall say that a base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of the uniformity $\mathcal{N}$ is a $\mathfrak{G}$-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. There is no loss of generality by assuming that each U_α is a symmetric vicinity.

Theorem 1. *The following statements are equivalent.*

(1) *There exists an admissible uniformity for X larger or equal than the Nachbin uniformity with a $\mathfrak{G}$-base.*
(2) *There exists a resolution on $C_c(X)$ consisting of equicontinuous sets.*

Proof. Assume that (1) holds. Let $\mathcal{N}$ denote a uniformity for X which contains the Nachbin uniform structure and let $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base of $\mathcal{N}$. In order to construct the desired resolution of $C_c(X)$ consisting of equicontinuous sets we need to encode in each index $\alpha \in \mathbb{N}^{\mathbb{N}}$ a whole sequence $\{\alpha_n\}_{n=1}^{\infty}$ of elements of $\mathbb{N}^{\mathbb{N}}$. A way to do this is to consider a bidimensional array whose i-th file is formed by the components $(\alpha_i(1), \alpha_i(2), \ldots, \alpha_i(n), \ldots)$ of α_i and define the index α through the short diagonals of the array by setting

$$\alpha(1) = \alpha_1(1), \alpha(2) = \alpha_1(2), \alpha(3) = \alpha_2(1), \alpha(4) = \alpha_1(3),$$
$$\alpha(5) = \alpha_2(2), \alpha(6) = \alpha_3(1), \alpha(7) = \alpha_1(4), \alpha(8) = \alpha_2(3), \ldots$$

and so on. Conversely, we shall also assume that given $\alpha \in \mathbb{N}^{\mathbb{N}}$ we extract a sequence $\{\alpha_n\}_{n=1}^{\infty} \subseteq \mathbb{N}^{\mathbb{N}}$ from α as indicated above. Of course, this defines a one-to-one correspondence between $\mathbb{N}^{\mathbb{N}}$ and $\left(\mathbb{N}^{\mathbb{N}}\right)^{\mathbb{N}}$. Now we define

$$P_\alpha = \left\{ f \in C(X) : \sup_{(x,y)\in U_{\alpha_n}} |f(x) - f(y)| \le \frac{1}{n} \ \forall n \in \mathbb{N} \right\}.$$

Let us show that $\{P_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution of $C_c(X)$ consisting of equicontinuous sets. In fact, if $\beta \ge \alpha$ then clearly $\beta_n \ge \alpha_n$ for every $n \in \mathbb{N}$, so that if $f \in P_\alpha$ then

$$\sup_{(x,y)\in U_{\beta_n}} |f(x) - f(y)| \le \sup_{(x,y)\in U_{\alpha_n}} |f(x) - f(y)| \le \frac{1}{n}$$

for all $n \in \mathbb{N}$, which means that $f \in P_\beta$. Hence $P_\alpha \subseteq P_\beta$. On the other hand, if $f \in C(X)$, since $\mathcal{N}$ is larger than the Nachbin uniformity, f is $\mathcal{N}$-uniformly continuous on X. Hence, bearing in mind that $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base of $\mathcal{N}$, for each $n \in \mathbb{N}$ there exists $\alpha_n \in \mathbb{N}^{\mathbb{N}}$ such that $|f(x) - f(y)| \le 1/n$ whenever $(x,y) \in U_{\alpha_n}$, which shows that $f \in P_\alpha$ for α defined as above. Finally, let us see that each set P_α is equicontinuous. Indeed, given $\epsilon > 0$ take $n \in \mathbb{N}$ such that $1/n < \epsilon$. Then, according to the definition of P_α there is $\alpha_n \in \mathbb{N}^{\mathbb{N}}$, which we extract from α according to the procedure explained above, such that $|f(x) - f(y)| < \epsilon$ whenever $(x,y) \in U_{\alpha_n}$ *and this happens for every* $f \in P_\alpha$, which shows that P_α is uniformly equicontinuous, hence equicontinuous. The proof of (2) is complete.

Let us assume conversely that statement (2) holds. Suppose that $\{P_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution of $C_c(X)$ consisting of equicontinuous sets. Then for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ define

$$V_\alpha = \{(x,y) \in X \times X : \sup_{f\in P_\alpha} |f(x) - f(y)| < \alpha(1)^{-1}\}.$$

If $\beta \ge \alpha$ and $(x,y) \in V_\beta$ then $\sup_{f\in P_\beta} |f(x) - f(y)| < \beta^{-1}(1)$. Since $P_\alpha \subseteq P_\beta$ and $\beta^{-1}(1) \le \alpha^{-1}(1)$, it follows that

$$\sup_{f\in P_\alpha} |f(x) - f(y)| < \alpha(1)^{-1},$$

so that $(x,y) \in V_\alpha$. Hence $V_\beta \subseteq V_\alpha$. Next, let us see that the family $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of $X \times X$ is a base of some uniformity $\mathcal{N}$ for the set X.

First observe that the diagonal $\Delta(X) = \{(x,x) : x \in X\}$ is contained in every V_α, so that no V_α is empty. On the other hand, given $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ choose $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\gamma(i) = \max\{\alpha(i), \beta(i)\}$ for each $i \in \mathbb{N}$. In this case $V_\gamma \subseteq V_\alpha \cap V_\beta$, which shows that $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a filter-base. In addition, it holds obviously that $V_\alpha^{-1} = V_\alpha$, and if $\beta \in \mathbb{N}^{\mathbb{N}}$ satisfies that $\beta \ge \alpha$ with $\beta(1) \ge 2\alpha(1)$ we claim that $V_\beta \circ V_\beta \subseteq V_\alpha$. Indeed, if $(x,y) \in V_\beta \circ V_\beta$ there is $z \in X$ with $(x,z), (z,y) \in V_\beta$. Hence $|f(x) - f(z)| < \beta(1)^{-1}$ and $|f(z) - f(y)| < \beta(1)^{-1}$ for every $f \in P_\beta$. So, if follows that

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)| < 2\beta(1)^{-1} \le \alpha(1)^{-1}$$

for every $f \in P_\alpha \subseteq P_\beta$, which shows that $(x,y) \in V_\alpha$.

Let us check that $\mathcal{N}$ is an admissible uniformity for the space X, i.e. that $\tau_{\mathcal{N}}$ coincides with the original topology of X. In fact, since X is assumed to be completely regular it suffices to show that X and $(X, \tau_{\mathcal{N}})$ have the same continuous functions, that is to say, that $C(X) = C(X, \tau_{\mathcal{N}})$. To achieve this goal take $f \in C(X)$, pick an arbitrary point $x_0 \in X$ and choose $\epsilon > 0$. Then select $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $f \in P_\alpha$ and $\alpha(1)^{-1} < \epsilon$. Clearly

$$V_\alpha(x_0) = \{y \in X : (x_0, y) \in V_\alpha\}$$

is a $\tau_{\mathcal{N}}$-neighborhood of x_0, and since

$$|f(x) - f(y)| < \alpha(1)^{-1} < \epsilon$$

for every $(x, y) \in V_\alpha$, we have in particular that $|f(x_0) - f(y)| < \epsilon$ for all $y \in V_\alpha(x_0)$. This shows that f is continuous at x_0 under the uniform topology $\tau_{\mathcal{N}}$. Assume conversely that $f \in C(X, \tau_{\mathcal{N}})$ and fix $x_0 \in X$ and $\epsilon > 0$. Then there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ with

$$|f(x_0) - f(y)| < \epsilon \tag{2.1}$$

for every $y \in V_\alpha(x_0)$. But, since P_α is equicontinuous at x_0, there exists a neighborhood V of x_0 of the original topology of X such that

$$\sup_{h \in P_\alpha} |h(y) - h(x_0)| < \alpha(1)^{-1}$$

for every $y \in V$. Hence if $x \in V$ then $\sup_{h \in P_\alpha} |h(x) - h(x_0)| < \alpha(1)^{-1}$, which according to the definition of V_α means that $x \in V_\alpha(x_0)$. This shows that $V \subseteq V_\alpha(x_0)$ and consequently (2.1) yields that $|f(x_0) - f(y)| < \epsilon$ for all $y \in V$. Thus we have shown that f is continuous at x_0 under the original topology of X, so that $f \in C(X)$.

Let us finally show that the uniformity $\mathcal{N}$ generated by the base $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is larger than the Nachbin uniformity. We have to prove that every real-valued continuous function on X is $\mathcal{N}$-uniformly continuous. Now, given $f \in C(X)$ and $\epsilon > 0$, taking advantage of the fact that $\{P_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a resolution of $C(X)$, we can choose $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\gamma(1)^{-1} < \epsilon$ and $f \in P_\gamma$. Consequently, for each $(x, y) \in V_\gamma$ it happens that

$$|f(x) - f(y)| < \gamma(1)^{-1} < \epsilon$$

which shows that f is $\mathcal{N}$-uniformly continuous, as stated. This proves the statement (1) and ends the proof of the theorem. □

3. Some consequences

A number of consequences of main theorem are in order.

Corollary 2. *Let X be a k_R-space. If $C_c(X)$ is K-analytic then there exists an admissible uniformity for X, larger or equal than the Nachbin uniformity, with a $\mathfrak{G}$-base.*

Proof. If the space $C_c(X)$ is K-analytic, it has a resolution consisting of compact sets. Since X is a k_R-space, by Ascoli's theorem every compact set in $C_c(X)$ is equicontinuous (see [12, Theorem 5.1]). Consequently, Theorem 1 ensures that there exists an admissible uniformity $\mathcal{N}$ on X, larger or equal than Nachbin's, with a $\mathfrak{G}$-base. □

Corollary 3. *Let $(G, \cdot)$ be a k_R-space topological group. If $C_c(G)$ is K-analytic then G has a $\mathfrak{G}$-base.*

Proof. If $C_c(G)$ is K-analytic, Corollary 2 provides an admissible uniformity $\mathcal{N}$ on G with a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of (symmetric) vicinities. Hence, the family $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$, where $V_\alpha = \{x \in G : (x, \imath) \in U_\alpha\}$ for all $\alpha \in \mathbb{N}^{\mathbb{N}}$, is a $\mathfrak{G}$-base (of neighborhoods of the unit element $\imath$) for the group topology of G, although clearly neither the left uniformity on G nor the right uniformity need to coincide with $\mathcal{N}$. □

Corollary 4. *Let $(G, \cdot)$ be a Fréchet–Urysohn topological group. If $C_c(G)$ is K-analytic then G is metrizable.*

Proof. If $(G, \cdot)$ is a Fréchet–Urysohn topological group, G is a k-space. Hence if $C_c(G)$ is K-analytic, it has a $\mathfrak{G}$-base. But as mentioned in the preliminaries section, each Fréchet–Urysohn topological group with a $\mathfrak{G}$-base is metrizable. □

Corollary 5. *Let $(G, \cdot)$ be a topological group with property U. If G has a $\mathfrak{G}$-base, then $C_c(G)$ has a resolution consisting of equicontinuous sets.*

Proof. If $\{V_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a topological $\mathfrak{G}$-base of G, then $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ with

$$U_\alpha = \{(x, y) \in G \times G : x^{-1}y \in V_\alpha\} \tag{3.1}$$

is a $\mathfrak{G}$-base of the admissible left uniformity $\mathcal{U}_L$ of G. Since G has property U we may assume that every continuous real-valued function on G is $\mathcal{U}_L$-uniformly continuous, [3]. This shows that $\mathcal{U}_L$ is larger than the Nachbin uniform structure for G, so the corollary is a straightforward consequence of Theorem 1. □

Lemma 6. *Let $(X, \mathcal{N})$ be a uniform space. If $(X, \tau_{\mathcal{N}})$ is pseudocompact, then $\mathcal{N}$ contains the Nachbin uniformity.*

Proof. Let $f \in C(X, \tau_{\mathcal{N}})$. We have to show that f is $\mathcal{N}$-uniformly equicontinuous. Since $(X, \tau_{\mathcal{N}})$ is pseudocompact, then $(X, \mathcal{N})$ is precompact, [8, Problem 15Q]. Consequently the completion $(Y, \mathcal{M})$ of $(X, \mathcal{N})$ is a compact uniform space, and hence the uniformity $\mathcal{M}$ is unique. Since $(X, \tau_{\mathcal{N}})$ with $\tau_{\mathcal{N}} = \tau_{\mathcal{M}}|_X$ is pseudocompact and $(Y, \tau_{\mathcal{M}})$ is realcompact, then $\beta X = \upsilon X \subseteq Y$. But since βX and Y have only a unique admissible uniformity, concerning βX this uniformity must be $\mathcal{U} = (Y \times Y) \cap \mathcal{M}$. But then $(\beta X, \mathcal{U})$ is a complete uniform space such that $\mathcal{N} = (X \times X) \cap \mathcal{U}$, which implies that $\beta X = Y$. Finally, given that the Stone–Čech extension f^β of f to βX is $\mathcal{U}$-uniformly continuous by virtue of the compactness of βX, it follows that f is $\mathcal{N}$-uniformly continuous. □

Theorem 7. *Let $(X, \mathcal{N})$ be a uniform pseudocompact space. If $\mathcal{N}$ has a $\mathfrak{G}$-base, then $C_c(X, \tau_{\mathcal{N}})$ is K-analytic.*

Proof. By the previous lemma $\mathcal{N}$ contains the Nachbin uniformity. For the proof we may proceed as in the proof of the implication (1) ⇒ (2) of Theorem 1 adding to the definition of the set P_α the condition $\sup_{x \in X} |f(x)| \leq \alpha(1)$, that is

$$P_\alpha = \left\{ f \in C(X) : \sup_{(x,y) \in U_{\alpha_n}} |f(x) - f(y)| \leq \frac{1}{n} \ \forall n \in \mathbb{N} \wedge \sup_{x \in X} |f(x)| \leq \alpha(1) \right\}$$

As before $P_\alpha \subseteq P_\beta$ whenever $\alpha \leq \beta$ and $\bigcup \left\{P_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\} = C(X)$, so that $\left\{P_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ is a resolution of $C_c(X, \tau_{\mathcal{N}})$ consisting of $\mathcal{N}$-uniformly equicontinuous sets. But now the sets P_α are pointwise bounded (in fact, uniformly bounded), so Ascoli's theorem ensures that each P_α is relatively $\tau_{\mathcal{N}}$-compact. Hence, if K_α stands for the closure of P_α in $C_c(X, \tau_{\mathcal{N}})$, it turns out that the family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution of $C_c(X, \tau_{\mathcal{N}})$. According to [7, Corollary 5], this fact guarantees that the space $C_c(X, \tau_{\mathcal{N}})$ is K-analytic. □

Corollary 8. *Let $(G, \cdot)$ be a pseudocompact topological group. If G has a $\mathfrak{G}$-base, then $C_c(G)$ is K-analytic.*

Proof. If $\mathcal{U}_L$ stands for the admissible left uniformity for G then $(G, \mathcal{U}_L)$ fulfills the hypotheses of Theorem 7, so that $C_c(G)$ must be K-analytic. □

Remark 9. *The space $C_c(G)$ of the previous corollary is even analytic.* According to [3, Theorem 1.1] every pseudocompact topological group is totally bounded and, according to [3, Theorem 1.5], every pseudocompact group has property U. Consequently, every pseudocompact group G is totally bounded and has property U. Since, by [9, Corollary 3.11] every totally bounded topological group with a $\mathfrak{G}$-base is metrizable, it follows that every pseudocompact group G with a $\mathfrak{G}$-base is metrizable and has property U. Consequently, by virtue of [3, Lemma 2.5], G must be compact or discrete. In the first case $C_c(G)$ is clearly analytic. In the second case, G being pseudocompact and discrete, must be finite. So in this latter case $C_c(G) = \mathbb{R}^G$ is also analytic. Hence in both cases $C_p(G)$ is analytic. Note in passing that, according to the classic Calbrix theorem [10, Theorem 9.7], this fact forces G to be σ-compact.

Example 10. *Under CH the equicontinuous resolution of Corollary 5 need not be formed by compact sets.* Let I be an index set of cardinality $\aleph_1$ and assume that G consists of all elements x of the direct product $\{-1,1\}^I$ of $\aleph_1$ copies of the Abelian multiplicative group $G_{\mathfrak{n}} = \{-1,1\}$ such that $x(\mathfrak{n}) = 1$ for all but finitely many coordinates $\mathfrak{n} \in I$. Denoting by ω_1 the first uncountable ordinal, let us identify the index set I with the well ordered ordinal interval $I = [0, \omega_1)$. For each $\mathfrak{n} \in I$, let $H_{\mathfrak{n}} = \{x \in G : x(\mathfrak{m}) = 1 \ \forall \mathfrak{m} < \mathfrak{n}\}$ and observe that $H_{\mathfrak{m}} \subseteq H_{\mathfrak{n}}$ whenever $\mathfrak{n} \leq \mathfrak{m}$. The system $\{H_{\mathfrak{n}} : \mathfrak{n} \in I\}$ is a base of neighborhoods of the identity $\imath = (1,1,1,\ldots)$ of a group topology τ on G. In [3, Example 3.2] is shown that under this topology G is a nondiscrete P-space enjoying property U. Let us prove that under CH the group G has a $\mathfrak{G}$-base. In fact, under this assumption, is known that the ordinal interval $[0, \omega_1)$ has a resolution $\{I_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ consisting of compact sets. If $\kappa_\alpha = \max I_\alpha$ define $U_\alpha = H_{\kappa_\alpha}$ and consider the family $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$. Observe in first place that if $\alpha \leq \beta$ then $I_\alpha \subseteq I_\beta$ and consequently $\kappa_\alpha \leq \kappa_\beta$, which implies that

$$U_\beta = H_{\kappa_\beta} \subseteq H_{\kappa_\alpha} = U_\alpha.$$

On the other hand, given $\mathfrak{n} \in I$ choose $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $\mathfrak{n} \in I_\gamma$. Consequently $\mathfrak{n} \leq \kappa_\gamma = \max I_\gamma$ and therefore

$$U_\alpha = H_{\kappa_\alpha} \subseteq H_{\mathfrak{n}},$$

which means that $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base of the group topology of G. According to Corollary 5 the space $C_c(G)$ has a resolution consisting of equicontinuous sets. Let us show next that $C_c(G)$ is not K-analytic. Otherwise $C_p(G)$ would be a Lindelöf space. But due to the fact that G is a P-space, [5, Theorem 1 (3)] assures that every bounded set in $C_p(G)$ is relatively countable compact; particularly, every pointwise closed bounded set of $C(X)$ is countably compact. So in case that $C_c(G)$ were K-analytic, every closed bounded set in $C_p(G)$ would be compact, hence complete. But this ensures that $C_p(G)$ is quasicomplete, which forces G to be discrete (see [1]), a contradiction. So we must conclude that $C_c(G)$ has a non-compact resolution $\mathcal{M}$ consisting of pointwise closed equicontinuous sets. Furthermore, it can be shown that G is even a Lindelöf space [3], which is known to imply that $C_p(G)$ is a Fréchet–Urysohn space. Since the pointwise topology coincides with the compact-open topology on the equicontinuous sets, it follows that the closed and equicontinuous sets of the resolution $\mathcal{M}$ of $C_c(G)$ are even Fréchet–Urysohn. Clearly G is not a k-space, since otherwise each set of $\mathcal{M}$ would be compact. Of course, neither G is pseudocompact, since it is infinite.

Remark 11. In [9, Remark 2.2] the notion of (local) G-base is extended to that of (local) I-base, i.e. a (local) base of neighborhoods of a topological space indexed by a partially ordered set I. On the other hand, in [9, Remark 3.2] the notion of (compact) resolution is generalized to that of an I-increasing family of sets. This allows to extend the main theorem of [4] to those classes of families of sets (cf. [9, Theorem 4.8]). We mention here the possibility that Theorem 1 could be generalized in such a way.

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