Two classes of metrizable spaces ℓ_c-invariant

Manuel López Pellicer (IUMPA, UPV)

Joint work with Jerzy Kakol and Oleg Okunev

April 15, 2014
Prague, Institute of Mathematics
Academy of Sciences of the Czech Republic

Project Prometeo I/2013/058

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

GENERALITAT VALENCIANA
CONSELLERIA D'EDUCACIÓ, CULTURA I ESPORT

Outline

1. Introduction
2. Michael's results on $\aleph_0$-spaces
3. Preservation of $\aleph_0$ and Polish spaces by ℓ_c-equivalence
4. Extension to spaces of pointwise countable type

Outline

1 Introduction

Introduction

Many properties of a Tychonoff space X have been characterized by properties of $C_p(X)$ or $C_c(X)$.

- Arhangel'skii question: Is metrizability ℓ_p-invariant in the class of first countable spaces?
- Partially answered by Valov for first countable Čech complete spaces.
- Complete metrizability is ℓ_p-invariant in the class of metrizable spaces (Baars, de Groot and Pelant).

The ℓ_c-invariance of separable metrizability and separable complete metrizability for spaces of pointwise countable type has been considered very recently by Kąkol, L-P and Okunev (JMAA 2014).

t-equivalence

ω (or $\mathbb{N}^+$) denotes the smallest infinite ordinal with the discrete topology and with the usual order.

Theorem

Two spaces X and Y are homeomorphic if and only if the topological rings $C_p(X)$ and $C_p(Y)$ are isomorphic.(Nagata)

Definition

X and Y are *t-equivalent* if $C_p(X)$ and $C_p(Y)$ are homeomorphic.

A property $\mathcal{P}$ is preserved by *t*-equivalence if whenever X and Y are *t*-equivalent and X has $\mathcal{P}$ then Y has also $\mathcal{P}$.
Velichko's theorem implies that *finite cardinality* is preserved by *t*-equivalence.

ℓ_p-equivalence

Definition

X and Y are *ℓ_p-equivalent* if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.

$\mathcal{P}$ is invariant by ℓ_p-equivalence if when X has property $\mathcal{P}$ and X is ℓ_p-equivalent to Y then Y has the property $\mathcal{P}$ too.
$\mathcal{P}$ is preserved by ℓ_P-equivalence if and only if $\mathcal{P}$ *admits a description in terms of the linear topological structure of* $C_p(X)$.

- The properties *hemicompact*, $\aleph_0$, *Lindelöf*, *Lindelöf-Σ*, *K-analytic* and *analytic* are preserved by ℓ_p-equivalence.
- *Metrizability*, *local compactness*, *countable weight*, *normality* and *paracompactness* are not ℓ_p-invariant.

ℓ_c-*equivalence*

Definition

X and Y are ℓ_c-*equivalent* if $C_c(X)$ and $C_c(Y)$ are linearly homeomorphic.

$\mathcal{S}$ is ℓ_c-invariant if and only if $\mathcal{S}$ in X *is characterized by a property of* $C_c(X)$.

- For instance, Nachbin -Shirota theorem states that μ-*spaces* are preserved by ℓ_c-equivalence.

Definition

X ℓ_c-*covers* a space Y if there is a continuous open linear map from $C_c(X)$ onto $C_c(Y)$.

If X and Y are ℓ_c-equivalent, then each of the two covers the other.

ℓ_p-eq $\Longrightarrow$ ℓ_c-eq for μ-spaces

Theorem

For μ-spaces ℓ_p-equivalence implies ℓ_c-equivalence.

Proof.

$h : C_p(X) \to C_p(Y)$ lin hom $\Longrightarrow$ $h' : L_p(Y) \to L_p(X)$ lin hom.
Then $B \subset L_p(Y)$ is bounded $\Longleftrightarrow$ $h'(B)$ is bounded.
Then $h : (C(X), \beta(C_p(X), L_p(X))) \to (C(Y), \beta(C_p(Y), L_p(Y)))$ is lin hom.
Let Z be a μ-space and $\tau_c(Z)$ the top of $C_c(Z)$ then:

$$\tau_c(Z) \leqslant \beta(C_p(Z), L_p(Z)) \leqslant \beta(C_c(Z), C_c(Z)') = \tau_c(Z)$$

□

ℓ_p-eq $\not\Leftarrow$ ℓ_c-eq for μ-spaces

The converse is not true: (Milyutin and Pestov)

- $[0,1]$ and $[0,1] \times [0,1]$ are ℓ_c-equivalent

but

- $[0,1]$ and $[0,1] \times [0,1]$ are not ℓ_p-equivalent.

Outline

2. Michael's results on $\aleph_0$-spaces

$\aleph_0$-spaces: Elementary properties

Definition

$\mathcal{N}$ is a k-network in X if $K \subset U$ (K compact, U open)$\Longrightarrow K \subset \cup\{F : F \in \mathcal{F}\} \subset U$ for some finite family $\mathcal{F} \subset \mathcal{N}$.
X is $\aleph_0$-space if it is regular and it has a countable k-network.

- Every $\aleph_0$-space X is Lindelöf, hereditary separable and every open subset is an F_σ
- First-countable or locally compact $\aleph_0$-spaces are separable and metrizable.
- $\aleph_0$-space are hereditay by subspaces or countable products.

k-equivalence

Definition

X and Y are k-equivalent if there exists a bijection $f : X \to Y$ with f and f^{-1} preserving compact subsets. Two topologies on X are *k-equivalent* if they yield the same compact subsets.

- For (X, τ), τ is k-equivalent to $\tau_k := \sup\{\tau_i : \tau_i$ k-equi. to $\tau\}$ (Tychonoff pr. th.). τ_k ($X_k := (X, \tau_k)$) is the k-topology (space) associated to τ ((X, τ)).
- (X, τ) is a k-space k-space if $\tau = \tau_k$ (if and only if A is closed when $C \cap A$ is closed for each compact subset C)
- $f : (X, \tau) \to Y$ is k-continuous if $f : (X, \tau_k) \to Y$ is continuous.
- If (X, τ) is a k-space k-equivalent to Y and f^{-1} is continuous then f is a homeomophism.

Heritability of $\aleph_0$-spaces by k-equivalence

Proposition

If $\mathcal{J}_1$ and $\mathcal{J}_2$ are two k-equivalent topologies, $(X, \mathcal{J}_2)$ has a countable k-network if and only if $(X, \mathcal{J}_1)$ does.

Proof.

Let $\mathcal{P}$ be a $\mathcal{J}_1$ (not $\mathcal{J}_2$) countable k-network closed under finite intersections. Let (compact) $C \subset U_2$ ($\mathcal{J}_2$-open) such that if $(P_n)_n$ is $\{P \in \mathcal{P} : C \subset P\}$, then there exists $x_n \in P_n' \backslash U_2$, with $P_n' = \cap\{P_m : 1 \leqslant m \leqslant n\}$, for each n.
If $C \subset U$ ($\mathcal{J}_1$ open) there exists $P'_{n(U)}$ such that $C \subset P'_{n(U)} \subset U$.
$\{x_m : n(U) \leqslant m\} \subset U \Longrightarrow C \cup \{x_n : n \in \omega\}$ is a compact set.
Contradiction with U_1 ($\mathcal{J}_1$ open) such that

$$U_1 \cap [C \cup \{x_n : n \in \omega\}] = U_2 \cap [C \cup \{x_n : n \in \omega\}] .$$

$C_c(X)$ $\aleph_0$-spaces: Lema 1

If X is a Hausdorff space with countable k-network and if Y is and $\aleph_0$-space then $C_c(X, Y)$ is and $\aleph_0$-space (Michael).

Lemma

M (compact)$\subset X$. Then $\psi : M \times C_c(X) \to C_c(X)$ such that $\psi(x, f) = f(x)$ is continuous.

Proof.

Fix $(x, f) \in M \times C_c(X)$.
Let V be a neighborhood of $f(x)$ and D be a closed neighborhood of x in M such that $f(D) \subset V$.
>From $\psi((D \times W(D, V))) \subset V$, follows the continuity in (x, f). □

$C_c(X)$ $\aleph_0$-spaces: Lema 2

Lemma

K (compact)$\subset C_c(X)$ and U (open)$\subset \mathbb{R}$. If X is a k-space then $V := \{x \in X : K(x) \subset U\}$ is open.

Proof.

Let M(compact)$\subset X$. For

$$x_0 \in V \cap M = \{x \in M : K(x) \subset U\}$$

and $f \in K$ there exists a M-neighborhood D_f of x_0 and a neighborhood W_f of f such that $W_f(D_f) \subset U$.
$K \subset \cup \left\{W_{f_i} : 1 \leqslant i \leqslant n\right\} \Longrightarrow K\left(\cap D_{f_i}\right) \subset U$ and then $\cap D_{f_i} \subset V \cap M$. $V \cap M$ is open in M. □

A property of normal spaces

Proposition

If $\cup\{A_i : 1 \leqslant i \leqslant n\}$ *is an open covering of a normal space* X*, there exists and open covering* $\cup\{B_i : 1 \leqslant i \leqslant n\}$ *such that*

$$\overline{B}_i \subset A_i,\ 1 \leqslant i \leqslant n$$

Proof.

A_1 is an open neighborhood of the closed set $X \setminus \cup \{A_i : 2 \leqslant i \leqslant n\}$.
There exists B_1 open: $X \setminus \cup \{A_i : 2 \leqslant i \leqslant n\} \subset B_1 \subset \overline{B}_1 \subset A_1$.
Hence $X = B_1 \cup \{A_i : 2 \leqslant i \leqslant n\}$.
After n-steps of an obvious induction process we get the $B_i, 1 \leqslant i \leqslant n$. □

$C_c(X)$ $\aleph_0$-spaces: Lema 3

Lemma

Let $\mathcal{S}$ be a subbase of the Hausdorff topological space X and let $\mathcal{P}$ be a collection of subsets of X such that for each compact set C and each open set $U \in \mathcal{S}$ there exists $P \in \mathcal{P}$ such that $C \subset P \subset U$. Then X has a countable k-network.

$C_c(X)$ $\aleph_0$-spaces: Lema 3

Proof.

Let $\mathcal{R}$ be the family of finite unions of finite intersections of elements of $\mathcal{P}$.
Let C be a compact subset of X and let U be an open neighborhood of C.
If $U = \cap\{S_i : 1 \leqslant i \leqslant n\}$, each $S_i \in \mathcal{S}$, then
$C \subset P_i \subset S_i \Longrightarrow C \subset \cap\{P_i : 1 \leqslant i \leqslant n\} \subset U$.
The base $\mathcal{B}$ generated by $\mathcal{S}$ is the family formed by finite intersections of elements of $\mathcal{S}$.
By compactness $C \subset \cup\left\{A_j : 1 \leqslant j \leqslant p\right\}$, with $A_j \in \mathcal{B}$ and $A_j \subset U$ for each $1 \leqslant j \leqslant p$.
$C = \cup\left\{\overline{B}_j : 1 \leqslant j \leqslant p\right\}$, with $\overline{B}_j$(compact)$\subset A_j$.
From $R_j \in \mathcal{R}$, with $\overline{B}_j \subset R_j \subset A_j$ for each j, follows $C \subset R \subset U$, with $R = \cup\left\{R_j : 1 \leqslant j \leqslant p\right\} \in \mathcal{R}$. □

$C_c(X)$ $\aleph_0$-spaces

Theorem

If X is an $\aleph_0$-space then $C_c(X)$ is an $\aleph_0$-space.

Proof.

kX is an $\aleph_0$-space and $C_c(X) \hookrightarrow C_c(kX)$. We may suppose that X is a k-space.
Let $\{Q_m : m \in \omega\}$ be a countable k-network in X and let $\mathcal{B} = \{B_n : n \in \omega\}$ be a base of the topology of $\mathbb{R}$.
For each compact subset K of $C_c(X)$, each compact subset M of X and each $B_n \in \mathcal{B}$ such that $K \subset W(M, B_n)$ we have $M \subset \{x \in X : K(x) \subset B_n\}$ and then there exists Q_m such that $M \subset Q_m \subset \{x \in X : K(x) \subset B_n\}$.
Hence $K \subset W(Q_m, B_n) \subset W(M, B_n)$.
Apply subbase lemma to $\mathcal{P} := \{W(Q_m, B_n) : (m, n) \in \omega^2\}$. □

Outline

3. Preservation of $\aleph_0$ and Polish spaces by ℓ_c-equivalence
   - Preservation of $\aleph_0$-spaces by ℓ_c-equivalence
   - Preservation of Polish spaces by ℓ_c-equivalence

Introduction

- Main result in this section: $\aleph_0$-spaces are preserved by ℓ_c-equivalence (Kąkol, L-P and Okunev 2014).
- Elementary properties:

1. $C_p(C_p(X)) \hookrightarrow C_p(C_c(X))$ is an embedding. Therefore $\hat{\Delta}_X : X \to \hat{X} := \hat{\Delta}_X(X)$ $(\subset C_p(C_c(X)))$ is an homeomorphism.
2. Recall $\hat{\Delta}_X(x) = \hat{x}$ with $\hat{x}(f) = f(x)$ for each $f \in C(X)$
3. The map $\tilde{\Delta}_X : X \to \tilde{X} := \tilde{\Delta}_X(X)(\subset C_c(C_c(X)))$ may be not continuous (top $C_c(C_c(X)) \succ$ top $C_p(C_c(X))$).
4. The map $\tilde{\Delta}_X$ is continuous$\Longleftrightarrow$ if it is an embedding.

k-continuity of $\tilde{\Delta}_X$

Lemma

The map $\tilde{\Delta}_X$ is k-continuous. $\hat{X}$ and $\tilde{X}$ are k-equivalents (by $i : i(\hat{x}) = \tilde{x}$, $x \in X$). If X is a k-space then $\tilde{\Delta}_X$ is an embedding.

Proof.

Let M be a compact subset of X and
$F = C_c(C_c(X)) \setminus W(K, V)$, where K is a compact subset of $C_c(X)$ and V is an open subset of $\mathbb{R}$.
Let $x = \lim_{l \in L} x_l$, with each $x_l \in M \cap \tilde{\Delta}_X^{-1}(F)$.
For each $l \in L$ fix $f_l \in K$ with $f_l(x_l) \notin V$.
We may suppose (compactness) that $(f_l : l \in L)$ converges uniformly on M to $f \in K$.
$\lim_{l \in L} f_l(x_l) = f(x) \Longrightarrow f(x) \notin V \Longrightarrow x \in M \cap \Delta_X^{-1}(F)$.
The map $\tilde{\Delta}_X$ is k-continuous, and the remaining is obvious. □

$\aleph_0$-property is ℓ_c-invariante

The first part of the next theorem extends Theorem 21 in Kąkol, L-P and Okunev.
We will refer this theorem as Main Theorem

Theorem

If X is an $\aleph_0$-space and $h : C_c(X) \longrightarrow C_c(Y)$ is continuous linear and onto, then Y is an $\aleph_0$-space. $\aleph_0$-property is ℓ_c-invariant.

$\aleph_0$ property in not preserved by open maps.

$\aleph_0$-property is ℓ_c-invariante

Proof.

$h^* : C_p(C_c(Y)) \to C_p(C_c(X))$ $(h^*(\hat{Y}))$ is an embedding.
$h^* : C_c(C_c(Y)) \to C_c(C_c(X))$ $(h^*(\tilde{Y}))$ is continuous.
$j : h^*(\hat{Y}) \to h^*(\tilde{Y})$ $(j(h^*(\hat{y})) = h^*(\tilde{y})$, for each $y \in Y)$ has inverse map j^{-1} continuous. $\hat{Y}$ and $\tilde{Y}$ are k-equivalent (by $i : \hat{Y} \to \tilde{Y}$, with $i(\hat{y}) = \tilde{y}$ for each $y \in Y$). By definition $h^* i = jh^*|_{\hat{Y}}$. Therefore
M (compact)$\subset \hat{Y} \Longrightarrow h^* i(M)$ (compact)$\subset h^*(\tilde{Y})$.
K (compact)$\subset h^*(\tilde{Y}) \Longrightarrow (jh^*|_{\hat{Y}})^{-1}(K)$ (compact)$\subset \hat{Y}$.
Therefore $h^*(\tilde{Y})$ is k-equivalent to $\hat{Y}$ and (homeomofism) to Y.
$C_c(C_c(X))$ is an $\aleph_0$-space (Michael)$\Longrightarrow h^*(\tilde{Y})$ is $\aleph_0$-space$\Longrightarrow$(k-equivalence) Y is an $\aleph_0$-space. □

ℓ_c-invariance of metrizability for first countable spaces

Let $\mathfrak{D}$ be the class of first countable or locally compact spaces. As $\aleph_0$-spaces in $\mathfrak{D}$ are metrizable (Michael) Kąkol, L-P and Okunev obtained:

Corollary

If X is an $\aleph_0$-space, $Y \in \mathfrak{D}$ and there exists a continuous linear map h from $C_c(X)$ onto $C_c(Y)$, then Y is metrizable and separable. In particular, the property of being metrizable and separable is preserved by ℓ_c-equivalence in the class $\mathfrak{D}$.

This result applies when X is second countable (extending Corollay 23 of Arhangel'skii's Survey of C_p-theory).

Compact resolutions in X and $\mathfrak{G}$-bases

Definition

A resolution is a family $\mathcal{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ of subsets of X such that $X = \cup\{K_\alpha : \alpha \in \omega^\omega\}$ and $K_\alpha \subset K_\beta$ if $\alpha \leqslant \beta$.

- A resolution $\mathcal{K}$ is *compact* if each K_α is a compact subset of X. If, additionally, for each compact subset K of X there exists $K_\alpha \in \mathcal{K}$ such that $K \subset K_\alpha$, then $\mathcal{K}$ is a *compact resolution swallowing compact subsets of* X.

Definition

A base $\{K_\alpha : \alpha \in \omega^\omega\}$ of neighborhoods of zero in the topological space E is a $\mathfrak{G}$-base if $U_\beta \subset U_\alpha$ whenever $\alpha \leqslant \beta$.

- Quotient maps preserve $\mathfrak{G}$-bases.

Compact resolutions in X and $\mathfrak{G}$-bases in $C_c(X)$

Proposition

X has a compact resolution $\mathcal{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ swallowing compact sets if and only if $C_c(X)$ has a $\mathfrak{G}$-base.

Compact resolutions in X and $\mathfrak{G}$-bases in $C_c(X)$

Proof.

Let $\mathcal{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ be a crscs. As for $K \in \mathcal{K}$ and $\varepsilon > 0$ there exists $K_\alpha \supset K$, with $a_1^{-1} < \varepsilon$, then $U_\alpha := W(K_\alpha, (-a_1^{-1}, a_1^{-1})) \subset W(K, \varepsilon)$ and $\mathcal{U} = \{U_\alpha : \alpha \in \omega^\omega\}$ is a $\mathfrak{G}$-base in $C_c(X)$.

Let $\mathcal{U} = \{U_\alpha : \alpha \in \omega^\omega\}$ be a $\mathfrak{G}$-base of $C_c(X)$. For U_α there exists $K \in \mathcal{K}$ and $\varepsilon > 0$ with $W(K, (-\varepsilon, \varepsilon)) \subset U_\alpha$. Then
$U_\alpha \subset W(K_\alpha, [-a_1, \ a_1])$, with
$K_\alpha := \{x \in X : |f(x)| \leq a_1, f \in U_\alpha\}$.
$W(K, (-\varepsilon, \varepsilon)) \subset W(K_\alpha, [-a_1, \ a_1]) \Longrightarrow (K_\alpha \subset K) K_\alpha$ compact.
For D (compact) $\subset X$ there exists $U_\alpha \subset W(D, [-1, 1]\}$. Then $x \in D$ and $f \in U_\alpha \Longrightarrow |f(x)| \leq 1 \ (\leq a_1)$, hence, by definition, $x \in K_\alpha$. From $D \subset K_\alpha$ follows $\mathcal{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ is a crscs of X. □

Christensen theorem

Proposition

A separable and complete metric space (X, d) has a compact resolution $\mathcal{K}$ swallowing compact sets.

Proof.

By hypothesis $X = \overline{\{x_m : m \in \omega\}}$. For $a_n \in \omega$ let $K_{a_n} := \cup \left\{ B(x_m; n^{-1}) : m \leqslant a_n \right\}$ and for $\alpha = (a_n)_n \in \omega^\omega$ let $K_\alpha = \cap \{K_{a_n} : n \in \omega\}$. K_α is closed and precompact and for each compact subset K of X and each n there exists $b_n \in \omega$ such that $K \in K_{b_n}$. Then $\mathcal{K} = \{K_\alpha : \alpha \in \omega^\omega\}$. □

Theorem

A metrizable space (X, d) has a compact resolution swallowing compact sets if and only if (X, d) is a Polish space.

Preservation of Polish spaces by ℓ_c-equivalence in $\mathfrak{D}$.

$\mathfrak{D}$ is the class of first-countable or locally compact spaces.

Theorem

Let X be a Polish space and $Y \in \mathfrak{D}$. If X ℓ_c-cover Y then Y is a Polish space. Therefore the property of being Polish space is invariant by ℓ_c-equivalence in the class $\mathfrak{D}$.

Proof.

By the Main Theorem Y is $\aleph_0$-space. Then Y is metrizable (Michael).
As X has a compact resolution swallowing compact sets then $C_c(X)$ has a $\mathfrak{G}$-base, hence $C_c(Y)$ has also a $\mathfrak{G}$-base.
This proves that Y has a compact resolution swallowing compact sets and then Christensen's Theorem implies that Y is a Polish space. □

Outline

4. Extension to spaces of pointwise countable type
   - Spaces of pointwise countable type and submetrizability
   - ℓ_c-equivalence in spaces of pointwise countable type

Spaces of pointwise countable type

Definition

The topological space Y is *of pointwise countable type* if for every $y \in Y$ there exists a compact set K that contains y such that K has a countable base of neighborhoods in Y.

It is easy to prove that:

- If Y is a space of pointwise countable type then Y is a k-space.
- The first countable spaces and the locally compact spaces are spaces of pointwise countable type.

Spaces of pointwise countable type

Proposition

If Y is a Čech-complete space then Y is a space of countable type.

Proof.

Let $Y = \cap O_n$, with each O_n open subset in βY of Y.
For each $y \in Y$ and each n there exists in βY a closed neighborhood A_n of y such that $A_n \subset O_n$.
The set $K := \cap A_n$ is a compact subset of X. We may suppose that $A_{n+1} \subset A_n$ for each $n \in \omega$.
Clearly $y \in K$ and (by compactness) if A is an open neighborhood of K in βY there exists $n \in \omega$ such that $A_n \subset A$.
Then $\{A_n \cap X : n \in \omega\}$ is a countable base of neighborhoods of K and hence Y is a space of pointwise countable type. □

Results on submetrizability.

Definition

X is submetrizable if there exists a continuous bijective map $\Phi : X \to M$, with M metrizable.

Proposition

If X has a dense σ-compact subset A then $C_c(X)$ is submetrizable.

Proposition

If X is submetrizable then $C_c(X)$ has a dense σ-compact subset.

Results on submetrizability.

Proposition

Let Y be a submetrizable space of pointwise countable type. Then Y is first countable.

Proof.

Let $y \in K$(compact)$\subset Y$. $(V_n)_n \downarrow$ base closed neigh. of K. $(W_n[\subset V_n])_n \downarrow$ sequence closed neigh. of y such that $(W_n \cap K)_n$ base K-neigh. of y.
If V is an open neighborhood of y there exist m such that $W_m \cap K \subset V$. If for each $m \in \omega$ there exists $x_m \in W_m \setminus V$ then $\{x_p : p \geqslant m\} \subset V_m$, hence $K \cup \{x_m : m \in \omega\}$ is compact. If x is adh. to $(x_m)_m \Longrightarrow x \in W_n$, $n \in \omega$, $\Longrightarrow x \in \cap\{V_n : n \in \omega\} = K$. Hence $x \in W_m \cap K \subset V$. But $x_n \in W_n \setminus V \subset E \setminus V$, $n \in \omega, \Longrightarrow x \in E \setminus V$ (contr.). Hence there exists $W_p \subset V$. □

Preservation of second countability

Proposition

Let X be a submetrizable space and let Y be a space of pointwise countable type. If there exists a continuous map from $C_c(X)$ onto $C_c(Y)$ then Y is submetrizable and first countable.

Proof.

$C_c(X)$ (and then $C_c(Y)$) has a dense σ-compact subset. Hence $C_c(C_c(Y))$ (and then Y -embeds in $C_c(C_c(Y))$-) is submetrizable. We get that Y is first countable. □

Preservation of second countability

Corollary

If X is metrizable and separable, Y is a space of pointwise countable type and h is a continuous linear mapping from $C_c(X)$ onto $C_c(Y)$, then Y is metrizable and separable (second countability is preserved by ℓ_c-equivalence for spaces of pointwise countable type).

Proof.

From first countability of Y follows metrizability and separability of Y. □

Preservation of Polish spaces by ℓ_c-equivalence

The property that Polish spaces are preserved by ℓ_c-equivalence in the spaces of countable type may be obtained by a combination of results of Baars, de Groot, Pelant and Valov on preservation of complete metrizability by ℓ_p-equivalence in the class of metrizable spaces. The next simple corollary is due to Kąkol, L-P and Okunev.

Corollary

Let X be a Polish space that ℓ_c-covers the space Y. If Y is of pointwise countable type then Y is a Polish space. Thus the property of being a Polish space is preserved by ℓ_c-equivalence for spaces of pointwise countable type.

Proof.

Like Y is first countable then it is a Polish space. □