Georgian Math. J. **20** (2013), 247–254
DOI 10.1515/gmj-2013-0022

07

On precompact sets in spaces $C_c(X)$

Juan Carlos Ferrando and Jerzy Kąkol

Abstract. We show that the fact that X has a compact resolution swallowing the compact sets characterizes those $C_c(X)$ spaces which have the so-called $\mathfrak{G}$-base. So, if X has a compact resolution which swallows all compact sets, then $C_c(X)$ belongs to the class $\mathfrak{G}$ of Cascales and Orihuela (a large class of locally convex spaces which includes the (*LM*) and (*DF*)-spaces) for which all precompact sets are metrizable and, conversely, if $C_c(X)$ belongs to the class $\mathfrak{G}$ and X satisfies an additional mild condition, then X has a compact resolution which swallows all compact sets. This fully applicable result extends the classification of locally convex properties (due to Nachbin, Shirota, Warner and others) of the space $C_c(X)$ in terms of topological properties of X and leads to a nice theorem of Cascales and Orihuela stating that for X containing a dense subspace with a compact resolution, every compact set in $C_c(X)$ is metrizable.

Keywords. Compact resolution, space $C_c(X)$, $\mathfrak{G}$-base, class $\mathfrak{G}$, K-analytic space, W-space, quasibarrelled space.

2010 Mathematics Subject Classification. 54D30, 46E10, 46A03.

1 Preliminaries

In what follows, unless otherwise stated, X will be a Hausdorff completely regular space and $C_p(X)$ and $C_c(X)$ will denote the space $C(X)$ of all real-valued continuous functions defined on X provided with the pointwise convergence topology and with the compact-open topology, respectively.

Let us recall that a family $\mathcal{A} = \{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of a set X is called a *resolution* of X if $\bigcup\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\} = X$ and $A_\alpha \subseteq A_\beta$ for $\alpha \leq \beta$ (coordinatewise), see [9, Chapter 3]. A resolution $\mathcal{A}$ of a topological space X is called *compact* if it consists of compact sets.

A locally convex space (lcs) E is said to have a $\mathfrak{G}$-base (or a $\mathfrak{G}$-basis) (see [9, Chapter 1]) if there exists a basis $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of (absolutely convex) neighborhoods of the origin in E, such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$.

An lcs E is said to belong to the *class* $\mathfrak{G}$ if its topological dual E' has a resolution $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that for every $\alpha \in \mathbb{N}^\mathbb{N}$ each sequence in A_α is

The research was supported for the second named author by National Center of Science, Poland, Grant No. N N201 605340.

equicontinuous [4] (see also [9, Chapter 11]). Clearly, if an lcs E has a $\mathfrak{G}$-base, then E belongs to the class $\mathfrak{G}$. More precisely, as shown in [3, Lemma 2], a quasi-barrelled lcs E has a $\mathfrak{G}$-base if and only if E belongs to the class $\mathfrak{G}$. The class $\mathfrak{G}$ is large; it contains 'almost' all important classes of lcs, for example (*LM*)-spaces (hence metrizable lcs) and (*DF*)-spaces. Many important properties about spaces in the class $\mathfrak{G}$ (see [9] for details) have been discovered during last twenty years, mostly due to Cascales, Orihuela, Saxon, López Pellicer and others. The following remarkable result due to Cascales and Orihuela motivates our work

Theorem 1 ([4]). *Let E be an lcs in the class $\mathfrak{G}$. Every precompact set in E is metrizable. A weakly compact set Y in E is weakly metrizable if and only if Y is contained in a weakly separable set.*

On the other hand, although spaces $C_p(X)$ are not in $\mathfrak{G}$ for uncountable X, see [3], Cascales and Orihuela proved in [5] that for web-compact spaces X every separable compact set in $C_p(X)$ is metrizable.

The corresponding problem for spaces $C_c(X)$ seems to be more involved. We start by showing that the fact that X has a compact resolution swallowing the compact sets characterizes those $C_c(X)$ spaces that have a $\mathfrak{G}$-base. This implies that

($*$) if X has a compact resolution which swallows all compact sets, then $C_c(X)$ belongs to the class $\mathfrak{G}$ and, conversely, if $C_c(X)$ belongs to the class $\mathfrak{G}$ and X satisfies a mild condition (weaker than that of being realcompact), then X has a compact resolution which swallows all compact sets.

This fact also supplements some other results of [14], where Warner collected together several locally convex properties of $C_c(X)$ in terms of X. As a nice application of ($*$) we propose Corollary 5 (originally due to Cascales and Orihuela, see [4, Corollary 1.1 (i)]), stating that if X contains a dense subspace with a compact resolution, then every compact set in $C_c(X)$ is metrizable.

Let us recall that a space X is called a *Lindelöf* Σ-space [1] (or K-countably determined) if there is an upper semi-continuous compact-valued map from a non-empty subspace Ω of the product space $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}$ is equipped with the discrete topology, into X covering X. If the same holds for $\Omega = \mathbb{N}^{\mathbb{N}}$, then X is called *K-analytic*. On the other hand, X is *quasi-Suslin* if there exists a set-valued map T from $\mathbb{N}^{\mathbb{N}}$ into X covering X which is quasi-Suslin, i.e., such that if $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ then $\{x_n\}_{n=1}^{\infty}$ has a cluster point in $T(\alpha)$, see [13].

Note that K-analytic $\Leftrightarrow$ (Lindelöf $\wedge$ quasi-Suslin), and K-analytic $\Rightarrow$ Lindelöf Σ. A topological space X is *analytic* if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$. Every K-analytic space has a compact resolution, see [12] or [2], and every angelic space with a compact resolution is K-analytic, see [2, Corollary 1.1].

2 $C_c(X)$ spaces in class $\mathfrak{G}$

Recall that a subset B of a topological space X is called *bounding* [resp. *b-bounding*] if $f(B)$ is a bounded subset of $\mathbb{R}$ for all $f \in C(X)$ [for every bounded set M in $C_c(X)$ one has that $\sup\{|f(t)| : t \in B, f \in M\} < \infty$], see [11, Definition 10.1.16]. A topological space X is a *μ-space* [resp. *W-space*] if every bounding [b-bounding] set is relatively compact. Every realcompact space is a μ-space, and every μ-space is a W-space. Note also that if X is a hemicompact space, i.e., if X has an increasing sequence $\{K_n : n \in \mathbb{N}\}$ of compact sets that swallow all compact sets in X, then the sets $A_\alpha := K_{n_1}$ for each $\alpha = (n_k) \in \mathbb{N}^\mathbb{N}$ compose a compact resolution swallowing the compact sets. By Christensen [6, Theorem 3.3], a second countable regular space X is completely metrizable if and only if X has a compact resolution swallowing the compact sets. It is also worth while to recall that a compact space X is metrizable if and only if $(X \times X) \setminus \Delta$ has a compact resolution which swallows all compact sets, where $\Delta := \{(x,x) : x \in X\}$, see [4]. On the other hand, it is well known by Arens' theorem that $C_c(X)$ is metrizable if and only if X is hemicompact, see [14]. We start with the following variant of Arens and Tkachuk's theorems for spaces $C_c(X)$, where τ_p and τ_c are the pointwise and the compact-open topologies on $C(X)$, respectively.

Theorem 2. *$C_c(X)$ has a $\mathfrak{G}$-base if and only if X has a compact resolution that swallows all compact sets.*

Proof. For each compact $K \subseteq X$ and $\epsilon > 0$ define

$$[K,\epsilon] = \left\{f \in C(X) : \sup_{x \in K} |f(x)| \le \epsilon\right\}.$$

If $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution of X, set

$$U_\alpha = [A_\alpha, \alpha(1)^{-1}]$$

for $\alpha \in \mathbb{N}^\mathbb{N}$ and put $\mathfrak{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Clearly, $\mathfrak{U}$ is a family of absolutely convex and absorbing sets in $C(X)$ such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$, composing a filter base. The reader may easily check that $\mathfrak{U}$ is a $\mathfrak{G}$-base for a locally convex topology τ on $C(X)$ with $\tau_p \le \tau \le \tau_c$. Now assume that $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows all compact sets and let V be a neighborhood of the origin of $C_c(X)$. If K is a compact set in X with $[K,\epsilon] \subseteq V$ for some $\epsilon > 0$, choosing $\gamma \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq A_\gamma$ and $\gamma(1)^{-1} < \epsilon$, then $U_\gamma \subseteq [K,\epsilon] \subseteq V$. This shows that $\tau = \tau_c$, so $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base for $C_c(X)$.

Conversely, suppose that $C_c(X)$ has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. For every set U in $C(X)$ define the corresponding set $U^\diamond$ in X by writing

$$U^\diamond = \left\{x \in X : |f(x)| \le 1 \; \forall f \in U\right\}.$$

Clearly, $U^{\diamond}$ is closed in X and $U \subseteq V$ implies that $U^{\diamond} \supseteq V^{\diamond}$. If K is compact and $\epsilon > 0$, then $[K,\epsilon]^{\diamond} \subseteq K$, since if $x \in X \setminus K$ there is $f \in C(X)$ with $f(x) = 2$ and $f(K) = \{0\}$, so that $f \in [K,\epsilon]$ and $x \notin [K,\epsilon]^{\diamond}$. If K is compact and $0 < \epsilon \leq 1$, then $K \subseteq [K,\epsilon]^{\diamond}$, hence $[K,\epsilon]^{\diamond} = K$. In addition, if U is a neighborhood of the origin in $C_c(X)$, then $U^{\diamond}$ is compact. Indeed, if K is a compact set in X such that $[K,\epsilon] \subseteq U$ for some $\epsilon > 0$, by the previous observations $U^{\diamond} \subseteq [K,\epsilon]^{\diamond} \subseteq K$ and hence $U^{\diamond}$ is a closed set of a compact set.

Now for any K compact in X there is some $U_\alpha \subseteq [K,1]$, which means that $K = [K,1]^{\diamond} \subseteq U_\alpha^{\diamond}$. This shows that the family $\mathcal{A} = \{U_\alpha^{\diamond} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallows all compact sets, so, in particular, it covers X. As in addition $U_\alpha^{\diamond} \subseteq U_\beta^{\diamond}$ for $\alpha \leq \beta$, it follows that $\mathcal{A}$ is a compact resolution of X. □

Corollary 1. *If $C_c(X)$ has a $\mathfrak{G}$-base, then υX is K-analytic.*

Proof. If $C_c(X)$ has a $\mathfrak{G}$-base, Theorem 2 provides a compact resolution for X. Hence X is a quasi-Suslin space and [7, Lemma 29] ensures that υX is K-analytic. □

If $\mathfrak{b}$ denotes the *bounding cardinal*, defined as the least cardinality for unbounded subsets of the quasi-ordered space $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, where $\alpha \leq^* \beta$ if $\alpha(n) \leq \beta(n)$ for almost all $n \in \mathbb{N}$, and ω_1 denotes the first ordinal of uncountable cardinality $\aleph_1$, the following consequence of Theorem 2 holds.

Corollary 2. *The ordinal space $[0,\omega_1)$ has a compact resolution which swallows all compact sets if and only if $\aleph_1 = \mathfrak{b}$.*

Proof. According to [8, Theorem 3], the space $C_c([0,\omega_1))$ admits a $\mathfrak{G}$-base if and only if we assume that $\aleph_1 = \mathfrak{b}$. □

Example 1. *The converse in Corollary* 1 *fails in general.* Indeed, it is clear that $\upsilon([0,\omega_1)) = \beta([0,\omega_1))$, but $C_c([0,\omega_1))$ has a $\mathfrak{G}$-base if and only if $\aleph_1 = \mathfrak{b}$, see again [8, Theorem 3].

Theorem 3. *If X has a compact resolution which swallows all compact sets, then $C_c(X)$ belongs to the class $\mathfrak{G}$. Conversely, if $C_c(X)$ belongs to the class $\mathfrak{G}$ and in addition X is a W-space, then X has a compact resolution which swallows all compact sets.*

Proof. If X has a compact resolution which swallows all compact sets, we have seen that the space $C_c(X)$ has a $\mathfrak{G}$-base, consequently it belongs to the class

$\mathfrak{G}$. Conversely, if X is a W-space, then $C_c(X)$ is quasibarrelled, see [11, Theorem 10.1.12]. If in addition $C_c(X)$ belongs to the class $\mathfrak{G}$, then by [3, Lemma 2] the space $C_c(X)$ has a $\mathfrak{G}$-base. Hence Theorem 2 applies. □

Corollary 3. *A metric separable space X is completely metrizable if and only if $C_c(X)$ belongs to the class $\mathfrak{G}$.*

Proof. Because X is a second countable and a W-space (in fact, a μ-space), Christensen's afore-mentioned theorem [6, Theorem 3.3] and our Theorem 3 apply. □

3 $C_c(X)$ spaces whose compact sets are metrizable

First we note the following fact, which can be deduced from [5, Theorem 3].

Proposition 1. *Let X be a web-compact space and let K be a weakly compact set in $C_c(X)$, i.e., in the weak topology of $C_c(X)$. Then K is weakly metrizable if and only if K is contained in a weakly separable subset of $C_c(X)$.*

The class of web-compact spaces (introduced by Orihuela [10]) includes, for example, spaces with dense σ-compact subspaces, hence separable spaces and spaces with dense K-analytic subspaces, and is much larger than the class of spaces admitting a compact resolution swallowing all compact sets. Distinguishing examples of (even nonseparable) K-analytic spaces not admitting compact resolutions swallowing compact sets can be found in [9]. We propose a nice application of the former two theorems (cf. Corollary 5), where the most simple examples of X illustrating this result are separable spaces. First we note

Corollary 4. *If X contains a compact resolution swallowing compact sets, then every precompact set in $C_c(X)$ is metrizable.*

Proof. This follows from Theorems 1 and 3. □

Corollary 5 (Cascales–Orihuela). *If X contains a dense subspace with a compact resolution, then every compact set in $C_c(X)$ is metrizable. In particular, if X contains a dense K-analytic subspace, then all compact sets in $C_c(X)$ are metrizable.*

Proof. Let Y be a dense subspace of X with a compact resolution and let $T : C_c(X) \to C_c(Y)$ be the restriction map defined by $Tf = f|_Y$. Clearly, T is a continuous linear injection. Set $Z := T(C(X))$ and let ξ be the locally convex topology on Z that makes T a linear homeomorphism onto (Z, ξ). Clearly, the compact-open topology of $C(Y)$ restricted to Z is weaker than ξ. If τ_p and τ_c

denote the pointwise and the compact-open topology on $C(Y)$, respectively, according to the first part of the proof of Theorem 2 there exists a locally convex topology τ on $C(Y)$ such that $\tau_p|_Z \leq \tau|_Z \leq \tau_c|_Z \leq \xi$ and $(C(Y), \tau)$ has a $\mathfrak{G}$-base. Moreover, since $(C(Y), \tau)$ belongs to the class $\mathfrak{G}$ we see that $(Z, \tau|_Z)$, as a subspace of $(C(Y), \tau)$, also belongs to the class $\mathfrak{G}$.

If P is a compact set of $C_c(X)$, it is homeomorphic to a compact subset Q of (Z, ξ). Since $(Z, \tau|_Z)$ is Hausdorff, both topologies $\tau|_Q$ and $\xi|_Q$ coincide on Q, so we may apply Theorem 1 to deduce that $(Q, \tau|_Q)$ is metrizable. This implies that Q is metrizable in (Z, ξ) and, consequently, that P is metrizable in $C_c(X)$. □

Example 2. *For $C_p(X)$ the previous corollary does not hold.* If X is a non-metrizable Talagrand compact space, then $C_p(X)$ is K-analytic and $C_p(C_p(X))$ contains a non-metrizable compact set.

Corollary 6. *Let E be a barrelled lcs space in the class $\mathfrak{G}$ (for example the (LF)-space). If $X := (E', \sigma(E', E))$, then every precompact set in $C_c(X)$ is metrizable.*

Proof. Since E is barrelled in the class $\mathfrak{G}$, the space E has a $\mathfrak{G}$-basis, say $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. Hence the polars A_α^0 form a $\sigma(E', E)$-compact resolution in E'. If K is a $\sigma(E', E)$-compact subset of E', then, by the barrelledness of E, the set K is equicontinuous, so K^0 is a neighborhood of zero in E. Hence there exists $\gamma \in \mathbb{N}^\mathbb{N}$ with $A_\gamma \subseteq K^0$, which proves that $\{A_\alpha^0 : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution in X swallowing all compact sets. According to Theorem 3 the space $C_c(X)$ belongs to the class $\mathfrak{G}$ and consequently every precompact set in $C_c(X)$ is metrizable by Theorem 1. □

The proof of the latter proposition uses the fact that $C_c(E', \sigma(E', E))$ belongs to the class $\mathfrak{G}$. If we replace $(E', \sigma(E', E))$ by $C_p(X)$ with X countably tight, we have the following.

Corollary 7. *If X has countable tightness, then $C_c(C_p(X))$ belongs to the class $\mathfrak{G}$ if and only if X is countable and discrete.*

Proof. Since X has countable tightness, $C_p(X)$ is realcompact (see [1, Corollary II.4.17]) and hence a W-space. If $C_c(C_p(X))$ belongs to the class $\mathfrak{G}$, then $C_p(X)$ has a compact resolution which swallows all compact sets by Theorem 3 which, from Tkachuk's theorem [9, Theorem 9.14], implies that X is countable and discrete. Conversely, if X is countable and discrete, then $C_p(X)$ is homeomorphic to $\mathbb{R}^\mathbb{N}$. Consequently, $C_p(X)$ has a compact resolution which swallows the compact sets. Then $C_c(C_p(X))$ belongs to the class $\mathfrak{G}$ by Theorem 3. □

Question. We do not know whether Corollary 5 is true if we assume that X is a web-compact space in the sense of Orihuela.

Acknowledgments. We are indebted to the referee, whose accurate recommendations have highly improved the quality of the paper. Among them is the proof of the exact 'only if' part of Theorem 2.

Bibliography

[1] A. V. Arkhangel'skii, *Topological Function Spaces* (translated from the Russian), Math. Appl. (Soviet Series) 78, Kluwer, Dordrecht, 1992.

[2] B. Cascales, On K-analytic locally convex spaces, *Arch. Math. (Basel)* **49** (1987), no. 3, 232–244.

[3] B. Cascales, J. Kąkol and S. A. Saxon, Metrizability vs. Fréchet–Urysohn property, *Proc. Amer. Math. Soc.* **131** (2003), no. 11, 3623–3631.

[4] B. Cascales and J. Orihuela, On compactness in locally convex spaces, *Math. Z.* **195** (1987), no. 3, 365–381.

[5] B. Cascales and J. Orihuela, On pointwise and weak compactness in spaces of continuous functions, *Bull. Soc. Math. Belg. Ser. B* **40** (1988), no. 3, 331–352.

[6] J. P. R. Christensen, *Topology and Borel Structure. Descriptive Topology and Set Theory with Applications to Functional Analysis and Measure Theory*, North-Holland Math. Stud. 10, North-Holland, Amsterdam, 1974.

[7] J. C. Ferrando, Some characterizations for υX to be Lindelof Σ or K-analytic in terms of $C_p(X)$, *Topology Appl.* **156** (2009), no. 4, 823–830.

[8] J. C. Ferrando, J. Kąkol, M. López Pellicer and S. A. Saxon, Tightness and distinguished Frechet spaces, *J. Math. Anal. Appl.* **324** (2006), no. 2, 862–881.

[9] J. Kąkol, W. Kubiś and M. López Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Dev. Math. 24, Springer, New York, 2011.

[10] J. Orihuela, Pointwise compactness in spaces of continuous functions, *J. London Math. Soc. (2)* **36** (1987), no. 1, 143–152.

[11] P. Pérez Carreras and J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Math. Stud. 131, North-Holland, Amsterdam, 1987.

[12] M. Talagrand, Espaces de Banach faiblement $\mathcal{K}$-analytiques, *Ann. of Math. (2)* **110** (1979), no. 3, 407–438.

[13] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland Math. Stud. 67, North-Holland, Amsterdam, 1982.

[14] S. Warner, The topology of compact convergence on continuous function spaces, *Duke Math. J.* **25** (1958), 265–282.

Received June 22, 2012; revised February 12, 2013; accepted April 12, 2013.

Author information

Juan Carlos Ferrando, Centro de Investigación Operativa,
Universidad Miguel Hernández, 03202 Elche, Spain.
E-mail: jc.ferrando@umh.es

Jerzy Kąkol, Faculty of Mathematics and Informatics,
A. Mickiewicz University, 61-614 Poznań, Poland.
E-mail: kakol@amu.edu.pl