

A quantitative version of Krein's theorems for Fréchet spaces

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Abstract. For a Banach space E and its bidual space E'' , the following function $k(H) := \sup_{y \in \overline{H}^{\sigma(E'', E')}} \inf_{x \in E} \|y - x\|$ defined on bounded subsets H of E measures how far H is from being $\sigma(E, E')$ -relatively compact in E . This concept, introduced independently by Granero [10] and Cascales et al. [7], has been used to study a quantitative version of Krein's theorem for Banach spaces E and spaces $C_p(K)$ over compact K . In the present paper, a quantitative version of Krein's theorem on convex envelopes coH of weakly compact sets H is proved for Fréchet spaces, i.e. metrizable and complete locally convex spaces. For a Fréchet space E , the above function $k(H)$ reads as follows $k(H) := \sup\{d(h, E) : h \in \overline{H}^{\sigma(E'', E')}\}$, where $d(h, E)$ is the natural distance of h to E in the bidual E'' . The main result of the paper is the following theorem: *For a bounded set H in a Fréchet space E , the following inequality holds $k(coH) < (2^{n+1} - 2)k(H) + \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Consequently, this yields also the following formula $k(coH) \leq \sqrt{k(H)}(3 - 2\sqrt{k(H)})$.* Hence coH is weakly relatively compact provided H is weakly relatively compact in E . This extends a quantitative version of Krein's theorem for Banach spaces (obtained by Fabian, Hajek, Montesinos, Zizler, Cascales, Marciszewski, and Raja) to the class of Fréchet spaces. We also define and discuss two other measures of weak non-compactness $lk(H)$ and $k'(H)$ for a Fréchet space and provide two quantitative versions of Krein's theorem for both functions.

Mathematics Subject Classification (2010). Primary 46A50;
Secondary 54C35.

Keywords. Krein's theorem, Compactness, Fréchet space, Space of continuous functions.

The research was supported for C. Angosto by the project MTM2008-05396 of the Spanish Ministry of Science and Innovation, for J. Kąkol by National Center of Science, Poland, Grant No. N N201 605340, and for M. López-Pellicer by the project MTM2010-12374-E (complementary action) of the Spanish Ministry of Science and Innovation.

1. Introduction. In the last decade, several *quantitative* counterparts of some other classical results (including Gantmacher, Eberlein–Grothendieck, Grothendieck, Krein–Smulyan’ theorems) have been proved by several specialists (see for example [1–5, 8, 10, 11], and references). It turns out that these new versions strengthen the original results and provide new applications both in functional analysis and topology.

The classical Krein theorem for Banach spaces E states that for a weakly relatively compact set K in E , its closed convex envelope is weakly compact, i.e. compact in the weak topology $\sigma(E, E')$ of E , see for example [8, Theorem 3.5.8]. The following questions, which refers to this theorem, was formulated in [9]:

(*) Let H be a bounded set in a Banach space E , and let $B_{E''}$ be the closed unit ball in the bidual E'' of E . Assume that H is a ε -weakly relatively compact set (in short ε -WRK), i.e. $\overline{H}^{\sigma(E'', E')} \subset E + \varepsilon B_{E''}$ for some $\varepsilon \geq 0$. Does the same hold for its convex envelope $\text{co}H$?

Clearly, the answer is positive if $\varepsilon = 0$, which is the statement of Krein’s theorem. As mentioned in [9], problem (*) was motivated by some results about closed subspaces of Weakly Compactly Generated Banach spaces (see [9, Theorem 15]). This, and the fact that the class of Weakly Compactly Generated Fréchet spaces is sufficiently large and attracted also specialists, see for example the results of Khurana [13], motivates also the present work.

In [9, Theorem 2] it was proved by applying Ptak’s combinatorial lemma that whenever H is ε -WRK for some $\varepsilon > 0$, then $\text{co}H$ is 2ε -WRK. Moreover, if $B_{E''}$ is $\sigma(E'', E')$ -angelic, then $\text{co}H$ is ε -WRK. The latter result applies to separable Banach spaces, or more generally, Weakly Compactly Generated Banach spaces or even Weakly Lindelöf Determined Banach spaces (see again [8]).

In [7, Corollary 3.4] Cascales, Marciszewski, and Raja obtained a more general theorem stating that for a compact space K and an uniformly bounded $H \subset C(K)$, the following evaluation holds

$$\hat{d}(\overline{\text{co}H}^{\mathbb{R}^K}, C(K)) \leq 2\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)),$$

where \hat{d} is the Hausdorff non-symmetrized distance.

Let E be a Banach space, and let E'' be its bidual. Following [7, 10], define the function

$$k(H) := \sup_{y \in \overline{H}^{\sigma(E'', E')}} \inf_{x \in E} \|y - x\|$$

for any bounded set H in E . Clearly, $k(H)$ measures how far H is from being weakly relatively compact in E . The above result from [7, Corollary 3.4] implies that $k(\text{co}H) \leq 2k(H)$ for any bounded set H in a Banach space E (see also [1]). Note that the equality $k(\text{co}H) = k(H)$ fails in general (see [10, 11]).

In the present paper, we continue this line of research for the class of Fréchet spaces. The main result as stated in the abstract implies that if H is a bounded set in a Fréchet space E and if $k(H) < \varepsilon^2$, then $k(\text{co}H) < \varepsilon(3 - 2\varepsilon)$ for each

$\varepsilon > 0$, where the function $k(H)$ for the case E being a Fréchet space takes the form as mentioned in the abstract, see also below.

Let E be a Fréchet space, and let $(U_n)_n$ be the family of absolutely convex neighbourhoods of zero such that $(\frac{1}{n}U_n)_n$ is a basis of neighbourhoods of zero. By $(E', \beta(E', E))$ and $(E'', \beta(E'', E'))$, we mean the strong dual of E and $(E', \beta(E', E))$, respectively. By $\|h\|_n = \sup\{|h(u)| : u \in U_n^0\}$, we denote the seminorm in E'' associated with U_n^0 , and d_n means the pseudometric defined by $\|\cdot\|_n$. The restriction of $\|\cdot\|_n$ to E , also denoted by $\|\cdot\|_n$, is the seminorm defined by U_n . The topology of E can be defined by the F -norm

$$d(x, y) := \sum_n 2^{-n} \|x - y\|_n (1 + \|x - y\|_n)^{-1}$$

for $x, y \in E$. Also, the topology of the space $(E'', \beta(E'', E'))$ is defined by the F -norm

$$d(x^{**}, y^{**}) := \sum_n 2^{-n} \|x^{**} - y^{**}\|_n (1 + \|x^{**} - y^{**}\|_n)^{-1}$$

for all $x^{**}, y^{**} \in E''$. Additionally, without loss of generality, we will assume that $U_{n+1} \subset U_n$ for $n \in \mathbb{N}$; and this clearly implies that $\|x^{**}\|_n \leq \|x^{**}\|_{n+1}$ for $n \in \mathbb{N}$ and each $x^{**} \in E''$.

2. More about the measure of weak non-compactness $k(H)$. By a measure of the weak non-compactness, we mean a function μ defined on the family of bounded subsets of a Fréchet space E such that if $A, B \subset E$ are bounded, then (i) $\mu(A) = 0$ if and only if A is weakly relatively compact, (ii) if $A \subset B$, then $\mu(A) \leq \mu(B)$. If H is a bounded subset of E , then H^0 is a neighbourhood of zero in $(E', \beta(E', E))$ and the bipolar H^{00} is a compact subset of $(E'', \sigma(E'', E'))$ which is bounded in the strong topology $\beta(E'', E')$. Therefore a bounded subset H of E is weakly relatively compact if and only if $\overline{H}^{\sigma(E'', E')}$ is contained in E . In [6] we introduced the following two functions for Fréchet spaces E :

$$k(H) := \sup \left\{ d(h, E) : h \in \overline{H}^{\sigma(E'', E')} \right\},$$

$$k_n(H) := \sup \left\{ d_n(h, E) : h \in \overline{H}^{\sigma(E'', E')} \right\}.$$

Observe that $k(H)$ is a measure of weak non-compactness, and a bounded set $H \subset E$ is weakly relatively compact if and only if $k(H) = 0$ if and only if $k_n(H) = 0$ for each $n \in \mathbb{N}$.

We need the following lemma, which will be used for the proof of Theorem 3.5.

Lemma 2.1. *If H is a bounded subset of a Fréchet space E and $n \in \mathbb{N}$, then*

$$k(H) \leq \frac{2^n - 1}{2^n} \frac{k_n(H)}{1 + k_n(H)} + \frac{1}{2^n}.$$

Proof. Since the function $f(x) = x/(1+x)$ is strictly increasing and $d_m(h, e) \leq d_n(h, e)$ for each $h, e \in E'$ and $m \leq n$ (recall that $d_m(h, e) \leq d_{m+1}(h, e)$ for

all $m \in \mathbb{N}$), we derive that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{d_m(h, e)}{1 + d_m(h, e)} &< \sum_{m=1}^n \frac{1}{2^m} \frac{d_n(h, e)}{1 + d_n(h, e)} + \sum_{m=n+1}^{\infty} \frac{1}{2^m} \\ &= \left(1 - \frac{1}{2^n}\right) \frac{d_n(h, e)}{1 + d_n(h, e)} + \frac{1}{2^n} \\ &= (2^n - 1) \frac{1}{2^n} \cdot \frac{d_n(h, e)}{1 + d_n(h, e)} + \frac{1}{2^n}. \end{aligned}$$

Then

$$\begin{aligned} d(h, E) &= \inf_{e \in E} \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{d_m(h, e)}{1 + d_m(h, e)} \leq \frac{2^n - 1}{2^n} \frac{\inf_{e \in E} d_n(h, e)}{1 + \inf_{e \in E} d_n(h, e)} + \frac{1}{2^n} \\ &= \frac{2^n - 1}{2^n} \cdot \frac{d_n(h, E)}{1 + d_n(h, E)} + \frac{1}{2^n}. \end{aligned}$$

Finally, if we take the supremum over all $h \in \overline{H}^{\sigma(E'', E')}$, we have the following

$$k(H) \leq \frac{2^n - 1}{2^n} \sup_{h \in \overline{H}^{\sigma(E'', E')}} \frac{d_n(h, E)}{1 + d_n(h, E)} + \frac{1}{2^n} = \frac{2^n - 1}{2^n} \frac{k_n(H)}{1 + k_n(H)} + \frac{1}{2^n}.$$

□

3. Quantitative version of Krein’s theorem. In this section we prove a promised quantitative version of Krein’s theorem for Fréchet spaces. Our approach will use the concept of ε -interchange of limits. This notion, originally introduced by Grothendieck [12] for $\varepsilon = 0$, was extended for $\varepsilon > 0$ in [9]. If $\varepsilon \geq 0$, we say that H ε -interchanges limits with a subset B of E' if

$$\left| \lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m) \right| \leq \varepsilon$$

for all sequences $(u_p)_p \subset B$ and $(h_m)_m \subset H$ provided the involved limits exist. For $\varepsilon = 0$ we say H interchanges limits with B . Fix $n \in \mathbb{N}$, and let H be a bounded subset of a Fréchet space E . Denote

$$\gamma_n(H) := \inf\{\varepsilon \geq 0 : H \text{ } \varepsilon\text{-interchanges limits with } U_n^0\}.$$

We need the following two results from [6, 7], respectively.

Proposition 3.1. [6, Proposition 5] *If H is a bounded subset of a Fréchet space E and $n \in \mathbb{N}$, then*

$$k_n(H) \leq \gamma_n(H) \leq 2k_n(H).$$

In particular, H is weakly relatively compact if and only if $\gamma_n(H) = 0$ for all $n \in \mathbb{N}$.

Lemma 3.2. [7, Lemma 3.2] *Let (I_n) be a sequence of pairwise disjoint finite nonempty sets, and let μ_n be a probability measure on $\mathcal{P}(I_n)$ for each n . Let (A_k) be a sequence of subsets of $I = \bigcup_{n \in \mathbb{N}} I_n$ such that, for some $\delta > 0$, $\liminf_n \mu_n(A_k \cap I_n) > \delta$ holds for every $k \in \mathbb{N}$. Then there is a subsequence (A_{k_i}) such that $\bigcap_{i \leq j} A_{k_i} \neq \emptyset$ for each $j \geq 1$.*

We need also the following somewhat technical lemma. Its proof uses some ideas similar to those that are applied in the proof of [7, Theorem 3.3].

Lemma 3.3. *Fix $n \in \mathbb{N}$. If H is a bounded subset of a Fréchet space E , then*

$$\gamma_n(H) = \gamma_n(\text{co}H).$$

Proof. Choose sequences $(u_p) \subset U_n^0$ and $(h_m) \subset \text{co}H$ such that the involved limits exist, and

$$d = \lim_m \lim_p u_p(h_m) - \lim_p \lim_m u_p(h_m) > 0.$$

Since $h_m \in \text{co}H$, for each m we have $h_m = \sum_{i \in I_m} t_i k_i$, where $k_i \in H$, and I_m is a finite set, $0 \leq t_i \leq 1$ for all $i \in I_m$ and $\sum_{i \in I_m} t_i = 1$. Without loss of generality, we may assume that the sets I_m are pairwise disjoint. Define $I = \bigcup_m I_m$. Since the set H is bounded, there exists $M > 0$ such that $|u_p(k_i)| < M$ for all $p, i \in \mathbb{N}$. Consequently, we may also assume that for each $i \in \mathbb{N}$, there is some $x_i \in [-M, M]$ such that

$$\lim_p u_p(k_i) = x_i. \tag{3.1}$$

For each $m \in \mathbb{N}$ define

$$y_m = \lim_p u_p(h_m) = \lim_p \sum_{i \in I_m} t_i u_p(k_i) = \sum_{i \in I_m} t_i x_i. \tag{3.2}$$

Then

$$\begin{aligned} d &= \lim_m \lim_p u_p(h_m) - \lim_p \lim_m u_p(h_m) = \lim_m y_m - \lim_p \lim_m u_p(h_m) \\ &= \lim_p \lim_m (y_m - u_p(h_m)). \end{aligned}$$

Fix $\varepsilon > 0$. We may assume that

$$\lim_m (y_m - u_p(h_m)) > d - \varepsilon$$

for every p . Then for each $p \in \mathbb{N}$ there exists $m_p \in \mathbb{N}$ such that if $m > m_p$, then

$$y_m - u_p(h_m) > d - \varepsilon. \tag{3.3}$$

For every $m \in \mathbb{N}$ define μ_m , the probability measure on I_m , as

$$\mu_m(A) = \sum_{i \in A} t_i.$$

Then we define

$$A_p = \{i \in I : x_i - u_p(k_i) > d - 2\varepsilon\}. \tag{3.4}$$

Observe that the following holds.

$$\begin{aligned}
 d - \varepsilon &\stackrel{(3.3)}{<} y_m - u_p(h_m) \stackrel{(3.2)}{=} \sum_{i \in I_m} t_i x_i - \sum_{i \in I_m} t_i u_p(k_i) \\
 &= \sum_{i \in I_m \cap A_p} t_i (x_i - u_p(k_i)) + \sum_{i \in I_m \setminus A_p} t_i (x_i - u_p(k_i)) \\
 &\stackrel{(3.4)}{\leq} \sum_{i \in I_m \cap A_p} t_i 2M + d - 2\varepsilon = 2M\mu_m(I_m \cap A_p) + d - 2\varepsilon,
 \end{aligned}$$

so $\mu_m(I_m \cap A_p) > \varepsilon/(2M)$, and then

$$\liminf_m \mu_m(I_m \cap A_p) \geq \varepsilon/(2M).$$

By Lemma 3.2 there exists a subsequence $(A_{p_k})_k$ such that $\bigcap_{k \leq l} A_{p_k} \neq \emptyset$ for each $l \in \mathbb{N}$. So, by the definition of A_p , for each l there exists an i_l such that

$$x_{i_l} - u_{p_k}(k_{i_l}) > d - 2\varepsilon \tag{3.5}$$

for all $k \leq l$. Choosing subsequences we may assume that $(u_{p_k}(k_{i_l}))_l$ converges to some $z_k \in \mathbb{R}$ for all k and that $(x_{i_l})_l$ converges to some $x \in \mathbb{R}$. We may also assume that the sequence $(z_k)_k$ converges to some $z \in \mathbb{R}$. Then

$$\lim_l \lim_k u_{p_k}(k_{i_l}) \stackrel{(3.1)}{=} \lim_l x_{i_l} = x$$

and

$$\lim_k \lim_l u_{p_k}(k_{i_l}) = \lim_k z_k = z.$$

Inequality (3.5) implies that

$$x - z_k = \lim_l (x_{i_l} - u_{p_k}(k_{i_l})) \geq d - 2\varepsilon,$$

so $x - z = \lim_k (x - z_k) \geq d - 2\varepsilon$. Then we derive that $\gamma_n(H) \geq d - 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain that

$$\gamma_n(H) \geq d = \lim_m \lim_p u_p(h_m) - \lim_p \lim_m u_p(h_m),$$

and this holds for all sequences $(u_p) \subset U_n^0$ and $(h_m) \subset coH$ such that the involved limits exist. Then we conclude that

$$\gamma_n(H) \geq \gamma_n(coH) \geq \gamma_n(H), \tag{3.6}$$

and the proof is finished. □

From Lemma 3.3 and Proposition 3.1, we obtain the following corollary.

Corollary 3.4. *If H is a bounded subset of a Fréchet space E and $n \in \mathbb{N}$, then*

$$k_n(coH) \leq 2k_n(H).$$

Now we are ready to prove the main result of the paper.

Theorem 3.5. *If H is a bounded subset of a Fréchet space E and $n \in \mathbb{N}$, then*

$$k(coH) < (2^{n+1} - 2)k(H) + \frac{1}{2^n}. \tag{3.7}$$

Proof. If $k(H) = 0$, then $k_n(H) = 0$ for all $n \in \mathbb{N}$, so by Corollary 3.4 we have $k(\text{co}H) = 0$ and inequality (3.7) holds. Thus, we may assume that $k(H) > 0$. By Lemma 2.1 and Corollary 3.4 with the fact that the function $f(x) = x/(1+x)$ is strictly increasing, we derive that

$$k(\text{co}H) \leq \frac{2^n - 1}{2^n} \frac{k_n(\text{co}H)}{1 + k_n(\text{co}H)} + \frac{1}{2^n} \leq \frac{2^n - 1}{2^n} \frac{2k_n(H)}{1 + 2k_n(H)} + \frac{1}{2^n}. \tag{3.8}$$

To finish the proof, we need only to prove that

$$\frac{1}{2^n} \frac{2k_n(H)}{1 + 2k_n(H)} < 2k(H). \tag{3.9}$$

If $k_n(H) = 0$, inequality (3.9) holds indeed. Suppose that $k_n(H) > 0$. Then

$$\rho := \sup_{h \in \overline{H}^{\sigma(E'', E')}} \inf_{e \in E} \frac{1}{2^n} \frac{d_n(h, e)}{1 + d_n(h, e)} = \frac{1}{2^n} \frac{k_n(H)}{1 + k_n(H)} > 0,$$

where to fix the last equality we applied the fact that the function $f(x)$ is strictly increasing. Fix $h_0 \in \overline{H}^{\sigma(E'', E')}$ such that

$$\alpha := \inf_{e \in E} \frac{d_n(h_0, e)}{1 + d_n(h_0, e)} > 2^{n-1} \rho.$$

Since $d_m(h, e) \leq d_{m+1}(h, e)$ for $m \in \mathbb{N}$ and $h, e \in E''$, we have

$$\begin{aligned} \inf_{e \in E} \sum_{m=n}^{\infty} \frac{1}{2^m} \frac{d_m(h_0, e)}{1 + d_m(h_0, e)} &\geq \inf_{e \in E} \sum_{m=n}^{\infty} \frac{1}{2^m} \frac{d_n(h_0, e)}{1 + d_n(h_0, e)} \\ &= \sum_{m=n}^{\infty} \frac{1}{2^m} \alpha = \frac{1}{2^{n-1}} \alpha > \rho. \end{aligned}$$

Therefore

$$k(H) = \sup_{h \in \overline{H}^{\sigma(E'', E')}} \inf_{e \in E} \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{d_m(h, e)}{1 + d_m(h, e)} > \rho.$$

Consequently, we note that

$$\frac{1}{2^n} \frac{2k_n(H)}{1 + 2k_n(H)} \leq \frac{1}{2^n} \frac{2k_n(H)}{1 + k_n(H)} = 2\rho < 2k(H),$$

so inequality (3.9) holds indeed, and this completes the proof. □

Corollary 3.6. *If H is a bounded set in a Fréchet space E such that $k(H) > 0$, then*

$$k(\text{co}H) < \sqrt{k(H)}(3 - 2\sqrt{k(H)}). \tag{3.10}$$

Proof. If $1/4 < k(H) \leq 1$, the inequality (3.10) holds because

$$\sqrt{k(H)}(3 - 2\sqrt{k(H)}) \geq 1 > k(\text{co}H).$$

Observe that, since $\text{co}H$ is a bounded set, the inequality $k(\text{co}H) < 1$ holds. Therefore we may assume that $0 < k(H) \leq 1/4$. Then there exists $n \in \mathbb{N}$ such that

$$\frac{1}{4^{n+1}} < k(H) \leq \frac{1}{4^n}.$$

By Theorem 3.5 we conclude that

$$k(\text{co}H) < (2^{n+1} - 2)k(H) + \frac{1}{2^n}. \tag{3.11}$$

For each $n \in \mathbb{N}$ define two real functions

$$f_n(x) = (2^{n+1} - 2)x + \frac{1}{2^n} \quad \text{and} \quad g(x) = \sqrt{x}(3 - 2\sqrt{x}).$$

Then

$$f_n(1/4^n) = g(1/4^n), \quad f_n(1/4^{n+1}) = g(1/4^{n+1}).$$

Since $(g - f_n)'' < 0$ in $[1/4^{n+1}, 1/4^n]$, we deduce that $g(x) \geq f_n(x)$ in $[1/4^{n+1}, 1/4^n]$. Then by (3.11) we conclude

$$k(\text{co}H) < f_n(k(H)) \leq g(k(H)) = \sqrt{k(H)}(3 - 2\sqrt{k(H)}).$$

□

Remark 3.7. *If $k(H) = 1/4^n$ for some $n \in \mathbb{N}$, then the upper bounds for $k(\text{co}H)$ from Theorem 3.5 and Corollary 3.6 are the same. Indeed, if we denote $f_n(x) = (2^{n+1} - 2)x + \frac{1}{2^n}$, then $f_n(x) \leq f_{n+1}(x)$ if and only if $x \geq 1/4^{n+1}$, so*

$$\inf_{m \in \mathbb{N}} f_m(x) = f_n(x) \quad \text{if} \quad \frac{1}{4^{n+1}} \leq x \leq \frac{1}{4^n}.$$

Then the upper bound for the function $k(\text{co}H)$ from Theorem 3.5 is $f_n(1/4^n)$, that is equal to $\sqrt{1/4^n}(3 - 2\sqrt{1/4^n})$.

Corollary 3.8. (Krein) *If H is a weakly relatively compact set in a Fréchet space E , then $\text{co}H$ is a weakly relatively compact set in E .*

4. Two additional measures of weak non-compactness $lk(H)$ and $k'(H)$.

Next, we start with the definition of two measures of weak non-compactness for Fréchet spaces.

Definition 4.1. *Let H be a bounded subset of a Fréchet space E . Set*

$$lk(H) := \sup_{h \in \overline{H}^{\sigma(E'', E')}} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(h, E)}{1 + d_n(h, E)}, \quad k'(H) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{k_n(H)}{1 + k_n(H)}.$$

The next proposition shows the relation between the new defined measures and the measure $k(H)$ and provides their equality for the case E being a Banach space.

Proposition 4.2. *If H is a bounded subset of a Fréchet space E , then $lk(H) \leq k(H)$ and $lk(H) \leq k'(H)$. If E is a Banach space and $U_n = U_1$ is the unit ball for all $n \in \mathbb{N}$, then $k'(H) = k(H) = lk(H)$.*

Proof. Since the function $f(x) = x/(1 + x)$ is strictly increasing in $[0, +\infty)$, for a bounded set $A \subset [0, +\infty)$, we have

$$\sup_{x \in A} f(x) = f(\sup_{x \in A} x) \quad \text{and} \quad \inf_{x \in A} f(x) = f(\inf_{x \in A} x). \tag{4.1}$$

Then

$$\begin{aligned} \sup_{h \in \overline{H}^{\sigma(E'', E')}} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(h, E)}{1 + d_n(h, E)} &\leq \sum_{n=1}^{\infty} \sup_{h \in \overline{H}^{\sigma(E'', E')}} \frac{1}{2^n} \frac{d_n(h, E)}{1 + d_n(h, E)} \\ &\stackrel{(4.1)}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{k_n(H)}{1 + k_n(H)}, \end{aligned}$$

so $lk(H) \leq k'(H)$. Next we show that $lk(H) \leq k(H)$. For this observe that

$$\begin{aligned} d(h, E) &= \inf_{e \in E} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(h, e)}{1 + d_n(h, e)} \geq \sum_{n=1}^{\infty} \inf_{e \in E} \frac{1}{2^n} \frac{d_n(h, e)}{1 + d_n(h, e)} \\ &\stackrel{(4.1)}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\inf_{e \in E} d_n(h, e)}{1 + \inf_{e \in E} d_n(h, e)} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(h, E)}{1 + d_n(h, E)}. \end{aligned}$$

Taking the supremums over all $h \in \overline{H}^{\sigma(E'', E')}$, we conclude that $lk(H) \leq k(H)$. Finally, if E is a Banach space and $U_n = U_1$ is the unit ball for all $n \in \mathbb{N}$, then $d_n = d_m$ and $k_n(H) = k_m(H)$ for all $n, m \in \mathbb{N}$. Hence

$$\begin{aligned} d(h, E) &= \inf_{e \in E} d(h, e) = \inf_{e \in E} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_1(h, e)}{1 + d_1(h, e)} \\ &= \sum_{n=1}^{\infty} \inf_{e \in E} \frac{1}{2^n} \frac{d_n(h, e)}{1 + d_n(h, e)} \stackrel{(4.1)}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(h, E)}{1 + d_n(h, E)}, \end{aligned}$$

and then $k(H)$ equals to

$$\begin{aligned} \sup_{h \in \overline{H}^{\sigma(E'', E')}} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(h, E)}{1 + d_n(h, E)} &= \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{h \in \overline{H}^{\sigma(E'', E')}} \frac{d_1(h, E)}{1 + d_1(h, E)} \\ &\stackrel{(4.1)}{=} \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{k_n(H)}{1 + k_n(H)}. \end{aligned}$$

This consequently yields the promised equalities $k(H) = lk(H) = k'(H)$. The proof is completed. □

For $x^{**} \in E''$ we have $d(x^{**}, E) = 0$ if and only if $x^{**} \in E$ if and only if $d_n(x^{**}, E) = 0$ for $n \in \mathbb{N}$. This provides the following relations.

Proposition 4.3. *For a bounded subset H of a Fréchet space E , the set H is weakly relatively compact if and only if $k(H) = 0$ if and only if $lk(H) = 0$ if and only if $k'(H) = 0$ if and only if $k_n(H) = 0$ for all $n \in \mathbb{N}$.*

Note also the following relations between functions $k(H)$ and $k'(H)$ with $lk(H)$.

Proposition 4.4. *If H is a bounded subset of a Fréchet space E and $n \in \mathbb{N}$, then*

$$k(H) \leq (2^n - 1) \cdot lk(H) + \frac{1}{2^n}.$$

Proof. The function $f(x) = x/(1 + x)$ is strictly increasing and $d_m(h, e) \leq d_n(h, e)$ for each $h, e \in E''$ and $m \leq n$, so we derive that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{d_m(h, e)}{1 + d_m(h, e)} &\leq \sum_{m=1}^n \frac{1}{2^m} \frac{d_n(h, e)}{1 + d_n(h, e)} + \sum_{m=n+1}^{\infty} \frac{1}{2^m} \\ &\leq \left(1 - \frac{1}{2^n}\right) \frac{d_n(h, e)}{1 + d_n(h, e)} + \frac{1}{2^n} \\ &= (2^n - 1) \frac{1}{2^n} \cdot \frac{d_n(h, e)}{1 + d_n(h, e)} + \frac{1}{2^n}. \end{aligned}$$

Then

$$\begin{aligned} d(h, E) &= \inf_{e \in E} \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{d_m(h, e)}{1 + d_m(h, e)} \leq (2^n - 1) \frac{1}{2^n} \frac{\inf_{e \in E} d_n(h, e)}{1 + \inf_{e \in E} d_n(h, e)} + \frac{1}{2^n} \\ &\leq (2^n - 1) \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{d_m(h, E)}{1 + d_m(h, E)} + \frac{1}{2^n}. \end{aligned} \tag{4.2}$$

This yields the following inequality when the supremum is taken over all $h \in \overline{H}^{\sigma(E'', E')}$.

$$\begin{aligned} k(H) &\leq (2^n - 1) \sup_{h \in \overline{H}^{\sigma(E'', E')}} \sum_{m=1}^{\infty} \frac{1}{2^m} \cdot \frac{d_m(h, E)}{1 + d_m(h, E)} + \frac{1}{2^n} \\ &= (2^n - 1)lk(H) + \frac{1}{2^n}. \end{aligned} \tag{4.3}$$

The proof is completed. □

Proposition 4.5. *If H is a bounded subset of a Fréchet space E and $n \in \mathbb{N}$, then*

$$k'(H) < n \cdot lk(H) + \frac{1}{2^n}.$$

Proof. Fix $m \in \mathbb{N}$. Then

$$\frac{1}{2^m} \frac{k_m(H)}{1 + k_m(H)} = \sup_{h \in \overline{H}^{\sigma(E'', E')}} \frac{1}{2^m} \frac{d_m(h, E)}{1 + d_m(h, E)} \leq lk(H). \tag{4.4}$$

Since H is a bounded set, we note the following $k_m(H) < +\infty$ for all $m \in \mathbb{N}$. Consequently, we have

$$\begin{aligned} k'(H) &= \sum_{m=1}^n \frac{1}{2^m} \frac{k_m(H)}{1 + k_m(H)} + \sum_{m=n+1}^{\infty} \frac{1}{2^m} \frac{k_m(H)}{1 + k_m(H)} \\ &\stackrel{(4.4)}{<} \sum_{m=1}^n lk(H) + \sum_{m=n+1}^{\infty} \frac{1}{2^m} = n \cdot lk(H) + \frac{1}{2^n}, \end{aligned}$$

and the proof is finished. □

The following proposition is a consequence of Proposition 3.1 and Lemma 3.3.

Proposition 4.6. *Let H be a bounded set in a Fréchet space E . Then $k'(coH) \leq 2 \cdot k'(H)$.*

We complete the paper with the following quantitative versions of Krein’s theorem for the function $lk(H)$. First we prove the following

Proposition 4.7. *If H is a bounded set in a Fréchet space, then*

$$lk(coH) < 2n \cdot lk(H) + \frac{1}{2^{n-1}}$$

for all $n \in \mathbb{N}$.

Proof. By Propositions 4.2, 4.5, and 4.6, we conclude that

$$lk(coH) \leq k'(coH) \leq 2k'(H) < 2(n \cdot lk(H) + \frac{1}{2^n}) = 2n \cdot lk(H) + \frac{1}{2^{n-1}}.$$

□

Corollary 4.8. *If H is a bounded set in a Fréchet space such that $lk(H) > 0$, then*

$$lk(coH) < (2 \log_{1/2} lk(H) + 2)lk(H).$$

Proof. If $1/2 < lk(H) \leq 1$, the inequality holds because then

$$(2 \log_{1/2} lk(H) + 2) lk(H) \geq 1 > lk(coH).$$

Therefore we can assume that $0 < lk(H) \leq 1/2$. Then there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2^{n+1}} < lk(H) \leq \frac{1}{2^n}.$$

By Proposition 4.7 we conclude that

$$lk(coH) < 2n \cdot lk(H) + \frac{1}{2^{n-1}}. \tag{4.5}$$

Let $f_n(x) = 2nx + \frac{1}{2^{n-1}}$ and $g(x) = (2 \log_{1/2} x + 2)x$ for each $n \in \mathbb{N}$. Then

$$f_n(1/2^n) = g(1/2^n), \quad f_n(1/2^{n+1}) = g(1/2^{n+1}).$$

Since $(g - f_n)'' < 0$ in $[1/2^{n+1}, 1/2^n]$, we deduce that $g(x) \geq f_n(x)$ in $[1/2^{n+1}, 1/2^n]$ for each $n \in \mathbb{N}$. Then, by inequality (4.5) we have

$$lk(coH) < f_n(lk(H)) \leq g(lk(H)) = (2 \log_{1/2} lk(H) + 2) lk(H).$$

□

Remark 4.9. *If $lk(H) = 1/2^n$ for some $n \in \mathbb{N}$, then the upper bounds for $lk(coH)$ from Proposition 4.7 and Corollary 4.8 are the same. Indeed, if we denote $f_n(x) = 2nx + \frac{1}{2^{n-1}}$, then $f_n(x) \leq f_{n+1}(x)$ if and only if $x \geq 1/2^{n+1}$, so*

$$\inf_{m \in \mathbb{N}} f_m(x) = f_n(x) \quad \text{if} \quad \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}.$$

Then the upper bound for the function $lk(coH)$ from Proposition 4.7 is $f_n(1/2^n)$, that is equal to $(2 \log_{1/2} 1/2^n + 2)1/2^n$.

We have also the following

Proposition 4.10. *If H is a bounded set in a Fréchet space, then*

$$lk(coH) \leq k(coH) < (2^{n+1} - 2)lk(H) + \frac{1}{2^n}$$

for all $n \in \mathbb{N}$

Proof. By Proposition 4.2, Lemma 2.1, and Corollary 3.4, we conclude that

$$\begin{aligned} lk(coH) &\leq k(coH) \leq \frac{2^n - 1}{2^n} \frac{k_n(coH)}{1 + k_n(coH)} + \frac{1}{2^n} \\ &\leq \frac{2^n - 1}{2^n} \frac{2k_n(H)}{1 + 2k_n(H)} + \frac{1}{2^n} < (2^{n+1} - 2) lk(H) + \frac{1}{2^n}, \end{aligned}$$

where the proof of the last inequality is very similar to the proof of (3.9). \square

Using the same proof as in Corollary 3.6, we deduce the following corollary that provides a better bound for $lk(coH)$ for the case when $lk(H) > \frac{1}{16}$.

Corollary 4.11. *If H is a bounded set in a Fréchet space and $lk(H) > 0$, then*

$$lk(coH) \leq k(coH) < \sqrt{lk(H)} (3 - 2\sqrt{lk(H)}).$$

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Received: 13 February 2013