THE WEAK TOPOLOGY OF A FRÉCHET SPACE

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1. A few general facts about Fréchet locally convex spaces

Fréchet spaces E, i.e. metrizable and complete locally convex spaces (lcs), have played an important role already from the beginning of Functional Analysis: Many vector spaces of holomorphic, differentiable or continuous functions arise as Fréchet spaces and are connected with certain problems in analysis and its applications. In particular, each Banach space is a Fréchet space and so has a countable basis of absolutely convex neighborhoods of zero. In the Banach case the basis can be obtained as multiples of the unit ball. Therefore the geometry of the unit ball is crucial in Banach space theory. Also, in Banach spaces the unit ball is a typical neighborhood of zero and a typical bounded set. If a Fréchet space has a bounded neighbourhood of zero it must be normable. Another important difference to the Banach space case is that the strong dual of a Fréchet space is not metrizable in general. Strong duals of Fréchet spaces are (DF)-spaces, a class introduced by Grothendieck. Countable (locally convex) inductive limits of Banach spaces, (LB)-spaces, are also (DF)-spaces.

Köthe echelon and co-echelon spaces are among the most important examples of Fréchet and (DF)- spaces, respectively. Many spaces of functions or distributions, for example $H(\Omega)$, $D(\Omega)$, $D'(\Omega)$, $C^\infty(\Omega)$, for an open subset $\Omega \in R^n$ are topologically isomorphic to (products of) echelon or co-echelon spaces. If X is a hemicompact space not compact, the space $C_c(X)$ is a Fréchet space which is not normed.

Topological invariants (studied by Meise, Vogt) are essential in the structure theory of Frécht spaces, They also have many interesting applications to problems arising in analysis, e.g. the existence of extension operators on compact or closed subsets of RN, the surjectivity or the existence of solution operators for convolution operators or for linear partial differential operators with constant coefficients on spaces of analytic or (ultra-) differentiable functions or (ultra-) distributions.

If E is a Fréchet space, the topology (and the uniform structure) of E can be given by an increasing sequence $(p_n)_n$ of continuous seminorms. We may always assume that $2p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$. Clearly, the topology of E as a topological vector space, is determined by a basis of neighborhoods of zero. Since E is locally convex, the basis of neioghbourhoods of zero can be formed by absolutely convex neighbourhoods of zero. Since E is metrizable, this basis can be taken as countable $(U_n)_n$ such that $2U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. One has that

$$U_n := \{x \in E : p_n(x) \leq 1\}$$

for all $n \in \mathbb{N}$. Each Fréchet space E is the projective limit of the projective sequence of its local Banach spaces E_n . Let E_n be the completion of the normed space $E/p_n^{-1}(0)$ (endowed with the natural normed topology). The canonical map $q_n : E_n \to E$ for each $n \in \mathbb{N}$ with the spaces $(E_n)_n$ forms the projective system related with E. Especially important are the linking maps $q_{nm} : E_m \to E_n$ for $m > n$.

Recall that a Fréchet space E is *Schwarz* if the linking maps q_{nm} are compact. E is called *nuclear* if the linking maps q_{nm} are nuclear (equivalently, absolutely summable).

Since any Fréchet space is a locally convex space, the dual E' can be endowed with many different topologies; for example the weak $*$-topology $\sigma(E',E)$, the strong topology $\beta(E',E)$ or the Mackey topology $\mu(E',E)$. Note that the strong dual $E'_\beta := (E', \beta(E',E)$ is complete, but is E is not normable, the space E'_β is not metrizable. Nevertheless, it admits a fundamental system of bounded sets

$$U_n^\circ := \{f \in E' : |f(x)| \le 1, x \in U_n\}.$$

Moreover, E'_β enjoys the property that every countable bornivorous intersection of absolutely convex neighborhoods of zero in E'_β is also a neighborhood of zero. The locally convex spaces E which have these two properties (like the strong duals of Fréchet spaces) are called (after Grothendieck) (DF)-spaces. Every countable (locally convex) inductive limit of Banach spaces is a (DF)-space.

A few examples:

(1) Let Ω be an open subset of C^n and $(v_n)_n$ a decreasing sequence of positive continuous function on Ω with $\lim_{z\to\delta\Omega} v_{n+1}(z)(v_n(z))^{-1} = 0$ for all n. Define the space

$$C_n := \{f \in C(\Omega) : \|f\|_n := \sup_{z\in\Omega} |f(z)|v_n(z) < \infty\}$$

and $A_n := C_n \cap A(\Omega)$, where $A(\Omega)$ is the space of all holomorphic functions on Ω. Let $C := \bigcup_n C_n$ and $A := \bigcup_n A_n$ be endowed with the inductive limit topology. The spaces C and A are (DF)-spaces.

(2) A matrix $A = (a_{j,k})$ of non-negative numbers is called a Köthe matrix if (i) for each $j \in N$ there exists $k \in N$ with $a_{j,k} > 0$. (ii) $a_{j,k} \le a_{j,k+1}$ for all j,k.

For $1 \le p < \infty$ we define Fréchet spaces

$$\lambda^p(A) := \{x \in K^N : \|x\|_k := (\sum_j |x_j a_{j,k}|^p)^{1/p} < \infty, k \in N\};$$

and

$$\lambda^\infty := \{x \in K^N : \|x\|_k := \sup |x_j|a_{j,k} < \infty, k \in N\};$$

$$c_o(A) := \{x \in K^N : \lim_j x_j a_{j,k} = 0, k \in N\}.$$

2. Separability of E'_β for a Fréchet space E

When the strong dual of a Fréchet space is separable? It is well-known that for a normed space E the normed dual E' is separable iff every bounded set in E is $\sigma(E,E')$-metrizable. Recall that a topological vector space E is called trans-separable if for every neighbourhood of zero U in E there exists a countable set D in E such that $E = U + D$. Clearly a metrizable topological vector space E is separable iff E is trans-separable.

Theorem 2.1. *(G-K-Ku-P). Assume $(MA + \neg CH)$. Let E be a quasibarrelled lcs in class $\mathfrak{G}$ which is trans-separable. Then $(E, \sigma(E, E'))$ is cosmic. In particular, E is separable.*

Theorem 2.2. *(F-K-P). The strong dual of a lcs E is trans-separable iff every bounded set in E is $\sigma(E, E')$-metrizable.*

Theorem 2.3. *(G-K-Ku-P) Let E be a Fréchet space. The following conditions are equivalent:*
(i) E'_β is separable.
(ii) E'_β is trans-separable and every bounded set in E'_β is metrizzble.
(iii) E'_β is trans-separable and has countable tightness.
(iv) E'_β is trans-separable and every bounded set in E'_β has countable tightness.

3. classes of Fréchet spaces

Dieudonne and Schwartz called a Fréchet E *distinguished* if E'_β is bornological, i.e. every bounded linear map on E'_β to any lcs is continuous. The latter condition is equivalent to the barrelledness or quasibarrelledness of E'_β for a Fréchet space E. Horvath mentioned in his book:

It is important to know if the strong duals of the function spaces which appear in the theory of distributions have good locally convex properties. Indeed, if this is the case, one could apply to them the Closed Graph or the Open Mapping Theorem or the Uniform Boundedness Principle.

Note that the first example of a non-distinguished Fréchet space was given by Grothendieck and Köthe, and it was the Köthe echelon space $\lambda^1(A)$ for a certain Köthe matrix A. The Köthe echelon spaces which are distinguished were characterized by Bierstedt, Bonet and Meise in the late 80' s. Taskinen showed that the Fréchet space $C(R) \cap L_1(R)$ endowed with the natural intersection topology is not distinguished. More general fact:

Theorem 3.1. *(Taskinen) For every open subset G of R^N the intersection space $C^\infty(G) \cap L_1(G)$ is not distinguished*

Grothendieck proved that the non-distinguished Köthe echelon space E mentioned above even has the property that there is a discontinuous linear form on E'_β which is bounded on the bounded subsets of E'_β. A simple proof of this fact has been proved also by Kąkol and Saxon by using the concept of countable tightness. Valdivia proved that if E is a separable Fréchet space which does not contain a copy of ℓ_1 then $(E', \mu(E', E''))$ is bornological.

Question 3.2. *Is every separable Fréchet space not containing ℓ_1 distinguished?*

Diaz answered this question in the negative. Note that if E'_β is separable, then E is distinguished (Grothendieck). Grothendieck proved that if the bounded subsets in E'_β are metrizable, then E is distinguished. The class of Fréchet spaces E for which the strong dual has metrizable bounded sets coincides with the class of Frćhet spaces satisfying the *Heinrich density condition* and contains the class of Montel spaces and quasinormable spaces.

A Fréchet space E is *reflexive* iff every bounded subset of E is relatively $\sigma(E, E')$-compact. E is called *Montel* (shortly (FM)-space), if each bounded subset of E is relatively compact. Every (FM)-space reflexive. Kothe and Grothendieck gave examples of (FM)-spaces with a quotient topologically isomorphic to ℓ_1, hence not reflexive. A Fréchet space E is called *totally reflexive* if every quotient of E is reflexive. Valdivia proved: A Fréchet space E is totally reflexive if and only if E is the projective limit of a sequence of reflexive Banach spaces.

Recall that the theorem Josefson and Nissenzweig states that :

A Banach space E is finite dimensional iff every null-sequwence in $(E', \sigma(E', E))$ which converges to zero for the norm topology of E'.

How to extend this theorem to Fréchet spaces?

Theorem 3.3. *(Bonet, Lindström, Schlumprecht, Valdivia) Let E be a Fréchet space.*

(i) E is Montel iff $\sigma(E', E)$-null sequence in E' is strongly convergent to zero.

(ii) E does not contain a copy of ℓ_1 iff every null sequence in $(E', \sigma(E', E))$ strongly converges to zero.

The latter condition has been extended in 2014 by Ruess to sequentially complete lcs whose bounded sets are metrizable.

4. A few words about bases in Fréchet spaces

Every Fréchet nuclear space (shortly (FN)-space) has the approximation property. For a long time it was an open problem whether there exists a (FN)-space without a basis. The first example of such a space was provided by Mitjagin and Zobin. Grothendieck asked if every (FN)-space has the bounded approximation property. Dubinsky answered in the negative.

(i) It is a classical problem, but still open, whether every complemented subspace of an (FN)-space with a basis must itself have a basis.

(ii) Taskinen (applying methods due to Mitjagin, Zobin and Peczynski) constructed an (FS)-space with a basis and with a complemented subspace which is (FN) and does not have a basis.

(iii) Every nuclear space is Schwartz. In 1973 Hogbe-Nlend used Enflos example to construct a Fréchet Schwartz space without the approximation property. Peris gave an example of a Frechet Schwartz space with the approximation property, but without approximable linking maps, thus answering a problem of Ramanujan in the negative.

(iv) Domanski and Vogt show that if G is an open subset of R^n, then the separable, complete and nuclear space $A(G)$ of all real analytic functions on G, endowed with its natural topology, does not have a Schauder basis.

(v) Floret and Moscatelli proved : Every Fréchet space with an unconditional basis is topologically isomorphic to a countable product of Fréchet spaces with a continuous norm and unconditional basis.

(vi) A Fréchet space E is called a *quojection* if it is the projective limit of a sequence of Banach spaces with surjective linking maps or, equivalently, if every quotient with a continuous norm is a Banach space for the quotient topology. Every Fréchet space $C_c(X)$ of continuous functions is a quojection. Every quojection is a quotient of a countable product of Banach spaces.

(vii) It is well-known that every non-normable Fréchet space E admits a quotient isomophic to ω and E has a subspace topologically isomorphic to ω iff it does not admit a continuous norm. Bessaga, Pelczynski and Rolewicz showed that a Fréchet space contains a subspace which is topologically isomorphic to an infinite dimensional nuclear Fréchet space with basis and a continuous norm iff it is not isomorphic to the product of a Banach space and ω. As a consequence of the results mentioned above, every non-normable Fréchet space contains a subspace which can be written in the form $X \bigoplus Y$ for X and Y infinite dimensional.

5. the Heinrich density condition

Condition (ii) (second part) in Theorem 2.3 motivates this section to say a few words about very applicable concept (due to Heinrich) called the *density condition.*

Let E be a Fréchet space and let $B(E)$ be the set of all bounded sets in E. We say (Heinrich) that E satisfies the *density condition* if there exists a bounded set B in E such that for each $n \in N$ and each $C \in B(E)$ there exists $\lambda > 0$ such that

$$C \subset \lambda B + U_n.$$

Every quasinormable, as well as, every (FM)-space has the density condition. Every Fréchet space with the density condition has a total bounded set, or equivalently, E'_β admits a continuous norm. Every Fréchet space with the density condition is distinguished.

Theorem 5.1. *(Bierstedt-Bonet) Let E be a Fréchet space.*

(i) E has the density condition iff every bounded set in E'_β is metrizable.

(ii) E has the density condition iff and only if the Fréchet space $\ell_1(E)$ is distinguished.

6. $\aleph_0$-spaces and the weak topology of a Fréchet space

We are partially motivated by question 3.2 of Valdivia mentioned above about distinguished spaces and the following remarkable example of James: The (James) tree JT is a separable Banach space whose dual E' is nonseparable and JT contains a copy of ℓ_1. Note also the following easy fact due to Michael and Tanaka: *Let E be a regular topological space. Then E is a continuous image of a compact-covering map from a metric space iff every compact set in E i metrizable.*

Corollary 6.1. *Let E be a metrizable and separable lcs. Then $(E, \sigma(E, E'))$ is a continuous image of a compact-covering map from a metric space.*

The class of $\aleph_0$-spaces in sense of Michael is the most immediate extension of the class of separable metrizable spaces. A regular space E is called an $\aleph_0$-space it it is a continuous image of a compact-covering map from a metric separable space. Michael proved that $(E, \sigma(E, E'))$ is an $\aleph_0$-space if E' is separable. If X is an uncountable compact space then the Banach space $C(X)$ endowed with their weak topology is not an $\aleph_0$-space (Corson). If E is a metrizable lcs or a (DF)-space whose dual E'_β is separable, then $(E, \sigma(E, E'))$ is an $\aleph_0$-space (G-K-Ku-P).

If E is a Banach space with the Schur property, then every infinite subspace of E contains ℓ_1 (Rosenthal). But then, if E is additionally separable, the space $(E, \sigma(E, E'))$ is an $\aleph_0$-space. What is the situation if E does not contain ℓ_1?

Theorem 6.2. *(G-K-Ku-P). Let E be a separable Fréchet space not containing ℓ_1 and satisfying the density condition. Then $(E, \sigma(E, E'))$ is an $\aleph_0$-space iff E'_β is separable.*

Corollary 6.3. *Let E be a separable Banach space not containing ℓ_1.*

(A) E' is separable iff $(E, \sigma(E, E'))$ is an $\aleph_0$-space.

(B) E' is separable and B_E is a Polish space in the topology $\sigma(E, E')$ iff $(E, \sigma(E, E'))$ is a continuous image of a compact-covering map from a Polish space.

Part (A) has been also observed by T. Banakh (for Banach spaces). Since for a Banach space E whose bidual E'' is separable the unit ball B_E is Polish in $\sigma(E, E')$ (Godefroy) we derive:

Corollary 6.4. *Let E be a Banach space with E'' separable. Then $(E, \sigma(E, E'))$ is a continuous image of a compact-covering map from a Polish space.*

Remark 6.5. *Note that the sequence space $E := c_0$ endowed with the weak topology is not a continuous image of a compact-covering map from a Polish space. Indeed, otherwise, since E' is separable B_E is a metrizable and separable space in $\sigma(E, E')$, so B_E would be a Polish space by applying the Christensen's theorem. Nevertheless, E with the weak topology is an $\aleph_0$-space. Note also that if E is a Fréchet space, then $(E, \sigma(E, E'))$ admits a compact resolution swallowing the compact sets iff there exists a usco map $T : N^N \to \mathcal{K}(E, \sigma(E, E'))$ compact-covering.*

Question 6.6. *Characterize those Banach spaces which under their weak topology are continuous images of a compact-covering map from a Polish space.*

Question 6.7. *Let X be a regular space. Is it rue that X has a compact resolution swallowing the compact sets iff X is continuous images of a compact-covering map from a Polish space?*

7. A few words about the proof of Theorem 6.2

$\Rightarrow$ Assume that E is a Fréchet space whose dual E'_β is separable. Then E is also separable and $(E'', \sigma(E'', E'))$ is submetrizable. Recall that E'_β is a (DF)-space, and since E'_β is separable, every bounded set in E'_β is metrizable; so E is distinguished. Consequently E'_β is barrelled. We apply Cascales-Kakol-Saxon theorem (PAMS) implying that then E'_β admits so-called a $\mathfrak{G}$-base, i.e. a basis $\{U_\alpha : \alpha \in N^N\}$ of absolutely convex neighbourhoods of zero such that $U_\alpha \subset U_\beta$ whenever $\beta \leq \alpha$. For each $\alpha \in N^N$ let W_α be the polar of the set U_α for each $\alpha \in N^N$. Then $\{W_\alpha : \alpha \in N^N\}$ forms a $\sigma(E'', E')$-compact resolution whose union is E" and (by barreledness of E'_β) every $\sigma(E'', E')$-compact set in E'' is contained in some W_α (we say: this resolution swallows compact sets). Now a very recent result of Cascales-Orihuela-Tkachuk applies to deduce that $(E'', \sigma(E'', E'))$ is an $\aleph_0$-space. Hence $(E, \sigma(E, E'))$ is an $\aleph_0$-space.

$\Leftarrow$ Assume that E does not contain ℓ_1 and $(E, \sigma(E, E'))$ is an $\aleph_0$-space. We need the following lemmas (extending recent results of Kalenda and all)

Lemma 7.1. *Let E be a topological vector space (resp. topological group) such that every bounded (resp. precompact) set is Fréchet-Urysohn. Then, every bounded (resp. precompact) set has the property (α_4) and therefore it is strongly Fréchet-Urysohn.*

A lcs E will be said to have *the Rosenthal property* if every bounded sequence in E either (R_1) has a subsequence which is Cauchy in the weak topology $\sigma(E, E')$, or (R_2) has a subsequence which is equivalent to the unit vector basis of ℓ_1. Recently, Ruess proved the following

Lemma 7.2. *(Ruess) Every sequentially complete lcs E whose every bounded set is metrizable has the Rosenthal property.*

Lemma 7.3. *Let E be a metrizable lcs. Then every bounded, separable set of E having the Rosenthal property (R_1) is Fréchet-Urysohn in the weak topology.*

Note that the above lemma also holds for any lcs in class $\mathfrak{G}$ (which includes all metrizable lcs).

Corollary 7.4. *Let E be the strong dual of a metrizable lcs F with the Heinrich density condition. Then the following assertions are equivalent:*

(i) *Every bounded set is Fréchet-Urysohn in the topology $\sigma(E, E')$.*
(ii) *Every bounded sequence in E contains a weakly Cauchy subsequence.*
(iii) *E does not contain ℓ_1.*

Now, if E is Fréchet space and $(E, \sigma(E, E'))$ is an $\aleph_0$-space and E does not contain ℓ_1. We apply Lemma 7.3 to see that every bounded set in E is Fréchet-Urysohn in $\sigma(E, E')$. By Lemma 7.1 we deduce that every bounded set B in $\sigma(E, E')$ is strongly Fréchet-Urysohn; consequently by Michael's theorem B is second countable, hence metrizable. We showed that E'_β is trans-separable. Finally Theorem 2.3 yields that E' is separable

8. WCG Fréchet spaces

We call a Fréchet space E a *Weakly Compactly Generated* (WCG) if E admits a sequence $(G_n)_n$ of weakly compact sets in E whose union is dense in the weak topology. We may assume that sets G_n are absolutely convex.

Theorem 8.1. *(Khurana) Every WCG Fréchet space is weakly K-analytic.*

Corollary 8.2. *(K-P) Let E be a Baire WCG lcs. Then E is a Fréchet space iff $(E, \sigma(E, E'))$ is K-analytic.*

Next fact is a version of the Amir-Lindestrauss theorem for WCG Banach spaces. More general cases were studied in the class $\mathfrak{G}$ by Cascales, Orihuela, Canela, Kakol.

Theorem 8.3. *For a WCG Fréchet space E we have $d(E, \sigma(E, E')) = d(E', \sigma(E', E))$.*