Journal of Convex Analysis
Volume 20 (2013), No. 1, 233–242

Non-Archimedean Quantitative Grothendieck and Krein's Theorems

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Received: May 5, 2011
Revised manuscript received: May 20, 2012

We show that the non-archimedean version of Grothendieck's theorem about weakly compact sets for $C(X,\mathbb{K})$, the space of continuous maps on X with values in a locally compact non-trivially valued non-archimedean field $\mathbb{K}$, fails in general. Indeed, we prove that if X is an infinite zero-dimensional compact space, then there exists a relatively compact set $H := \{g_n : n \in \mathbb{N}\} \subset C(X,\mathbb{K})$ in the pointwise topology τ_p of $C(X,\mathbb{K})$ which is not $w-$relatively compact, i.e. compact in the weak topology of $C(X,\mathbb{K})$, such that all $\|g_n\| = 1$ and $\gamma(H) := \sup\{|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| : (f_m)_m \subset B, (x_n)_n \subset H\} > 0$, where B is the closed unit ball in the dual $C(X,\mathbb{K})^*$ and the involved limits exist. The latter condition $\gamma(H) > 0$ shows in fact that a quantitative version of Grothendieck's theorem for real spaces (due to Angosto and Cascales) fails in the non-archimedean setting. The classical Krein and Grothendieck's theorems ensure that for any compact space X every uniformly bounded set H in a real (or complex) space $C(X)$ is τ_p-relatively compact if and only if the absolutely convex hull $\operatorname{aco} H$ of H is τ_p-relatively compact. In contrast, we show that for an infinite zero-dimensional compact space X the absolutely convex hull $\operatorname{aco} H$ of a τ_p-relatively compact and uniformly bounded set H in $C(X,\mathbb{K})$ needs not be τ_p-relatively compact for a locally compact non-archimedean $\mathbb{K}$. Nevertheless, our main result states that if $H \subset C(X,\mathbb{K})$ is uniformly bounded, then $\operatorname{aco} H$ is τ_p-relatively compact if and only if H is w-relatively compact.

Keywords: Grothendieck's theorem, Krein's theorem, locally compact non-archimedean field, cmpactness, space of continuous functions

1991 Mathematics Subject Classification: 46S10, 46A50, 54C35

1. Introduction

Let X be a compact Hausdorff space and $C(X) := C(X,\mathbb{R})$ the Banach space of real-valued continuous functions on X. If $H \subset C(X) \subset \mathbb{R}^X$ is a pointwise

*The research of was supported for the first named author by National Center of Science, Poland, grant no. N N201 605340.

ISSN 0944-6532 / $ 2.50

bounded set, then the closure $\mathrm{cl}_{\mathbb{R}^X}(H)$ is compact in the topology τ_p of pointwise convergence in $\mathbb{R}^X$.

If $\hat{d}(\mathrm{cl}_{\mathbb{R}^X}(H), C(X)) := \sup\{d(f, C(X)) : f \in \mathrm{cl}_{\mathbb{R}^X}(H)\}$, where d is the standard supremum metric, then $\hat{d} = 0$ iff $\mathrm{cl}_{\mathbb{R}^X}(H) \subset C(X)$ iff H is τ_p-relatively compact in $C(X)$. Therefore the relation $\hat{d} > 0$ provides a measure of non-τ_p-compactness for H in $C(X)$. Let $\gamma(H)$ be defined as in the abstract. It is known $k(H) \leq \gamma(H) \leq 2k(H)$, where $k(H) := \sup_{x^{**} \in \mathrm{cl}_{\omega^*}(H)} d(x^{**}, C(X, \mathbb{R}))$, and d is the usual inf distance for sets associated to the natural norm in the bidual of $C(X, \mathbb{R})^{**}$, see [1, Theorem 2.3].

Grothendieck proved that a bounded set H in the Banach space $C(X)$ is relatively compact in the pointwise topology τ_p iff it is relatively compact in the weak topology w of $C(X)$, see [4], [3, Theorem 4.2]. Angosto and Cascales proved in [1, Theorem 3.5] a quantitative version of this result by showing the inequalities

$$\gamma_X(H) \leq \gamma(H) \leq 2\gamma_X(H), \qquad (*)$$

where $\gamma_X(H) := \sup\{|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| : (f_m)_m \subset H, (x_n)_n \subset X\}$, provided the iterated limits exist. Since $\gamma_X(H) = 0$ iff H is τ_p-relatively compact and $\gamma(H) = 0$ iff H interchanges limits with the dual ball B (see [2, Corollary 2.5]), the above inequalities yield the Grothendieck's theorem.

The following question is natural: Are the inequalities $(*)$ still true for spaces $C(X, \mathbb{K})$ of $\mathbb{K}$-valued continuous maps on a compact X for any non-trivially valued complete field $\mathbb{K}$?

Throughout this note by $\mathbb{K}$ we mean a non-trivially valued non-archimedean locally compact field. Recall that a complete valued field $(\mathbb{K}, |.|)$ not topologically isomorphic to the field $\mathbb{R}$ or $\mathbb{C}$ is non-archimedean, i.e. $|x + y| \leq \max\{|x|, |y|\}$, $x, y \in \mathbb{K}$, see [10]. Theorem 2.1 shows that if X is a zero-dimensional compact space the above question for $C(X, \mathbb{K})$ has a negative answer.

The classical Krein's theorem yields (see [3, Theorem 7.1]) that in a real or complex Banach space E the absolutely convex hull $\mathrm{aco}\, H$ is $w-$relatively compact for a $w-$relatively compact set $H \subset E$. This, combined with Grothendieck's theorem for $C(X)$ over compact X allows to show that every uniformly bounded set $H \subset C(X)$ is τ_p-relatively compact iff $\mathrm{aco}\, H$ is τ_p-relatively compact. A quantitative version of this theorem has been proved recently by Cascales, Marciszewski and Raja (see [2, Theorem 3.3]). In contrast to the real case, being motivated by the first part, we show that for an infinite zero-dimensional compact space X the absolutely convex hull $\mathrm{aco}\, H$ of a uniformly bounded τ_p-relatively compact set H in $C(X, \mathbb{K})$ needs not be τ_p-relatively compact. In fact we prove also that if X is an ultrametrizable infinite compact space and $(g_n)_n \subset C(X, \mathbb{K})$ is an orthogonal sequence for which $||g_n|| = 1$ for all $n \in N$, then there exist sequences $(w_m)_m \subset X$ and $(f_n)_n \subset \mathrm{aco}\,\{g_1, g_2, ...\}$ such that $\lim_n \lim_m f_n(w_m) \neq \lim_m \lim_n f_n(w_m)$, see Theorem 2.6. Fortunately, our main Theorem 2.8 states that if X is an infinite zero-dimensional compact space and $H \subset C(X, \mathbb{K})$ is uniformly bounded, then $\mathrm{aco}\, H$ is τ_p-relatively compact iff H is $w-$relatively compact.

Let $B_{\mathbb{K}}$ denote the set $\{\alpha \in \mathbb{K} : |\alpha| \leq 1\}$. Let E be a linear space. A set $A \subset E$ is *absolutely convex* if for any $\alpha, \beta \in B_{\mathbb{K}}$ and $x, y \in A$ we have $\alpha x + \beta y \in A$. If $A \subset E$, then the set $\operatorname{aco} A = \{\sum_{i=1}^{n} \alpha_i a_i : n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in B_{\mathbb{K}}, a_1, \ldots, a_n \in A\}$ is called the absolutely convex hull of A. A metric space (X, d) is called *ultrametric* if $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y \in X$. Note that every ultrametric space is zero-dimensional. Recall that a clopen set $B(x_0, r) = \{x \in X : d(x, x_0) \leq r\}$ is called a "closed" ball. Note that any two balls in X are either disjoint, or one is contained in the other.

2. Results and Proofs

First we prove the following promised

Theorem 2.1. *Let X be an infinite compact zero-dimensional space. Then there exists a τ_p−relatively compact set $H := \{g_n : n \in \mathbb{N}\}$, not relatively weakly compact in $C(X, \mathbb{K})$, such that all $\|g_n\| = 1$ and $\gamma(H) > 0$.*

Proof. Since X is compact and infinite, there exists $x \in X$ which is not isolated. Let $U_1 := U$ be a clopen neighbourhood of x. Since $U \neq \{x\}$, there are an $x_1 \in U \backslash \{x\}$ and a clopen neighbourhood U_2 of x such that $U_2 \subset U$ and $x_1 \in U \backslash U_2$. Then $U_2 \neq \{x\}$ and we find a clopen neighbourhood U_3 of x with $U_3 \subset U_2$ and an $x_2 \in U_2 \setminus U_3$. Continuing this procedure we construct a sequence $x_1, x_2, \ldots$ in X and a decreasing sequence $(U_n)_n$ of clopen subsets of X such that $x_n \in U_n \setminus U_{n+1}$ for all $n \in \mathbb{N}$.

Since each set U_n is clopen, for each $n \in \mathbb{N}$ the function $f_n : X \to \mathbb{K}$ defined by $f_n(x) := \mathbf{1}_{U_n}(x)$, $x \in X$, is continuous. If $x \in \bigcap_n U_n$, then $f_n(x) \to 1$. If $x \notin \bigcap_n U_n$, then $f_n(x) \to 0$. For every $n \in \mathbb{N}$ set $g_n(x) := f_n(x) - f_{n+1}(x)$, $x \in X$. Then $g_n \to 0$ for each $x \in X$. Moreover, $1 \geq \|g_n\| \geq |f_n(x_n) - f_{n+1}(x_n)| = 1$, so $\|g_n\| = 1$ for all $n \in \mathbb{N}$. Set $H := \{g_n : n \in \mathbb{N}\}$. The only cluster point of H in $\mathbb{K}^X$ is a zero function, obviously continuous; hence, H is τ_p−relatively compact. But, H is not relatively compact in the weak topology of $C(X, \mathbb{K})$. Indeed, otherwise $g_n \to 0$ in the weak topology of $C(X, \mathbb{K})$. Since in $C(X, \mathbb{K})$ every weakly converging sequence converges in the norm (see [5, Corollary 2.5]), we reach a contradiction as $\|g_n\| = 1$ for each $n \in \mathbb{N}$.

Let D be the linear span of H in $C(X, \mathbb{K})$. Define $g_n^* \in D'$ by $g_n^*(g_m) := 1$ if $n = m$ and $g_n^*(g_m) := 0$ if $n \neq m$. Using [10, Corollary 3.18], for every $n \in N$ we extend g_n^* to the whole $C(X, \mathbb{K})$. For every $n \in N$ define a continuous linear functional $h_n^* := g_1^* + \ldots + g_n^*$ on $C(X, \mathbb{K})$. Observe that $h_n^*(g_m) = 1$ if $m \leq n$ and $h_n^*(g_m) = 0$ if $m > n$, so for every $m \in N$ we have $\lim_n h_n^*(g_m) = 1$ and $\lim_m h_n^*(g_m) = 0$ for each $n \in N$. Thus $\lim_n \lim_m h_n^*(g_m) \neq \lim_m \lim_n h_n^*(g_m)$, so $\gamma(H) > 0$. □

The first observation concerning a non-archimedean Krein's theorem is the following

Proposition 2.2. *Let E be a Banach space over $\mathbb{K}$ and $A \subset E$ be a weakly relatively compact set. Then, $\operatorname{aco} A$ is weakly relatively compact.*

Proof. Assume that $A \subset E$ is weakly relatively compact. Then, by [9, Theorem 5.2] the set $\overline{A}$ is compact. By [8, Proposition 1.6] and [7, Proposition 2.1], $\overline{\text{aco}}A$ is precompact, thus compact. Hence, we conclude that $\overline{\text{aco}}A$, thus $\text{aco}\, A$, is weakly relatively compact. □

Example 2.3. If H is the set constructed in the proof of Theorem 2.1, then $\text{aco}\, H$ is not τ_p−relatively compact.

Proof. For every $m \in N$ define $h_m := g_1 + ... + g_m$; then, $(h_m)_m \subset \text{aco}\, H$. Choose a sequence $(x_n)_n \subset X$ with $x_n \in U_n \backslash U_{n+1}$ for each $n \in N$. Then $\lim_m h_m(x_n) = 1$ for all $n \in N$; hence $\lim_n \lim_m h_m(x_n) = 1$. Since $\lim_n h_m(x_n) = 0$ for all $m \in N$, we have $\lim_m \lim_n h_m(x_n) = 0$. By [2, Corollary 2.5], the set $\text{aco}\, H$ is not τ_p−relatively compact. □

Theorem 2.1 and Example 2.3 suggest the following question. Is $H \subset C(X, \mathbb{K})$ weakly relatively compact iff $\text{aco}\, H$ is τ_p−relatively compact? Theorem 2.8 provides an affirmative answer. To prove this result, we need a couple of additional facts of own interest.

Lemma 2.4. *Let (X, d) be an infinite ultrametric compact space, V be a closed ball in X and $x_0 \in V$.*

(*1*) *If x_0 is an isolated point of X, then there exists a finite partition $\{V_1, ..., V_n\}$ of $V \backslash \{x_0\}$ consisting of closed balls.*

(*2*) *If x_0 is an accumulation point of X and $(x_n)_n \subset V$ is a sequence convergent to x_0 for which $d(x_n, x_{n-1}) > d(x_{n+1}, x_n)$ for all $n \in N$, then*
 (*i*) *there exists an infinite partition $(V_j)_j$ of $V \backslash \{x_0\}$ consisting of closed balls such that for every $n \in N$ there exists $K_n \in N$ with $x_n \in V_{K_n}$, $K_p \neq K_q$ if $p \neq q$ and the radius of V_{K_n} is equal to $d(x_{n+1}, x_{n+2})$;*
 (*ii*) *there exist a maximal collection of closed balls $(U_n)_n \subset X$ for which $(\mathbf{1}_{U_n})_n$ is an orthonormal base of $C(X, \mathbb{K})$ and a subsequence $(p_m)_m \subset N$ such that*

$$x_0 \in U_1 \backslash \bigcup_{n>1} U_n, \quad x_m \in (U_1 \cap U_{p_m}) \backslash \bigcup_{n \in N \backslash \{1, p_m\}} U_n. \qquad (**)$$

Proof. (*1*): Suppose that x_0 is an isolated point. Then, there exists $r > 0$ such that $\{z \in X : d(z, x_0) \leq r\} = \{x_0\}$. By compactness of V we can choose a finite partition $\{V_0, V_1, ..., V_n\}$ of V consisting of balls with radius r. Without loss of generality, we can assume that $x_0 \in V_0$. Thus $\{V_1, ..., V_n\}$ is a required partition of $V \backslash \{x_0\}$.

(*2*): Assume that x_0 is an accumulation point. By compactness of V we find a finite partition $W_{1,0}, ..., W_{1,m_1}$ of V with $x_2 \in W_{1,0}$, where $W_{1,k}$ $(k = 0, ..., m_1)$ are closed balls with radius $d(x_2, x_3)$. Then, since

$$\begin{aligned} d(x_2, x_n) \leq\ & \max\{d(x_2, x_3), d(x_3, x_4), ..., d(x_{n-1}, x_n)\} \\ &= d(x_2, x_3),\ d(x_2, x_1) > d(x_2, x_3), \end{aligned}$$

we have $\{x_0, x_2, x_3, ...\} \subset W_{1,0}$ and $x_1 \notin W_{1,0}$. Clearly $x_1 \in W_{1,l_1}$ for some $l_1 \in \{1, ..., m_1\}$. Set $K_1 := l_1$ and $V_j := W_{1,j}$ for $j = 1, ..., m_1$. Continuing on this direction in the $n-$th step ($n > 1$) we choose a finite partition $W_{n,0}, ..., W_{n,m_n}$ of $W_{n-1,0}$ by balls with radius $r_n = d(x_{n+1}, x_{n+2})$ such that $x_{n+1} \in W_{n,0}$. Then, $x_n \in W_{n,l_n}$ for some $l_n \in \{1, ..., m_n\}$ and $\{x_0, x_{n+1}, x_{n+2}, ...\} \subset W_{n,0}$. We set $K_n := m_1 + ... + m_{n-1} + l_n$ and $V_{m_1+...+m_{n-1}+j} := W_{n,j}$ for $j = 1, ..., m_n$. This leads to an infinite sequence $(V_j)_j$ of closed balls, which is a partition of $U\backslash \{x_0\}$. Then $x_n \in V_{K_n}$, where V_{K_n} is a ball with radius $d(x_{n+1}, x_{n+2})$ and $n \in \mathbb{N}$. The proof of (a) is completed.

Finally, we prove (b). Applying (*2a*), we generate an infinite partition $(V_j)_j$ of $X\backslash \{x_0\}$, and a subsequence $(K_n)_n \subset N$ such that $x_n \in V_{K_n}$ for every $n \in N$ ($K_p \neq K_q$ if $p \neq q$); V_{K_n} is a ball with radius $d(x_{n+1}, x_{n+2})$. Define $J_0 := \{K_n : n \in N\}$ and $J := \{n \in N : x_n \text{ is an accumulation point in } X\}$. For every $n \in J$, selecting any sequence $(y^n_m)_m \subset X$ converging to x_n with $d(y^n_m, {y^n}_{m-1}) > d({y^n}_{m+1}, y^n_m)$ ($m \in N$) and applying again (*2a*), we form $(W^n_m)_m$, an infinite partition of $V_{K_n}\backslash \{x_n\}$. Using (*1*) for every $n \in N\backslash J$ we generate a finite partition $(W^n_m)^{M_n}_{m=1}$ of $V_{K_n}\backslash \{x_n\}$ for some $M_n > 0$. Define the collections of balls

$$\Im_1 := \{V_n : n \in N\backslash J_0\}, \qquad \Im_2 := \{V_n : n \in J_0\},$$
$$\Im_3 := \{W^n_m : n \in J,\ m \in N\}, \qquad \Im_4 := \{W^n_m : n \in N\backslash J,\ m = 1, ..., M_n\}.$$

Next, for each $S \in \Im_1 \cup \Im_3 \cup \Im_4$ we find $\Im_S$, a maximal collection of balls contained in S, such that $\{\mathbf{1}_S\} \cup \{\mathbf{1}_U : U \in \Im_S\}$ is linearly independent. Let $\Im' := \bigcup_{S\in\Im_1\cup\Im_3\cup\Im_4} \Im_S$. Since elements of $\Im_1 \cup \Im_3 \cup \Im_4$ are pairwise disjoint and $\{x_0, x_1, ...\} \cap \bigcup_{S\in\Im_1\cup\Im_3\cup\Im_4} S = \emptyset$, the set

$$\{\mathbf{1}_U : U \in \Im' \cup \Im_1 \cup \Im_2 \cup \Im_3 \cup \Im_4\} \cup \{\mathbf{1}_X\}$$

is also linearly independent. Moreover,

$$\Im := \Im' \cup \Im_1 \cup \Im_2 \cup \Im_3 \cup \Im_4 \cup \{X\}$$

is a maximal collection of closed balls of X such that the set of characteristic functions $\{\mathbf{1}_U : U \in \Im\}$ is linearly independent. Indeed, take a ball $V \subset X$, $V \notin \Im$ and suppose that $\{\mathbf{1}_U : U \in \Im\} \cup \{\mathbf{1}_V\}$ is linearly independent. Then, by the construction of $\Im'$, there is no $U \in \Im' \cup \Im_1 \cup \Im_3 \cup \Im_4$ such that $V \subset U$. If $x_0 \in V$, then the collection $\{V\} \cup \Im_1 \cup \Im_2$ covers X. By compactness of X, we find $U_{p_1}, .., U_{p_n} \in \{V\} \cup \Im_1 \cup \Im_2$, a finite partition of X. Thus, the characteristic functions $\mathbf{1}_X, \mathbf{1}_{U_{p_1}}, ..., \mathbf{1}_{U_{p_n}}$ are not linearly independent. If $x_0 \notin V$, the collection $\Im_1 \cup \Im_2$ covers V. Since V is compact, we choose $U_{q_1}, .., U_{q_n} \in \Im_1 \cup \Im_2$, which form a finite partition of V; hence $\mathbf{1}_V, \mathbf{1}_{U_{q_1}}, ..., \mathbf{1}_{U_{q_n}}$ are not linearly independent, a contradiction.

By [6, Theorem 2.5.27], the family $\{\mathbf{1}_U : U \in \Im\}$ is an orthonormal base of $C(X, \mathbb{K})$. Since X is ultrametrizable, $\{\mathbf{1}_U : U \in \Im\}$ is countable by [6, Theorem 2.5.24]. We can write $\Im = \{U_1, U_2, ...\}$ and assume that $U_1 = X$. Clearly, we can select $(p_m)_m \subset N$ such that $\{U_{p_m} : m \in N\} = \Im_2$. Then, it follows from the above construction that $(**)$ is satisfied. □

Lemma 2.5. *Let X be an infinite ultrametrizable compact set and let $(f_n)_n \subset C(X,\mathbb{K})$ $(f_n \neq 0)$ be an orthogonal sequence. Then, there exists a sequence $(x_n)_n \subset X$ with $|f_n(x_n)| = ||f_n||$ and $x_n \neq x_m$ if $n \neq m$.*

Proof. Since X is compact, for every $n \in N$ there is $x \in X$ that $|f_n(x)| = ||f_n||$. Take $n \in N$ and suppose that $X_0 := \{x \in X : |f_k(x)| = ||f_k||,\ k = 1,...,n+1\}$ has at most n elements, say $X_0 = \{x_1,...,x_n\}$.

Without loss of generality we may assume that $||f_1|| = ... = ||f_{n+1}||$ and $|f_1(x_1)| = ||f_1||$. For each $k = 2,...,n+1$ set

$$f_k^1 := f_k - \frac{f_k(x_1)}{f_1(x_1)} f_1.$$

Then, for every $k \in \{2,...,n+1\}$ we get $f_k^1(x_1) = 0$,

$$||f_k^1|| = \max\left\{||f_k||, \left|\frac{f_k(x_1)}{f_1(x_1)}\right| \cdot ||f_1||\right\} = ||f_1||$$

by orthogonality of $(f_n)_n$ and $|f_k^1(x)| < ||f_k^1||$ if $x \notin X_0$. Next, for every $j = 2,...,n$ and $k = j+1,...,n+1$ define

$$f_k^j := f_k^{j-1} - \frac{f_k^{j-1}(x_{n_j})}{f_j^{j-1}(x_{n_j})} f_j^{j-1},$$

where $n_j := \min\left\{i \in \{1,...,n\} : \left|f_j^{j-1}(x_i)\right| = \left|\left|f_j^{j-1}\right|\right|\right\}$. Clearly, $\left|\left|f_k^j\right|\right| = ||f_1||$, $f_k^j(x_i) = 0$ for $i \in \{1,n_2,...,n_j\}$ and $\left|f_k^j(x)\right| < \left|\left|f_k^j\right|\right|$ if $x \notin X_0$. Hence, $f_{n+1}^n(x_i) = 0$ if $i \in \{1,...,n\}$ and $\left|f_{n+1}^n(x)\right| < \left|\left|f_{n+1}^n\right|\right|$ if $x \notin X_0$. But, $\sup_{x \notin X_0}\left|f_{n+1}^n(x)\right| < \left|\left|f_{n+1}^n\right|\right|$ since K is discretely valued, a contradiction. □

Theorem 2.6. *Let X be an ultrametrizable infinite compact set and let $(g_n)_n \subset C(X,\mathbb{K})$ be an orthogonal sequence for which $||g_n|| = 1$ for all $n \in N$. Then, there exist sequences $(w_m)_m \subset X$ and $(f_n)_n \subset \operatorname{aco}\{g_1,g_2,...\}$ such that*

$$\lim_n \lim_m f_n(w_m) \neq \lim_m \lim_n f_n(w_m).$$

Proof. First, applying Lemma 2.5, choose a sequence $(x_n)_n \subset X$ such that $|g_n(x_n)| = 1$ and $x_n \neq x_m$ if $n \neq m$ $(n,m \in N)$. Since X is compact, we can select a subsequence $(x_{n_k})_k$ of $(x_n)_n$, convergent to some $x_0 \in X$. $\mathbb{K}$ is discretely valued, thus, there exists $j_0 \in N$ for which $|g_{j_0}(x_0)| = \max_k |g_{n_k}(x_0)|$. Let $k_0 := \max\{k \in N : n_k \leq j_0\}$. For every $k > k_0$, $k \in N$, define

$$g'_{k-k_0} := g_{n_k} - \frac{g_{n_k}(x_0)}{g_{j_0}(x_0)} g_{j_0} \quad \text{and} \quad y_{k-k_0} := x_{n_k}.$$

Clearly, for every $n \in N$ we have $g'_n \in \operatorname{aco}\{g_1,g_2,...\}$, $||g'_n|| = 1$ by orthogonality of $(g_n)_n$ and $g'_n(x_0) = 0$. Using Lemma 2.4 (*2b*) for a sequence $\{x_0,y_1,y_2,...\}$ we get a sequence of closed balls $(U_n)_n$ and a sequence $(p_m)_m \subset N$ such that $(\xi_n)_n$, $\xi_n := \mathbf{1}_{U_n}$ $(n \in N)$, form an orthonormal base of $C(X, \mathbb{K})$, $\xi_n(x_0) = 1$ iff $n = 1$ and for every $m \in N$ $\xi_n(y_m) = 1$ iff $n \in \{1, p_m\}$; for every $n \in N$ we can write $g'_n = \sum_{k=1}^{\infty} \lambda_k^n \xi_k$ for some $(\lambda_k^n)_k \subset \mathbb{K}$.

First, we find m_1 such that $|\lambda_k^1| < 1$ for all $k \geq p_{m_1}$; clearly, $\left|g'_{m_1}(y_{m_1})\right| = 1$. Next, inductively we choose a subsequence $(m_i)_i \subset N$, selecting $m_i \in N$ in the $i-$th $(i > 1)$ step such that $p_{m_i} > p_{m_{i-1}}$ and

$$\left|\lambda_k^{m_j}\right| < 1 \quad (j = 1, ..., i-1)$$

for all $k \geq p_{m_i}$. Obviously, $\left|g'_{m_i}(y_{m_i})\right| = 1$. Since $g'_{m_i}(x_0) = \lambda_1^{m_i} = 0$, we get

$$g'_{m_j}(y_{m_i}) = \lambda_1^{m_j} + \lambda_{p_{m_i}}^{m_j} = \lambda_{p_{m_i}}^{m_j}$$

for all $i, j \in N$. Set $w_i := y_{m_i}$ for every $i \in N$, and define

$$g''_1 := g'_{m_1}, \qquad g''_n := g'_{m_n} - \sum_{k=1}^{n-1} \frac{g'_{m_n}(w_k)}{g''_k(w_k)} g''_k$$

for all $n > 1$ $(n \in N)$. Clearly, $g''_1 \in \operatorname{aco}\{g_1, g_2, ...\}$. It follows from $|g''_1(w_1)| = 1$ that $g''_2 \in \operatorname{aco}\{g_1, g_2, ...\}$. Since $|g''_1(w_1)| = 1$ and $|g''_1(w_2)| = \left|\lambda_{p_{m_2}}^{m_1}\right| < 1$, we note $\left|\frac{g''_1(w_2)}{g''_1(w_1)}\right| < 1$ and $|g''_2(w_2)| = 1$; thus, $g''_3 \in \operatorname{aco}\{g_1, g_2, ...\}$. A similar argument applies to show that for fixed $n \in N$ one has $\left|\frac{g''_k(w_n)}{g''_k(w_k)}\right| < 1$ for each $k = 1, ..., n-1$ and conclude that $|g''_n(w_n)| = 1$, thus $g''_{n+1} \in \operatorname{aco}\{g_1, g_2, ...\}$. Note that for a given $n \in N$

$$g''_n(w_m) = 0$$

if $m < n$.

Next, define

$$f_1 := g''_1, \qquad f_{n+1} := f_n + \frac{f_1(w_1) - f_n(w_{n+1})}{g''_{n+1}(w_{n+1})} g''_{n+1}$$

for all $n \in N$. Clearly, $(f_n)_n \subset \operatorname{aco}\{g_1, g_2, ...\}$ since $|g''_n(w_n)| = 1$ $(n \in N)$.

We obtain $f_n(w_m) = f_1(w_1)$ if $m \leq n$ $(n, m \in N)$. Hence, for every $m \in N$ we have $\lim_n f_n(w_m) = f_1(w_1) \neq 0$.

On the other hand, for every $n \in N$, writing $f_n = \sum_{k=1}^{\infty} \mu_k^n \xi_k$ for some $(\mu_k^n)_k \subset \mathbb{K}$, we have that $\lim_m |\mu_m^n| = 0$; hence, $f_n(w_k) = \mu_{p_{m_k}}^n$. Thus, for a given $n \in N$ we get $\lim_m f_n(w_m) = 0$.

Finally, we note that $\lim_n \lim_m f_m(w_n)$, $\lim_m \lim_n f_m(w_n)$ exist but $\lim_n \lim_m f_m(w_n) \neq \lim_m \lim_n f_m(w_n)$. □

Theorem 2.7. *Let X be an infinite zero-dimensional compact space and let $(h_i)_{i\in I}$ denote an orthonormal base of $C(X,\mathbb{K})$ consisting of characteristic functions of clopen sets $(V_i)_{i\in I}$. Let $J\subset I$ be a countable set. Then, there exists a closed $X_0\subset X$ such that $\overline{[(h_i)_{i\in J}]}$ and $C(X_0,\mathbb{K})$ are isometrically isomorphic.*

Proof. Suppose that $J=\{j_1,j_2,...\}$. We construct the set $X_0\subset X$ by induction as follows. First define $W_1^1:=V_{j_1}$, choose any $x_1\in W_1^1$ and define $X_1:=\{x_1\}$, $N_1:=1$. Assume that for $k=1,...,n-1$ we have done, i.e., $W_1^1,W_1^2,...,W_{N_2}^2,...,W_1^{n-1},...,W_{N_{n-1}}^{n-1}$ and $X_1,...,X_{n-1}$ have been chosen. In the $n-$th step we define clopen sets as follows

$$W_k^n:=W_k^{n-1}\backslash V_{j_n},\quad k=1,...,N_{n-1},$$
$$W_{N_{n-1}+k}^n:=W_k^{n-1}\cap V_{j_n},\quad k=1,...,N_{n-1},$$
$$W_{2N_{n-1}+1}^n:=V_{j_n}\backslash(W_1^{n-1}\cup...\cup W_{N_{n-1}}^{n-1})$$

and set $N_n:=2N_{n-1}+1$. For each $k=1,...,N_n$ define

$$x_k^n:=\begin{cases}x_1 & \text{if } W_k^n\backslash(X_1\cup...\cup X_{n-1})=\emptyset\\ x & \text{otherwise,}\end{cases}$$

where x is any element of $W_k^n\backslash(X_1\cup...\cup X_{n-1})$. Set $X_n:=\{x_k^n:k=1,...,N_n\}$.

Now, define $X_0=\overline{X_1\cup X_2\cup....}$ and a map $T:h_i\mapsto h_i|_{X_0}$, $i\in J$. Clearly, $||h_i||=||h_i|_{X_0}||$ for all $i\in J$. We prove that $(h_i|_{X_0})_{i\in J}$ is a maximal orthonormal sequence in $C(X_0,\mathbb{K})$. Then, [6, Theorem 2.5.4] applies to deduce that $(h_i|_{X_0})_{i\in J}$ is an orthonormal base of $C(X_0,\mathbb{K})$ and consequently T can be extended to the required isomorphism.

Take $i_1,...,i_m\in J$ and $\lambda_1,...,\lambda_m\in\mathbb{K}$, ($|\lambda_i|\in\{0,1\}, i=1,...,m$). Since $(h_i)_{i\in J}$ is orthonormal, there exists $x\in X$ such that

$$|\lambda_1 h_{i_1}(x)+...+\lambda_m h_{i_m}(x)|=1.$$

Let $J_1:=\{j\in\{i_1,...,i_m\}:h_j(x)=1\}$. Then,

$$x\in\bigcap_{j\in J_1}V_j\backslash\bigcup_{j\in\{i_1,...,i_m\}\backslash J_1}V_j$$

It follows from the construction of $(W_m^n)_{n,m}$ that there exists $k_0\in N$ such that $W_{k_0}^{i_m}\cap X_0$ is nonempty and

$$W_{k_0}^{i_m}\subset\bigcap_{j\in J_1}V_j\backslash\bigcup_{j\in\{i_1,...,i_m\}\backslash J_1}V_j.$$

Hence, for $x_0\in W_{k_0}^{i_m}\cap X_0$, we get

$$\left|\lambda_1 h_{i_1|_{X_0}}(x_0)+...+\lambda_m h_{i_m|_{X_0}}(x_0)\right|=1;$$

thus, we derive that $(h_{i|X_0})_{i\in J}$ is orthonormal.

Suppose that there exists $g \in C(X_0, \mathbb{K})$ with $||g|| = 1$, which is orthogonal to $(h_{i|X_0})_{i\in J}$; it means that for every finite $\{i_1, ..., i_m\} \subset J$ and $\lambda_1, ..., \lambda_m \in \mathbb{K}$ ($|\lambda_i| \in \{0, 1\}, i = 1, ..., m$) there exists $x_0 \in X_0$ such that

$$\left| g(x_0) + \lambda_1 h_{i_1|X_0}(x_0) + ... + \lambda_m h_{i_m|X_0}(x_0) \right| = 1.$$

By [6, Corollary 2.5.23], g can be extended to some $g' \in C(X, \mathbb{K})$ with the same norm. Thus, we can write $g' = \sum_{i\in I} \mu_i h_i$ for some $\mu_i \in \mathbb{K}$, $i \in I$. Since $\left(\sum_{i\in J} \mu_i h_i\right)|_{X_0} = g$, we deduce that $\left(\sum_{i\in J} \mu_i h_i\right)$ is orthogonal to $(h_i)_{i\in J}$, a contradiction. □

Theorem 2.8. *Let X be an infinite zero-dimensional compact space and let H be an uniformly bounded subset of $C(X, \mathbb{K})$. Then $\operatorname{aco} H$ is τ_p-relatively compact iff H is $w-$relatively compact.*

Proof. If H is w-relatively compact, by Proposition 2.2, the set $\operatorname{aco} H$ is $w-$relatively compact, thus $\operatorname{aco} H$ is τ_p-relatively compact. Conversely, assume that H is not $w-$relatively compact. By [9, Theorem 5.2], H, thus, $\operatorname{aco} H$ is not relatively compact. Applying [10, Theorem 4.38] we choose $(g_n)_n \subset H$, an orthogonal sequence such that $||g_n|| \geq \varepsilon$ for some $\varepsilon > 0$ and every $n \in N$. Since $\mathbb{K}$ is discretely valued, without loss of generality we can assume that $||g_n|| = \varepsilon$. Then, by [10, Theorem 4.38], $G := \operatorname{aco}\{g_1, g_2, ...\}$ is not relatively compact. By [6, Theorem 2.5.22], $C(X, \mathbb{K})$ has an orthonormal base $(h_i)_{i\in I}$ consisting of characteristic functions of clopen sets $(V_i)_{i\in I}$. For every $n \in N$ we choose a countable $J_n \subset I$ such that $g_n = \sum_{i\in J_n} \mu_i^n h_i$. Let $J := J_1 \cup J_2 \cup ....$ Then $J \subset I$ is countable and $\overline{[(h_i)_{i\in J}]}$ is a closed subspace of countable type, orthocomplemented in $C(X, \mathbb{K})$ by [6, Theorem 2.5.4]. Obviously, $G \subset \overline{[(h_i)_{i\in J}]}$. Using Theorem 2.7 we find a closed $X_0 \subset X$ such that there exists an isometrical isomorphism $T : \overline{[(h_i)_{i\in J}]} \to C(X_0, \mathbb{K})$. Since $C(X_0, \mathbb{K})$ is a Banach space of countable type, X_0 is ultrametrizable by [6, Theorem 2.5.23]. Applying Theorem 2.6, we find sequences $(x_n)_n \subset X_0$ and $(f_m)_m \subset T(G) \subset C(X_0, \mathbb{K})$ such that $\lim_n \lim_m f_m(x_n) \neq \lim_m \lim_n f_m(x_n)$. Since $\overline{[(h_i)_{i\in J}]}$ is orthocomplemented in $C(X, \mathbb{K})$, for every $n \in N$ there exists $f'_n \in G \subset C(X, \mathbb{K})$ such that $f'_n|_{X_0} = T^{-1}(f_n)$. Then, $\lim_n \lim_m f'_m(x_n) \neq \lim_m \lim_n f'_m(x_n)$. Clearly, $G \subset \operatorname{aco} H$; thus, X and $\operatorname{aco} H$ does not have the interchangeable double-limit-property and $\operatorname{aco} H$ is not τ_p-relatively compact (see [3, 1.4 Theorem]). □

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