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Some topological cardinal inequalities for spaces $C_p(X)$ [☆]

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ABSTRACT

Using the index of Nagami we get new topological cardinal inequalities for spaces $C_p(X)$. A particular case of Theorem 1 states that if $L \subseteq C_p(X)$ is a Lindelöf Σ-space and the Nagami index $Nag(X)$ of X is less or equal than the density $d(L)$ of L (which holds for instance if X is a Lindelöf Σ-space), then (i) there exists a completely regular Hausdorff space Y such that $Nag(Y) \leqslant Nag(X)$, $L \subset C_p(Y)$ and $d(L) = d(Y)$; (ii) Y admits a weaker completely regular Hausdorff topology τ' such that $w(Y, \tau') \leqslant d(Y) = d(L)$. This applies, among other things, to characterize analytic sets for the weak topology of any locally convex space E in a large class $\mathfrak{G}$ of locally convex spaces that includes (DF)-spaces and (LF)-spaces. The latter yields a result of Cascales–Orihuela about weak metrizability of weakly compact sets in spaces from the class $\mathfrak{G}$.

1. Introduction

A consequence of a remarkable theorem due to Amir–Lindenstrauss [1] stating that every weakly compactly generated (WCG) Banach space admits a continuous one-to-one linear operator into $c_0(\Gamma)$ for suitable Γ is that for a (WCG) Banach space E one has $d(E, \sigma(E, E')) = d(E', \sigma(E', E))$, hence E is weakly separable if and only if E' is weak* separable. This result has been extended to all K-analytic metrizable locally convex spaces by Canela [4, Proposition 7]. Later, Cascales and Orihuela [8, Theorem 13] extended the latter result to a large class $\mathfrak{G}$ of locally convex spaces E (containing all metrizable and (DF)-spaces) for which E under the weak topology $\sigma(E, E')$ is a Lindelöf Σ-space (see Section 3). In this paper, using the index of Nagami, we provide a general version of the mentioned consequence of the Amir–Lindenstrauss theorem for spaces $C_p(X)$ of real-valued continuous functions with the pointwise convergence topology (Theorem 1 and Corollary 1), which particularly contains the above results.

All spaces X are assumed to be completely regular and Hausdorff. By $\ell(X)$, $d(X)$, $hd(X)$, $w(X)$, $nw(X)$, $t(X)$ and $q(X)$ we denote the Lindelöf number, the density, the hereditarily density, the weight, the network weight, the tightness and the Hewitt–Nachbin number of X respectively. Note that $iw(X)$ means the smallest infinite cardinal λ for which there exists a continuous bijection from X onto a space Y with $w(Y) \leqslant \lambda$, see [2]. A compact-valued map $\phi : Y \to 2^X$ is said to be *usco* if it is upper semicontinuous.

The index of Nagami of a topological space X, denoted by $Nag(X)$, is the smallest infinite cardinal number λ for which there are a topological space Y of weight λ and an usco map $\phi : Y \to 2^X$ covering X. The number $Nag(X)$ measures how

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the space X is determined by its compact subsets via upper semicontinuous compact-valued maps defined on topological spaces, see [14] and also [7,24].

In [7] and [16] another useful cardinal function has been introduced which assigns to each topological space X the cardinal number $\ell\Sigma(X)$ of K-determination of X defined as the smallest infinite cardinal number λ for which there exist a metric space M with $w(M)=\lambda$ and an usco map from M into 2^X covering X. Both concepts have been essentially used to study and generalize several results from C_p-theory, see [3,7,17]. Note that the spaces X with $\ell\Sigma(X)=\aleph_0$ are precisely the Lindelöf Σ-spaces (also called K-determined spaces). If there exists an usco map from $\mathbb{N}^{\mathbb{N}}$ into 2^X covering X, then X is called K-*analytic*. Note that K-analytic $\Rightarrow$ Lindelöf Σ $\Rightarrow$ Lindelöf. We say that X is *analytic* if it is a continuous image of the space $\mathbb{N}^{\mathbb{N}}$. A topological space X is called a *quasi-Suslin* space if there exists a set-valued map T from $\mathbb{N}^{\mathbb{N}}$ into 2^X covering X such that if $\alpha_n\to\alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n\in T(\alpha_n)$ for each $n\in\mathbb{N}$, then the sequence $(x_n)_n$ has a cluster point in $T(\alpha)$, see [26, Chapter One, 4.2].

Recall some interesting relations for any completely regular Hausdorff space X.

(1) $t(C_p(X))\leqslant Nag(X)$, $\ell(X)\leqslant Nag(X)\leqslant \ell\Sigma(X)\leqslant |X|$,
(2) $Nag(X)=\max\{\Sigma(X),\ell(X)\}$,
(3) $Nag(X)\leqslant nw(X)$,

see [7, Proposition 7, Theorem 25, Corollary 27], where $\Sigma(X)$ is the Σ-degree in the sense of Hödel, [14].

Following [2], we say that a set $A\subset X$ is of type G_λ in X if there exists a family γ of open sets in X such that $A=\bigcap\gamma$ and $|\gamma|\leqslant\lambda$. Recall also that A is called λ-*placed* in X if for each $x\in X\setminus A$ there is a set P of type G_λ in X such that $x\in P\subset X\setminus A$. A Hewitt λ-extension of X is the subspace $\upsilon_\lambda X$ of βX consisting of all $x\in\beta X$ for which any set of type G_λ in βX containing x intersects X. If $\lambda:=\aleph_0$ the set $\upsilon_\lambda X$ is just the (Hewitt) realcompactification of X. Now define $q(X):=\min\{\lambda\geqslant\aleph_0\colon\ X \text{ is } \lambda\text{-placed in } \beta X\}$. If $q(X)=\aleph_0$, then X is called a *realcompact space*. It is clear that $\upsilon_\lambda X=X$ if $\lambda\geqslant q(X)$. The inclusions $X\subset\upsilon_\gamma X\subset\upsilon_\lambda X$ for $\lambda\leqslant\gamma$ are obvious, in fact, $\upsilon_\gamma X$ is C-embedded in $\upsilon_\lambda X$. In particular, X is C-embedded in $\upsilon_\lambda X$ for every infinite cardinal number λ. We recall that a subspace Y of a space X is C-embedded in X if every real-valued continuous function on Y can be extended to a real-valued continuous function on X. Observe also that if Y is a subspace of X which is C-embedded in X, then the restriction map, $\pi: C_p(X)\to C_p(Y)$, is a continuous onto map, hence $Nag(C_p(Y))\leqslant Nag(C_p(X))$. Note also the following inequalities, see [25, Theorem 1], [17, Proposition 4.3].

(1) $q(X)\leqslant d(C_p(X))\leqslant hd(C_p(X))$,
(2) $q(X)\leqslant Nag(X)$.

Another consequence of the above-mentioned theorem of Amir–Lindenstrauss is that a compact space X is Eberlein compact if and only if the Banach space $C(X)$ is (WCG). Since every closed linear subspace Y of a (WCG) Banach space is a Lindelöf Σ-space in the weak topology of Y, [11, Proposition 7.1.6], this may suggest the possibility to classify compact spaces X in terms of topological properties of $C(X)$ equipped with the weak topology. In regard to this, let us recall that a compact space X is said to be a Gul'ko compact if $C_p(X)$ is a Lindelöf Σ-space. This leads to an interesting variant of the Amir–Lindenstrauss theorem for Gul'ko compact spaces (see for example [11, Theorem 7.1.8]), which asserts that a compact space X is Gul'ko compact if and only if the Banach space $C(X)$ is a weakly Lindelöf Σ-space.

Consider the following concrete situation. If X and $C_p(X)$ are Lindelöf Σ-spaces, Talagrand's Theorem 2.4 in [23] assures that $d(X)=d(C_p(X))$. Consequently, if X is a Gul'ko compact space, so that both X and $C_p(X)$ are Lindelöf Σ-spaces, setting $Y:=C_p(X)$ and using the fact that $d(X)=iw(Y)$ as stated in [2, Theorem I.1.4], one gets $iw(Y)=d(Y)=d(X)$. This implies that there exists on Y a weaker completely regular topology τ' such that $w(Y,\tau')=d(Y)=d(X)$. These topological cardinal equalities motivate parts (2) and (5) of Theorem 1 and Corollary 1 below.

Theorem 1. *Let X be a topological space and $L\subset C_p(X)$. Then there exist a space Y and completely regular Hausdorff topologies $\tau'\leqslant\tau$ for Y such that*

(1) $Nag(Y,\tau)\leqslant Nag(X)$, $\ell\Sigma(Y,\tau)\leqslant\ell\Sigma(X)$, $d(Y,\tau)\leqslant d(X)$;
(2) $w(Y,\tau')\leqslant d(L)$;
(3) $nw(C_p(Y,\tau))=nw(Y,\tau)\leqslant\max\{Nag(X),d(L)\}$, $d(Y,\tau)\leqslant\max\{Nag(X),d(L)\}$, *hence* $d(C_p(Y,\tau))\leqslant\max\{Nag(X),d(L)\}$;
(4) *L is embedded into $C_p(Y,\tau)$;*
(5) $d(L)\leqslant\max\{Nag(L),d(Y,\tau)\}$. *Hence, if L is a Lindelöf Σ-space and $Nag(X)\leqslant d(L)$ (this holds for example if X is a Lindelöf Σ-space), then $d(L)=d(Y,\tau)$.*

Consequently, if X is a Lindelöf Σ-space and $Nag(C_p(X))\leqslant d(X)$, since X is embedded into $C_p(C_p(X))$ there exists a space Y admitting a weaker completely regular Hausdorff topology τ' such that X embeds into $C_p(Y)$ and $d(X)=d(Y)\geqslant w(Y,\tau')$.

Corollary 1. *Let X be a topological space and $\mathfrak{m}$ an infinite cardinal number with $\aleph_0\leqslant\mathfrak{m}\leqslant q(X)$. Let $L\subset C_p(X)$ a subspace of $C_p(X)$ such that $d(L)=\mathfrak{m}$. Then there exist a space Y and completely regular Hausdorff topologies τ and τ' on Y such that $\tau'\leqslant\tau$ and*

(1) $Nag(Y,\tau) \leqslant \min\{Nag(\upsilon_\lambda X)\colon \lambda \geqslant \mathfrak{m}\} \leqslant Nag(\upsilon_\mathfrak{m} X)$, $\ell\Sigma(Y,\tau) \leqslant \min\{\ell\Sigma(\upsilon_\lambda X)\colon \lambda \geqslant \mathfrak{m}\} \leqslant \ell\Sigma(\upsilon_\mathfrak{m} X)$;
(2) $w(Y,\tau') \leqslant d(L)$;
(3) $d(Y,\tau) \leqslant \min\{Nag(\upsilon_\lambda X)\colon \lambda \geqslant \mathfrak{m}\} \leqslant Nag(\upsilon_\mathfrak{m} X)$;
(4) *L is embedded into $C_p(Y,\tau)$.*

We provide applications of Theorem 1. Among the others, see Proposition 1, we characterize weakly analytic sets from the class $\mathfrak{G}$, and reprove the aforementioned generalization of Amir–Lindenstrauss theorem (Corollary 10) for spaces from the class $\mathfrak{G}$.

2. Proofs of Theorem 1 and corollaries

We are prepared to present a proof of Theorem 1.

Proof. Let D be a dense subset of L, such that $|D| = d(L)$. Let $\mathcal{T}_D$ and $\mathcal{T}_L$ be the weakest topologies on X that make continuous all those real-valued functions that belong to D or L, respectively. By density, $f(x) = f(y)$ for each $f \in D$ implies $f(x) = f(y)$ for each $f \in L$. Let $(\widehat{X}, \widehat{\mathcal{T}_D})$ and $(\widehat{X}, \widehat{\mathcal{T}_L})$ be the topological quotients of $(X, \mathcal{T}_D)$ and $(X, \mathcal{T}_L)$ with respect to the relation $x \sim y$ if and only if $f(x) = f(y)$ for all f of D and L, respectively. If we define the map $F : (X, \mathcal{T}_D) \to \mathbb{R}^D$ by $F(z) = \rho_z$, where $\rho_z(f) = f(z)$ for all $f \in D$, then clearly F is continuous and $x \sim y$ if and only if $F(x) = F(y)$. Hence, $(\widehat{X}, \widehat{\mathcal{T}_D})$ is homeomorphic to a subspace of $\mathbb{R}^D$ and consequently

$$w(\widehat{X}, \widehat{\mathcal{T}_D}) \leqslant w\big(\mathbb{R}^D\big) \leqslant |D| = d(L).$$

On the other hand, since $(\widehat{X}, \widehat{\mathcal{T}_L})$ is a continuous image of X, we note that

$$Nag(\widehat{X}, \widehat{\mathcal{T}_L}) \leqslant Nag(X), \qquad \ell\Sigma(\widehat{X}, \widehat{\mathcal{T}_L}) \leqslant \ell\Sigma(X), \qquad d(\widehat{X}, \widehat{\mathcal{T}_L}) \leqslant d(X),$$

see [7, Proposition 7(iv), Remark 8] and [10, Theorem 1.4.10]. Now, applying [18, Theorem 3.2] we have that

$$nw(\widehat{X}, \widehat{\mathcal{T}_L}) \leqslant \max\big\{Nag(\widehat{X}, \widehat{\mathcal{T}_L}), w(\widehat{X}, \widehat{\mathcal{T}_D})\big\} \leqslant \max\big\{Nag(X), d(L)\big\}.$$

On the other hand, by [2, Theorem I.1.3] we have

$$nw\big(C_p(\widehat{X}, \widehat{\mathcal{T}_L})\big) = nw(\widehat{X}, \widehat{\mathcal{T}_L}).$$

In particular, as the density is less or equal than the network weight we have that

$$d(\widehat{X}, \widehat{\mathcal{T}_L}) \leqslant \max\big\{Nag(X), d(L)\big\},$$
$$d\big(C_p(\widehat{X}, \widehat{\mathcal{T}_L})\big) \leqslant \max\big\{Nag(X), d(L)\big\}.$$

Therefore $(Y,\tau) := (\widehat{X}, \widehat{\mathcal{T}_L})$ is a completely regular Hausdorff space such that $d(Y,\tau) \leqslant \max\{Nag(X), d(L)\}$ and $d(C_p(Y,\tau)) \leqslant \max\{Nag(X), d(L)\}$. Now set $(Y,\tau') := (\widehat{X}, \widehat{\mathcal{T}_D})$. Clearly $\tau' \leqslant \tau$ on Y.

Setting

$$\widehat{x} := \big\{y \in X\colon\ f(y) = f(x) \text{ for all } f \in L\big\},$$

define $T : L \to C_p(Y,\tau)$ by $T(f) = \widehat{f}$, where $\widehat{f}(\widehat{x}) := f(x)$. Note that if $y \in \widehat{x}$, then

$$\widehat{f}(\widehat{y}) = f(y) = f(x) = \widehat{f}(\widehat{x}),$$

since $f \in L$, so that $\widehat{f}$ is well-defined. On the other hand, $\widehat{f} \in C_p(Y,\tau)$. Indeed, if $\widehat{x}_d \to \widehat{x}$ in (Y,τ), then

$$\widehat{f}(\widehat{x}_d) = f(x_d) \to f(x) = \widehat{f}(\widehat{x}).$$

If $\widehat{f}(\widehat{x}) = \widehat{g}(\widehat{x})$ for all $x \in X$ then $f(x) = g(x)$ for all $x \in X$, which implies that $f = g$ and consequently that $\widehat{f} = \widehat{g}$, so that T is injective. Moreover, if $f_p \to f$ in L then $f_p(x) \to f(x)$ for every $x \in X$, which implies that $\widehat{f}_p(\widehat{x}) \to \widehat{f}(\widehat{x})$. Hence, T is continuous. Finally, if $\widehat{f}_p \to \widehat{f}$ in $T(L)$ under the pointwise convergence topology, then $\widehat{f}_p(\widehat{x}) \to \widehat{f}(\widehat{x})$ and hence $f_p(x) \to f(x)$ for all $x \in X$. This shows that T is a homeomorphism from L into $C_p(Y,\tau)$.

Lastly, let B be a dense subset of (Y,τ) of cardinality $d(Y,\tau)$. The restriction map

$$j : C_p(Y,\tau) \to C_p(B)$$

is bijective and continuous. Since $w(C_p(B)) = |B|$, we deduce that $C_p(Y,\tau)$ admits a weaker Hausdorff topology ξ such that $w(C(Y,\tau),\xi) \leqslant |B|$. Hence, L admits a weaker topology ξ_L such that $w(L,\xi_L) \leqslant |B|$. By [7, Proposition 7(v) and Remark 8] we deduce that

$$d(L) \leqslant \max\{Nag(L), w(L, \xi_L)\} \leqslant \max\{Nag(L), |B|\}.$$

Since $Nag(L)$ is countable when L is Lindelöf Σ, the second part of (5) follows from (3) and from the previous inequality. □

We prove Corollary 1.

Proof. Let $D \subset L$ be a dense set in L with $|D| = d(L) = \mathfrak{m}$. Let λ be an infinite cardinal number such that $\lambda \geqslant \mathfrak{m}$ and $\psi : C_p(X) \to C_p(\upsilon_\lambda X)$ be defined by $\psi(f) = f^{\upsilon_\lambda}$ where f^{υ_λ} is the unique continuous extension of f to the whole $\upsilon_\lambda X$, [21, Proposition 2]. Since ψ is continuous on each set of cardinality λ, see [21, Proposition 1], $\psi(D)$ is a dense subset of $\psi(L)$ with $|\psi(D)| = d(L)$. On the other hand, as ψ^{-1} is continuous, there exists on $C(X)$ a topology ξ stronger than the pointwise convergence topology τ_p of $C_p(X)$ such that $C_p(\upsilon_\lambda X)$ is linearly homeomorphic to $(C(X), \xi)$. But $\tau_p|_D = \xi|_D$ (since $|D| \leqslant \lambda$, so ψ is a homeomorphism on D) so also $\tau_p|_L = \xi|_L$. Hence, $L \subset C_p(X)$ is homeomorphic to $\psi(L) \subset C_p(\upsilon_\lambda X)$. We apply Theorem 1 for each $\lambda \geqslant \mathfrak{m}$ and the proof finishes. □

Particular results applied to the countable case are obtained as corollaries. The first one follows from Corollary 1.

Corollary 2. *Let υX be a Lindelöf Σ-space. If $L \subset C_p(X)$ is separable, then there exists a separable submetrizable Lindelöf Σ-space Y such that L is embedded into $C_p(Y)$.*

Corollary 3. *Let X be a topological space such that $C_p(X)$ is a Lindelöf Σ-space. If $L \subset C_p(X)$ is separable, then there exists a separable submetrizable Lindelöf Σ-space (Y, τ) such that L is embedded into $C_p(Y)$.*

Proof. Let $\mathfrak{m} := \aleph_0$ in Corollary 1. Since $C_p(X)$ is a Lindelöf Σ-space then υX is a Lindelöf Σ-space, see [2, Theorem IV.9.5], now Corollary 2 applies. □

Corollary 4. *Let υX be a Lindelöf Σ-space and let L be a Lindelöf Σ-subspace of $C_p(X)$. Then L is separable if and only if there exists a separable and submetrizable Lindelöf Σ-space Y such that L is embedded into $C_p(Y)$.*

The following corollary can be found in [7, Proposition 15], that extends [23, Theorem 3.4] and [2, Corollary IV.2.10].

Corollary 5. *Let X be a compact space and λ an infinite cardinal number. Then $\ell\Sigma(C_p(X)) \leqslant \lambda$ if and only if there is a space Y such that $X \subset C_p(Y)$ (that is Y is homeomorphic to a subspace of $C_p(X)$ which separates points of X), and $\ell\Sigma(Y) \leqslant \lambda$.*

Proof. Assume that $\ell\Sigma(C_p(X)) \leqslant \lambda$. Since $X \subset C_p(C_p(X))$, by Theorem 1(4 and 1) there exists a space Y such that $X \subset C_p(Y)$ and $\ell\Sigma(Y) \leqslant \ell\Sigma(C_p(X)) \leqslant \lambda$. Conversely, if there exists a space Y such that $X \subset C_p(Y)$ and $\ell\Sigma(Y) \leqslant \lambda$, then Y is homeomorphic to a subspace of $C_p(X)$ and Y separates points of X. Then by [7, Proposition 14] we have that $\ell\Sigma(C_p(X)) \leqslant \lambda$. □

In [16] it is proved that if E is a Banach space and B^* is the unit ball in the dual E' with the weak*-topology, then $\ell\Sigma(E, \sigma(E, E')) = \ell\Sigma(C_p(B^*))$. Applying Corollary 5 we have the following general

Corollary 6. *For every Eberlein–Grothendieck space X there exists a compact space Y such that X embeds into $C_p(Y)$ and $\ell\Sigma(C_p(Y)) \leqslant \ell\Sigma(X)$.*

Proof. Since X is an *Eberlein–Grothendieck space*, there exists a compact space K such that X is homeomorphic to a subset of $C_p(K)$, see for example [2]. Let $\phi : K \to C_p(X)$ be the (continuous) map defined by $\phi(p)(f) := f(p)$ for each $f \in X$ and $p \in K$. Set $Y := \phi(K)$. Let $\phi^* : C_p(Y) \to C_p(K)$ be the map defined by $\phi^*(f) := f \circ \phi$. Then ϕ^* embeds $C_p(Y)$ into a closed subspace of $C_p(K)$, $C_p(Y)$ contains a subspace homeomorphic to X which separates points of Y, see [2, Corollary 0.4.8]. If λ is an infinite cardinal number with $\ell\Sigma(X) \leqslant \lambda$, by Corollary 5 we have that $\ell\Sigma(C_p(Y)) \leqslant \lambda$. Hence, $\ell\Sigma(C_p(Y)) \leqslant \ell\Sigma(X)$. □

A family $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of sets covering a set X is called a *resolution* of X if $A_\alpha \subset A_\beta$ whenever $\alpha \leqslant \beta$, $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. We call $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ a *compact resolution* in X if each set A_α is compact in X. Every K-analytic space has a compact resolution, see [23], or [5], and every angelic space with a compact resolution is K-analytic, see [5, Corollary 1.1].

Corollary 7. *Let υX be a Lindelöf Σ-space. Then a non-empty subset Y of $C_p(X)$ is analytic if and only if Y has a compact resolution and is contained in a separable subset of $C_p(X)$.*

Proof. If Y is contained in a separable subset of $C_p(X)$, then Y admits a weaker metric topology by Corollary 2. If additionally Y has a compact resolution, Y must be analytic by application [23] or [8, Theorem 15]. The converse is obvious. □

Corollary 8. *$C_p(X)$ is analytic if and only if $C_p(X)$ is separable and admits a compact resolution.*

Proof. Assume $C_p(X)$ is separable and has a compact resolution. Since υX is a Lindelöf Σ-space by [13, Corollary 23], we apply Corollary 7 with $Y := C_p(X)$. □

3. Application to locally convex spaces

In compact sets in locally convex spaces, two essential questions may naturally arise, the first one about the metrizability of such sets, the second relative to the weakly angelicity of whole space. We refer to [8] (and the references therein) where a list of positive results is provided (among others for (*LF*)-spaces and (*DF*)-spaces) concerning both questions. We only recall here that, inspired by some particular results about (*LF*)-spaces and (*DF*)-spaces, in [8] it is introduced and studied the so-called class $\mathfrak{G}$ of lcs, for which the two above-mentioned problems have positive answers.

An lcs E belongs to the class $\mathfrak{G}$ if there is a resolution $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in $(E', \sigma(E', E))$ (a $\mathfrak{G}$-representation of E') such that each sequence in any A_α is equicontinuous, [8]. The class $\mathfrak{G}$ is indeed large and contains "almost all" important locally convex spaces (including (*LF*)-spaces and (*DF*)-spaces), is stable by taking subspaces, Hausdorff quotients, countable direct sums and products. Nevertheless, as it is proved in [6], the space $C_p(X)$ belongs to the class $\mathfrak{G}$ if and only if $C_p(X)$ is metrizable.

A compact set K is a *Talagrand compact* if and only if it is homeomorphic to a weakly compact subset of an lcs from the class $\mathfrak{G}$, see [8, Theorem 12]. Therefore, one may ask when (weakly) compact sets in an lcs in class $\mathfrak{G}$ are (weakly) metrizable. Both questions were answered in [9] and [8], respectively. We prove the following more general case.

Proposition 1. *A subset Y of an lcs E from the class $\mathfrak{G}$ is $\sigma(E, E')$-analytic if and only if Y has a $\sigma(E, E')$-compact resolution and is contained in a $\sigma(E, E')$-separable subset.*

Proof. By [12, Corollary 1] the space $Z := (E', \sigma(E', E))$ is quasi-Suslin. Hence, there exists a quasi-Suslin map $T : \mathbb{N}^{\mathbb{N}} \to 2^Z$, $\alpha \mapsto T(\alpha)$. Note that υZ is K-analytic. Indeed, since every $T(\alpha)$ is countably compact, its closure $\overline{T(\alpha)}$ in υZ is compact. Then, $\alpha \mapsto \overline{T(\alpha)}$ is an usco map, so $W := \bigcup \overline{T(\alpha)}$ is K-analytic. Since $Z \subset W \subset \upsilon Z$, we have $W = \upsilon W = \upsilon Z$ is K-analytic. As $(E, \sigma(E, E')) \subset C_p(Z)$, we apply Corollary 7. □

Since every analytic compact set is metrizable, Proposition 1 yields [9, Theorem 10].

Corollary 9 *(Cascales–Orihuela). A $\sigma(E, E')$-compact set Y in an lcs E from the class $\mathfrak{G}$ is $\sigma(E, E')$-metrizable if and only if Y is contained in a $\sigma(E, E')$-separable subset of E.*

Note that Theorem 1 applies to show [8, Theorem 13].

Corollary 10 *(Cascales–Orihuela). Let E be an lcs from the class $\mathfrak{G}$ such that $(E, \sigma(E, E'))$ is a Lindelöf Σ-space, then $d(E, \sigma(E, E')) = d(E', \sigma(E', E))$.*

Proof. Since E belongs to $\mathfrak{G}$ and $L := (E, \sigma(E, E'))$ is a Lindelöf Σ-space, the space $X := (E', \sigma(E', E))$ is K-analytic by [22, Theorem 21] and [5, Corollary 1.1]. Clearly $L \subset C_p(X)$, so by Theorem 1(5, 1) there exists Y such that $d(L) = d(Y) \leqslant d(X)$. Note that $d(X) \leqslant d(L)$ also holds. Indeed, since $X \subset C_p(C_p(X))$, by Theorem 1(5, 1) there exists Z such that

$$d(X) \leqslant \max\{Nag(X), d(Z)\} = d(Z) \quad \text{and} \quad d(Z) \leqslant d\bigl(C_p(X)\bigr).$$

On the other hand, $d(C_p(X)) = iw(X)$, see [19] or [2, Theorem I.1.5]. If B is a dense subset of L, then $\sigma(E', B) \leqslant \sigma(E', E)$, so $iw(X) \leqslant d(L)$. Hence, $d(X) \leqslant d(L)$. □

Finally we provide a short proof of the following result of this type taken from [8]. We were informed by Prof. Cascales that a simple proof for E being an (*LF*)-space was already presented in the Meeting of Zaragoza (Spain) held in 1985.

Theorem 2 *(Cascales–Orihuela). A precompact set in an lcs from the class $\mathfrak{G}$ is metrizable.*

Proof. Since the completion of an lcs E in class $\mathfrak{G}$ belongs to the class $\mathfrak{G}$, we may assume that E is complete and a precompact set K is compact. Let $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-representation of E'. By τ we denote the topology of E. We say that a subset M of E' is K^0-separated if $(a + K^0) \cap M = \{a\}$ for each $a \in M$. By Zorn's lemma there exists a maximal K^0-separated subset M_1 of E'. Clearly $M_1 + K^0 = E'$. Note that M_1 is countable. Indeed, otherwise, since $E' = \bigcup\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ and $A_\alpha \subset A_\beta$ whenever $\alpha \leqslant \beta$, for α, β in $\mathbb{N}^{\mathbb{N}}$, we apply a standard argument providing a countable infinite subset P of M_1 and $\gamma \in \mathbb{N}^{\mathbb{N}}$ such that $P \subset A_\gamma$, see [20], or [5,15]. Since E belongs to $\mathfrak{G}$, P is equicontinuous, so P is precompact in the topology of uniform convergence on the τ-precompact subsets of E. Therefore there exists a finite set $\{a_i\colon 1 \leqslant i \leqslant k\} \subset P$

such that $P \subset \bigcup\{a_i + K^0\colon\ 1 \leqslant i \leqslant k\}$. Clearly there exists $1 \leqslant j \leqslant k$ such that the set $(a_j + K^0) \cap P$ is infinite, contradicting the hypothesis that M_1 $(\supset P)$ is K^0-separated. Let M_n be a maximal subset of E' that is $n^{-1}K^0$-separated, for each $n \in \mathbb{N}$. The set $M_0 := \bigcup\{M_n\colon\ n \in \mathbb{N}\}$ is countable. Let τ_{M_0} be the weakest topology on K that makes continuous the functions of M_0. If $x \neq y$ are two points of K then there exist $g \in E'$ and $n \in \mathbb{N}$ such that $|g(x) - g(y)| > 3n^{-1}$. Since $E' = M_n + n^{-1}K^0$, there exists $f \in M_n (\subset M_0)$ such that $g \in f + n^{-1}K^0$. Hence, $|f(x) - f(y)| > n^{-1}$. Therefore (K, τ_{M_0}) is metrizable, so K is metrizable. □

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