Angelicity and compactness in spaces $C(X)$

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Introduction. In the theory of lcs E two essential questions may arise:

1. **(weak) angelicity of E.**
2. **metrizability of (weakly) compact sets.**

Question (1) refers to a very useful concept of angelic spaces (introduced by **Fremlin**) for which several variants of compactness coincide. A number of results, including a remarkable and applicable **Orihuela's** theorem for spaces $C_p(X)$ over web-compact spaces X (providing angelicity) will be presented.

How to apply angelicity?.....

Positive answers for (2) were covered by results of

1. **Pfister** ((DF)-spaces),
2. **Valdivia** (dual metric spaces),
3. **Cascales-Orihuela** ((LF)-spaces and class $\mathfrak{G}$).

We characterize analytic sets in $C_p(X)$; this leads to generalizations of the earlier results about part (2). The concepts of **Eberlein, Talagrand, Gul'ko** and **Corson** compact sets will be discussed in the context of (1) and (2).

$X, Y, \ldots$- **completely regular Hausdorff spaces.** $E, F \ldots$- **locally convex spaces.** $C_p(X)$ **(resp** $C_c(X)$**) the space of all continuous real functions on** X **with the pointwise (compact-open) topology.**

A bit of the history.

1. **Šmulian** (and **Phillips**) proved that any relatively compact set is relatively sequentially compact for the weak topology of a Banach space and the both concepts coincide if the weak-*-topology is separable.
2. The latter fact extended by **Dieudonne** and **Schwartz** for submetrizable lcs. The converse to **Šmulian** theorem was discovered by **Eberlein**.
3. **Grothendieck** proved the above for $C_p(X)$, compact X.
4. This line of research (results) continued by **Fremlin, Pryce, De Wilde** allowed **Floret** to present a general version of the **Eberlein-Šmulian** theorem.

Nevertheless, theorems said **nothing about metrizability of compact sets.**

Angelic spaces, Fremlin's angelic lemma, Orihuela's theorem for spaces $C_p(X)$

Theorem 1 (Fremlin)

Let X and Y be topological spaces, where X regular, and let $\Phi : X \to Y$ be an injective and continuous map. If $A \subset X$ is relatively countably compact, and if for each $B \subset \Phi(A)$ and $y \in \overline{B}$ there exists a sequence $(y_n)_n$ in B converging to y, then $\Phi(\overline{A})$ is closed and $\Phi|\overline{A}$ is a homeomorphism.

X is called *angelic* if every relatively countably compact set K in X is relatively compact, and for every $x \in \overline{K}$ there exists a sequence in K converging to x **[Fremlin]**.

1. If X is an angelic space, then X endowed with any stronger regular topology is also angelic.

In angelic spaces the (relatively) countably compact, (relatively) compact, (relatively) sequentially compact sets are the same. **Classical examples**: spaces $C_p(X)$ with compact X, Banach spaces E with the weak topology.

Corollary 2 (Šmulian)

Let E be a lcs such that $(E', \sigma(E', E))$ is separable. Then for the space $(E, \sigma(E, E'))$ the following conditions hold.

1. *(relatively) compact, (relatively) countably compact, (relatively) sequentially compact sets are the same.*
2. *If A is relatively (countably) compact and $x \in \overline{A}^{\sigma}$, there exists a sequence in A converging to x.*
3. *Every relatively countably compact set that is sequentially closed is compact.*

The following concept **[Orihuela]** leads to generalizations of **Grothendieck, Fremlin, Pryce, De Wilde, Floret** results.

1. X is ***web-compact*** if there exists a non-empty subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of X such that, if $C_{n_1,n_2,\dots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \Sigma, m_j = n_j, j = 1, 2, \dots, k\}$ for any $\alpha = (n_k) \in \Sigma$, the following hold: **(i)** $\overline{\bigcup\{A_\alpha : \alpha \in \Sigma\}} = X$. **(ii)** If $\alpha = (n_k) \in \Sigma$ and $x_k \in C_{n_1,n_2,\dots,n_k}$ for all $k \in \mathbb{N}$, then $(x_k)_k$ has a cluster point in X.
2. Separable spaces are web-compact.
3. Spaces containing dense **Lindelöf Σ-spaces** (particularly **K-analytic spaces**) and also quasi-Suslin spaces are web-compact.

The weak*-dual of "almost all" important classes of spaces in functional analysis are web-compact (below + applications)!

X is ***K-analytic*** if it is the image under an upper semi-continuous compact-valued map T defined on $\mathbb{N}^{\mathbb{N}}$. If T is defined on $\emptyset \neq \Omega \subset \mathbb{N}^{\mathbb{N}}$, X is called a ***Lindelöf*** Σ***-space***. X is ***analytic*** if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$.

1. Above spaces are very applicable! Closed with resp. to countable operations, etc.....
2. If $C_p(X)$ is angelic, $C_p(X, Z)$ is angelic for any metric space Z [**Fremlin**].
3. If $X = \overline{\bigcup_n K_n}$, K_n are relatively countably compact, then $C_p(X)$ is angelic [**Eberlein-Šmulian-Floret**].

The next covers results of **Grothendieck, Fremlin, Pryce, De Wilde, Floret**.

Theorem 3 (Orihuela)

For a web-compact space X the space $C_p(X)$ is angelic.

Corollary 4

Let E be a lcs such that $(E', \sigma(E', E))$ is web-compact. Then $(E, \sigma(E, E'))$ is angelic. Consequently, this holds for any metrizable lcs E.... and more general cases.

More about angelic spaces $C_p(X)$ in terms of resolutions.

We need the following concept: A family $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of sets covering a space X is called a **resolution** if $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$.

1. If A_α are compact, countably compact, relatively countably compact, etc., $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is called a **compact, countably compact, relatively countably compact resolution**, respectively. A resolution in a lcs consisting of bounded sets is called **bounded**.

2. Any K-analytic space admits a compact resolution; the converse fails [**Talagrand**].
3. An angelic space X is K-analytic iff X has a compact resolution [**Cascales**].

Angelicity seems to be a good concept to study (and simplify solutions of some of) the following problems (some still open):

Problem 5

Characterize $C_p(X)$ as having a σ-compact covering (resp. σ-bounded covering, ...bounded resolution, compact resolution,....) by a natural topological property of $C_p(X)$ or X.

It turns out that spaces $C_p(X)$ with a bounded resolution are already angelic! Hence $C_p(X)$ admitting a stronger metrizable vector topology is angelic.

Proposition 6 (**Ferrando-Kąkol**)

The following conditions are equivalent:

1. $C_p(X)$ *admits a bounded resolution.*
2. $C_p(X)$ *is* K*-analytic-framed in* $\mathbb{R}^X$ *and* $C_p(X)$ *is angelic.*
3. $C_p(X)$ *is* K*-analytic-framed in* $\mathbb{R}^X$*, i.e. there exists a* K*-analytic space* Z *such that* $C_p(X) \subset Z \subset \mathbb{R}^X$.

Theorem 7 (Calbrix-Arkhangell'ski)

Let X *be a cosmic space (in particular, a separable metrizable space). The following are equivalent:*

1. X *is* σ*-compact.*
2. $C_p(X)$ *is* K*-analytic-framed in* $\mathbb{R}^X$.
3. $C_p(X)$ *is analytic-framed in* $\mathbb{R}^X$.

How to apply angelicity?

Corollary 8 (Talagrand)

Let X be a compact space. Then $C_p(X)$ is K-analytic iff $C_c(X)$ is weakly K-analytic.

Proof.

B closed unit ball in $C(X)$, $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ a compact resolution in $C_p(X)$. Show: $\{B \cap K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ a compact resolution in B in the weak topology of $C_c(X)$. Fix $\alpha \in \mathbb{N}^\mathbb{N}$. Since $C_p(X)$ is angelic, $B \cap K_\alpha$ is sequentially compact in pointwise topology. By Lebesque's theorem $B \cap K_\alpha$ is weakly sequentially compact in $C_c(X)$; hence compact (by weak angelicity of $C_c(X)$). Since $B = \bigcup\{B \cap K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and B is weakly angelic in $C_c(X)$, B is weakly K-analytic in $C_c(X)$. Hence $C_c(X) = \bigcup_n nB$ is weakly K-analytic. □

"More delicate" application yields

Corollary 9

Let X be a locally compact pseudocompact space. Then $C_p(X)$ is K-analytic iff $C_c(X)$ is weakly K-analytic.

Fact (3) and above ((1)$\Rightarrow$ (2)) can provide simpler proofs of

Theorem 10 (Tkachuk)

$C_p(X)$ admits a compact resolution iff it is K-analytic (such spaces are Lindelöf).

Theorem 11 (Velichko-Tkachuk-Shakhmatov)

$C_p(X)$ is covered by a sequence of compact (relatively countably compact) sets iff X is finite.

Class $\mathfrak{G}$, angelicity, Amir-Lindenstrauss theorem for spaces in $\mathfrak{G}$.

First motivating examples:

1. E **is an (LM)-space** of metrizable lcs E_n with a countable basis of absolutely convex neighbourhoods of zero $(U_k^n)_k$ in each E_n such that $U_{k+1}^n \subset U_k^n$, $k, n \in \mathbb{N}$. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $K_\alpha := \bigcap_k (U_{n_k}^k)^\circ$. The family $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a **relatively countably compact resolution** in $(E', \sigma(E', E))$ and each sequence in any K_α is equicontinuous.
2. E **is a (DF)-space** and $(S_n)_n$ is a fundamental sequence of bounded sets. For $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ set $K_\alpha := \bigcap_k n_k S_k^\circ$. Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is as above.

3. **The strong dual $(E', \beta(E', E))$ of a locally complete (LF)-space** E. Then E has a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of Banach discs and each bounded set in E is contained in some A_α. For $\alpha \in \mathbb{N}^\mathbb{N}$ set $K_\alpha := A_\alpha^{\circ\circ}$ in E'' and $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ in $(E'', \sigma(E'', E'))$ is as above. In particular this holds for the space of test functions $E := \mathfrak{D}(\Omega)$ over $\Omega \subset \mathbb{R}^n$ an open set, as well as for the space $A(\Omega)$ of real analytic functions on Ω.

The common topological structure that appears in the dual $(E', \sigma(E', E))$ of this examples (and results around problems (1) and (2) from Introduction) motivated Cascales and Orihuela to introduce the class $\mathfrak{G}$ of lcs.

A lcs E belongs to class $\mathfrak{G}$ if $F := (E', \sigma(E', E))$ admits a **resolution** $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that each sequence in any A_α is **equicontinuous.** Class $\mathfrak{G}$ enjoys good permanence properties.

First Observations and Facts for $E \in \mathfrak{G}$:

1. Each A_α is $\sigma(E', E)$-relatively countably compact.
2. F is **web-compact**, so $C_p(F)$ is angelic. Hence $(E, \sigma(E, E'))(\subset C_p(F))$ **is angelic!**
3. Weakly compact $A \subset E$ is **Fréchet-Urysohn**, i.e. if $B \subset A$ and $x \in \overline{B}$, there is a sequence in B converging to x.
4. $C_p(X) \in \mathfrak{G}$ iff X is countable **[Cascales-Kąkol-Saxon]**.

Next result is motivated by the following general problem due to **Corson** (still open):

Problem 12

Let E be a weakly Lindelöf Banach space. Is it true that the unit ball in E' in $\sigma(E', E)$ has countable tightness?

Seems to be more natural to consider property (C) of Corson rather than the weak Lindelöf itself (**Pol**). Another variant:

Problem 13

Let X be Lindelöf. Is it true that every compact subset in $C_p(X)$ has countable tightness?

Arkhangell'ski answered in positive with Proper Forcing Axiom (**Corson, Pol, Kunen, Plebanek, Frankiewicz, Nardzewski, Fremlin**). Nothing more seems to be known.

The following "inverse case" is easier!

Theorem 14 (Cascales-Kąkol-Orihuela-Saxon)

The following assertions are equivalent for $E \in \mathfrak{G}$:

1. $(E, \sigma(E, E'))$ *has countable tightness.*
2. $(E', \sigma(E', E))$ *is Lindelöf.*
3. $(E', \sigma(E', E))^n$ *is Lindelöf for each* $n \in \mathbb{N}$.
4. $(E', \sigma(E', E))$ *is K-analytic.*

If $E \in \mathfrak{G}$ *is quasibarrelled, both* E *and* $(E, \sigma(E, E'))$ *have countable tightness.*
If $E \in \mathfrak{G}$ *and* $(E, \sigma(E, E'))$ *is a Lindelöf* Σ*-space,* $dens(E, \sigma(E, E')) = dens(E', \sigma(E', E))$.

There is a another way (strategy) to study problems (1) and (2) (Introduction) in the frame of the "**descriptive theory of compactness**"– Theory about **Eberlein, Talagrand, Gul'ko, Corson**...... compact spaces.

"It turns out that compact parts of $C_p(X)$, where X is compact, have much better convergence properties than $C_p(X)$ itself" [**Arkhangell'ski**].

Corson and **Lindenstrauss** conjectured (1966) that every weakly compact set in a Banach space is homeomorphic to a weakly compact set in $c_0(\Gamma)$ over suitable Γ. THIS WAS THE BEGINNING......

Elementary facts about Eberlein, Talagrand, Gul'ko Corson compacta.

1. Compact K is *Eberlein* compact if K is homeomorphic to a weakly compact subset of a Banach space.[**A**-**L**]
2. K is Eberlein compact iff $C(K)$ is (WCG) iff K is homeomorphic to a weakly compact set in $c_0(\Gamma)$ for some Γ iff $(B_{C(K)'}, w^*)$ is Eberlein compact [**A**-**L**].
3. A compact K is Talagrand compact if $C(K)$ is a weakly K-analytic Banach space. Hence K is Talagrand compact iff $C_p(K)$ is K-analytic.

4. **K is Talagrand compact iff K is homeomorphic to a weakly compact set in a lcs in class $\mathfrak{G}$ [Cascales-Orihuela].**
5. Compact K such that $C(K)$ is (WCG) is Talagrand compact [**Talagrand**].
6. **Every dyadic Eberlein compact is metrizable** (since every compact dyadic space with countable tightness is metrizable [**Engelking**]). Hence every Eberlein compact group is metrizable.
7. **Eberlein compact satisfying the countably chain condition** (ccc) i.e. every pairwise disjoint collection of open sets is countable, **is second countable** (**Rosenthal**).

8. Compact K is *Corson* compact if K is homeomorphic to a compact subset of a Σ-product $\Sigma\mathbb{R}^{\tau}$ for some cardinal τ.
9. K **Corson compact, then** K **is Fréchet-Urysohn** and $C_p(K)$ is Lindelöf. The converse fails **[Gu'lko-Alster-Pol]**.
10. K compact and $C_p(X)$ is a Lindelöf Σ-space, then X is Corson compact **[Gul'ko]**.
11. $C_p(X)$ need not be Lindelöf Σ if X is Corson compact **[Sokolov]**.
12. Compact K is Gul'ko compact if $C_p(X)$ is a Lindelöf Σ-space.
13. **Eberlein (Gul'ko) compact** K **contains a dense** G_δ **metrizable subspace [Namioka (Gruenhage)]**. This fails for Corson compact spaces.

14. $C(X)$ is weakly Lindelöf for Corson compact X iff every Radom measure on X is separable **[A-M-N]**. **Proof depends on two facts**: $C_p(X)$ is Lindelöf for Corson compact X. If X is compact and every Radon probability measure on X has separable support, $P(X)$, the space of all Radom probabilistic measure on X is Corson compact.
15. Let A be the one-point compactification of the discrete space of cardinality $\mathfrak{c} = 2^{\aleph_0}$. $C_p(A)$ contains a compact subset K homeomorphic with A. K **is nonmetrizable Fréchet-Urysohn Eberlein compact**..... and **cannot be included in any separable part of any** $C_p(X)$ **over web-compact** X. $[0, \omega_1]$ **is not Eberlein** compact (as non-angelic)

Eberlein compact $\Rightarrow$ Talagrand compact $\Rightarrow$ Gul'ko compact $\Rightarrow$ Corson compact......... $\Rightarrow$......(and more!)

Metrizable (pre)compact sets in spaces $C_p(X)$.

Possible approach: Let $K \subset L \subset C_p(X)$, L separable, K compact and X possible nonseparable. If there exists separable completely regular Hausdorff Y with $L \subset C_p(Y)$, then K is metrizable. EASY! For many spaces X this is possible!

Proposition 15 (Kąkol-Lopez-Pellicer-Munoz)

Let υX or $C_p(X)$ be a Lindelöf Σ-space. If $L \subset C_p(X)$ is separable, there exists a separable submetrizable Lindelöf Σ-space Y such that L is embedded into $C_p(Y)$.

Proposition 16 (K-L-P-M)

Let υX be Lindelöf Σ. Then $Y \subset C_p(X)$ is analytic iff it has a compact resolution and is contained in a separable set of $C_p(X)$.

Corollary 17

$C_p(X)$ is analytic iff $C_p(X)$ is separable and admits a compact resolution (iff $C_p(X)$ is separable and K-analytic).

Previous fact and (to prove): If $C_p(X)$ has a compact resolution, then υX is Lindelöf Σ.

Corollary 18 (Cascales-Orihuela)

Let X be web-compact. Then a compact set $K \subset C_p(X)$ is metrizable iff it is contained in a separable subset of $C_p(X)$.

For (less technical) applications of the former results for class $\mathfrak{G}$ note the following useful simple fact:

1. If E is a lcs then $(E, \sigma(E, E')) \subset C_p(E', \sigma(E', E))$

Corollary 19

A subset Y of a lcs E in the class $\mathfrak{G}$ is $\sigma(E, E')$-analytic iff Y has a $\sigma(E, E')$-compact resolution and is contained in a $\sigma(E, E')$-separable subset.

Idea of the proof.

1. $Z := (E', \sigma(E', E))$ is quasi-Suslin (hard part!). Hence, there exists a quasi-Suslin map $T : \mathbb{N}^{\mathbb{N}} \to 2^Z$, $\alpha \mapsto T(\alpha)$.
2. υZ is K-analytic: Since every $T(\alpha)$ is countably compact, its closure $\overline{T(\alpha)}$ in υZ is compact. Then $\alpha \mapsto \overline{T(\alpha)}$ is an usco map, so $W := \bigcup \overline{T(\alpha)}$ is K-analytic.
3. Since $Z \subset W \subset \upsilon Z$, we have $W = \upsilon W = \upsilon Z$ is K-analytic. As $(E, \sigma(E, E')) \subset C_p(Z)$, apply Corollary 16.

As compact analytic spaces are metrizable, we summarize:

Corollary 20

Let E be a lcs in class $\mathfrak{G}$ and $K \subset E$.

1. *Weakly compact K is $\sigma(E, E')$-Talagrand compact.*
2. *Weakly compact K is $\sigma(E, E')$-Fréchet-Urysohn.*
3. *Weakly compact K contains a dense G_δ-metrizable subspace.*
4. *Precompact K is metrizable in E.*
5. *$\sigma(E, E')$-compact K is $\sigma(E, E')$-metrizable iff K is contained in a $\sigma(E, E')$-separable subset of E. Hence the closure of any countable subset in Talagrand compact is metrizable.*
6. *$(E, \sigma(E, E'))$ is K-analytic (analytic) iff $(E, \sigma(E, E'))$ has a compact resolution (and is separable).*

Problem 21

1. *Let X be Lindelöf such that $C_p(X)$ is K-analytic. Is X σ-compact? (**Calbrix-Arkhangell'ski**)*
2. *Characterize X as being analytic by a natural topological property of $C_p(X)$ (**Calbrix-Arkhangell'ski**)*
3. *Characterize $C_p(X)$ as having a bounded resolution in term of X.*
4. *Let K be a compact space. Is it rue that for $C(K)$ with Corson property (C) the space $P(K)$ has countable tightness ?(**Corson-Pol**)*
5. *Is the product of weakly Lindelöf Banach spaces by itself a weakly Lindelöf Banach space? Recall that finite products of Banach spaces with property (C) have property (C).*

We provide a short proof of part (4): **Every precompact set K in a lcs in $\mathfrak{G}$ is metrizable.**

Skech of Proof. Since the completion of E in class belongs $\mathfrak{G}$, we assume that E is complete and K is compact. Let $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-representation of E'. By τ we denote the topology of E. We say that a subset M of E' is K^0-separated if $(a + K^0) \cap M = \{a\}$ for each $a \in M$. By Zorn's lemma there exists a maximal K^0-separated subset M_1 of E'. Clearly $M_1 + K^0 = E'$. **Note that M_1 is countable.** Indeed, otherwise, since $E' = \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ and $A_\alpha \subset A_\beta$ whenever $\alpha \leq \beta$, for α, β in $\mathbb{N}^\mathbb{N}$, we choose (to prove!) a countable infinite subset P of M_1 and $\gamma \in \mathbb{N}^\mathbb{N}$ such that $P \subset A_\gamma$.

Since E belongs to $\mathfrak{G}$, P is equicontinuous, so P is precompact in the topology of uniform convergence on the τ-precompact subsets of E. Therefore there exists a finite set $\{a_i : 1 \leq i \leq k\} \subset P$ such that $P \subset \bigcup \{a_i + K^0 : 1 \leq i \leq k\}$. Clearly there exists $1 \leq j \leq k$ such that the set $(a_j + K^0) \cap P$ is infinite, contradicting the hypothesis that M_1 $(\supset P)$ is K^0-separated. Let M_n be a maximal subset of E' that it is $n^{-1}K^0$-separated, for each $n \in \mathbb{N}$. The set $M_0 := \bigcup \{M_n : n \in \mathbb{N}\}$ is countable.Let τ_{M_0} be the weakest topology on K that makes continuous the functions of M_0. If $x \neq y$ are two points of K then there exist $g \in E'$ and $n \in \mathbb{N}$ such that $|g(x) - g(y)| > 3n^{-1}$. Since $E' = M_n + n^{-1}K^0$, there exists $f \in M_n (\subset M_0)$ such that $g \in f + n^{-1}K^0$. Hence $|f(x) - f(y)| > n^{-1}$. Therefore (K, τ_{M_0}) is metrizable, so K is metrizable.

Many spaces in functional analysis enjoy the following property:

1. A lcs is said to have a $\mathfrak{G}$-base if E has a basis $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighbourhods of zero such that $U_\alpha \subset U_\beta$ if $\alpha \geq \beta$.
2. Every space with a $\mathfrak{G}$-base is in class $\mathfrak{G}$.
3. Every quasibarrelled lcs in class $\mathfrak{G}$ has a $\mathfrak{G}$-base (**Cascales–Kakol–Saxon**).

Theorem 22

Every lcs E with a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ has all precompact sets K metrizable.

Proof. We may assume that K is compact and absolutely convex. Let $\mathcal{T}_{pc}$ be the topology on E' of the uniform convergence on the precompact subsets of E. Each polar set U_α^0 is equicontinuous and $\sigma(E', E)$-compact, so $\{U_\alpha^0 : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact covering of $(E', \mathcal{T}_{pc})$. The map

$$\varphi : (E', \mathcal{T}_{pc}) \to (\varphi(E'), \|\|_\infty),$$

defined by $\varphi(f) := f|_K$ is continuous and $\{\varphi(U_\alpha^0) : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact covering of the normed space $(\varphi(E'), \|\|_\infty)$, where $\|\|_\infty$ is the supremum norm on K. The space $(\varphi(E'), \|\|_\infty)$ is analytic. Let F be a countable subset of E' such that $\varphi(F)$ is dense $(\varphi(E'), \|\|_\infty)$.

If $x \neq y$ in K, there exists $g \in E'$ such that $|g(x) - g(y)| > 3$. Then there exists $f \in F$ such that $\|\varphi(g) - \varphi(f)\|_\infty < 1$. Hence $|f(x) - f(y)| > 1$. Hence the restriction $\sigma(E, F)|K$ is metrizable (this restriction is the initial topology defined by F on K). By compactness the original topology of K is the topology $\sigma(E, F)|_K$. Hence K is metrizable.

1. If X has a compact resolution swallowing compact sets, then $C_c(X)$ has a $\mathfrak{G}$-base.
2. If Xc is a μ-space and $C_c(X)$ has a $\mathfrak{G}$-base, then X has a compact resolution swallowing compact sets.
3. A second countable X is completely metrizable iff X has a compact resolution swallowing compact sets (**Christensen**).

1. It is well known that if X **almost σ-compact** (i.e. X has a dense σ-compact subset), then $C_c(X)$ admits a weaker metric topology. Hence any compact set in such $C_c(X)$ is metrizable. The following extends this fact.
2. If X be a space having a compact resolution swallowing compact sets. Then every precompact set in $C_c(X)$ is metrizable. EVEN more general:
3. If X has a **dense subspace covered by a compact resolution**, then any precompact set in $C_c(X)$ is metrizable.

The best result related with Corollary 20 might be the following

Theorem 23 (Ferrando–Kakol-Lopez-Pellicer)

Precompact sets in a lcs E are metrizable iff E' endowed with the topology τ_p of uniform convergence on the precompact sets of E is trans-separable.

1. A lcs E is trans-separable iff E is isomorphic to subset of a product of separable metrizable spaces (**Pfister**).
2. Part (4) from Corollary 20 follows from above theorem: (i) If $E \in \mathfrak{G}$, then (E', τ_p) is quasi-Suslin. (ii) Every quasi-Suslin space is web-compact. (iii) Every web-compact lcs is trans-separable.