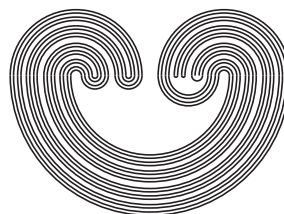


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## ON COMPACT SETS IN $C_b(X)$

by

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**ON COMPACT SETS IN  $C_b(X)$** 

JUAN CARLOS FERRANDO

**ABSTRACT.** Let  $X$  be a Tychonoff space and  $C_p(X)$  and  $C_b(X)$  denote the space  $C(X)$  of all real-valued continuous functions on  $X$  provided with the pointwise convergence and the compact-bounded topology, respectively. Let us call an unbounded subspace  $\Sigma$  of  $\mathbb{N}^{\mathbb{N}}$   $\sigma$ -complete if each bounded sequence  $\{\alpha_n\}_{n=1}^{\infty}$  in  $\Sigma$  verifies that  $\sup_{n \in \mathbb{N}} \alpha_n \in \Sigma$ , and let us call  $\sigma$ -complete every family  $\{A_\alpha : \alpha \in \Sigma\}$  of subsets of  $X$  such that  $\Sigma$  is  $\sigma$ -complete and  $A_\alpha \subseteq A_\beta$  whenever  $\alpha \leq \beta$ . We show that if there is a dense subspace of  $X$  covered by a  $\sigma$ -complete family consisting of bounded sets then  $C_p(X)$  is angelic. This is used to prove our main theorem, which asserts that if  $X$  has a dense subspace  $Y$  covered by a  $\sigma$ -complete family consisting of  $Y$ -bounded sets, then  $C_b(X)$  is angelic and every compact set in  $C_b(X)$  is metrizable.

**1. PRELIMINARIES**

Let us start by recalling that a subset  $A$  of a topological space  $X$  is called (functionally) *bounded* [1, Chapter 0] if  $f(A)$  is a bounded set in  $\mathbb{R}$  for every real-valued continuous function  $f$  on  $X$ . On the other hand, a subset  $B$  of a topological vector space  $E$  is (linearly) *bounded* [9, Chapter 3] if  $B$  is absorbed by every neighborhood of the origin. In what follows, unless otherwise stated,  $X$  will be a Hausdorff completely regular space and  $C_p(X)$ ,  $C_c(X)$  and  $C_b(X)$  will denote the space  $C(X)$  of all real-valued continuous functions defined on  $X$  equipped with the pointwise convergence topology, the compact-open and the compact-bounded topology, respectively. We denote by  $L(X)$  the topological dual of  $C_p(X)$  and by  $L_p(X)$  the linear space  $L(X)$  endowed with the weak\* topology.

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A locally convex space  $E$  belongs to the class  $\mathfrak{G}$  if its topological dual  $E'$  has a covering  $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  such that  $A_\alpha \subseteq A_\beta$  if  $\alpha \leq \beta$  and for each  $\alpha \in \mathbb{N}^{\mathbb{N}}$  every sequence in  $A_\alpha$  is equicontinuous [7, Chapter 11]. A space  $X$  is called *web-compact* if there is a mapping  $T$  from a subspace  $\Sigma$  of  $\mathbb{N}^{\mathbb{N}}$  into  $\mathcal{P}(X)$  such that  $\overline{\bigcup\{T(\alpha) : \alpha \in \Sigma\}} = X$  and if  $\alpha_n \rightarrow \alpha$  in  $\Sigma$  and  $x_n \in T(\alpha_n)$  for all  $n \in \mathbb{N}$  then  $\{x_n\}_{n=1}^\infty$  has a cluster point in  $X$  [7, Chapter 4]. It turns out that each *quasi-Suslin* space [7, Chapter 3] is web-compact, and every *Lindelöf  $\Sigma$ -space* [1, Chapter 0] is web-compact and Lindelöf. The space  $\mathbb{R}^{\mathbb{R}}$  is web-compact but not Lindelöf. A topological space  $X$  is *angelic* [5] if relatively countably compact sets in  $X$  are relatively compact and for every relatively compact subset  $A$  of  $X$  each point of  $\overline{A}$  is the limit of a sequence of  $A$ . In angelic spaces (relatively) compact sets, (relatively) countably compact sets and (relatively) sequentially compact sets are the same.

A uniform space  $(X, \mathcal{N})$  is called *trans-separable* if for every vicinity  $N$  of  $\mathcal{N}$  there is a countable subset  $Q$  of  $X$  such that  $\bigcup_{x \in Q} U_N(x) = X$ , where  $U_N(x) = \{y \in X : (x, y) \in N\}$ , see [7, Section 6.4]. Separable uniform spaces and Lindelöf uniform spaces are trans-separable but the converse statements are not true in general, although for uniform pseudometrizable spaces trans-separability is equivalent to separability. Equivalently, a (Hausdorff) uniform space  $(X, \mathcal{N})$  is trans-separable if it is uniformly isomorphic to a subspace of a uniform product of separable (pseudo) metric spaces. Each locally convex space in its weak topology is trans-separable when equipped with the (unique) translation-invariant uniformity associated to that topology. The class of trans-separable uniform spaces is hereditary, productive and closed under uniform continuous images. Moreover, it is shown in [3] that every uniform web-compact space is trans-separable, but if  $X := [0, \omega_1)$  (where  $\omega_1$  is the first ordinal of uncountable cardinality) and

$$N_\gamma := \{(\alpha, \beta) : \alpha = \beta \text{ or } (\alpha \geq \gamma \text{ and } \beta \geq \gamma)\} \subseteq X \times X$$

for  $0 \leq \gamma < \omega_1$ , then the family  $\{N_\gamma : 0 \leq \gamma < \omega_1\}$  is a base of a uniformity  $\mathcal{N}$  for  $X$  such that  $(X, \mathcal{N})$  is a trans-separable uniform space but  $(X, \tau_{\mathcal{N}})$ , where  $\tau_{\mathcal{N}}$  stands for the uniform topology of  $\mathcal{N}$ , is not a web-compact topological space. Regarding to the interest of this paper, let us recall that trans-separability is ‘dually’ related to metrizability of compact sets in the sense that trans-separability of the space  $C_c(X)$  is equivalent to metrizability of all compact subsets of  $X$  [7, Lemma 6.5] (see also [8, Theorem 2] for a generalization of this result to spaces of vector-valued continuous functions).

The following theorem extends some earlier results concerning metrizability of precompact sets in locally convex spaces due to Cascales, Orihuela, Pfister and Valdivia (see [7, Chapter 10] for details).

**Theorem 1.1** (Cascales-Orihuela). [7, Theorem 11.1] *Every precompact set in a locally convex space in the class  $\mathfrak{G}$  is metrizable.*

The best result on this subject is the next theorem, which characterizes those locally convex spaces whose precompact sets are metrizable in terms of trans-separability.

**Theorem 1.2** (Ferrando-Kąkol-López Pellicer). [7, Theorem 6.4] *Precompact subsets of a locally convex space  $E$  are metrizable if and only if  $E'$  endowed with the topology of uniform convergence on the precompact sets of  $E$  is trans-separable.*

Theorem 1.1 easily follows from Theorem 1.2 if one uses the following facts: (i) according to [4, Theorem 5], if  $E$  is a locally convex space in the class  $\mathfrak{G}$  and  $\tau_P$  denotes the topology on  $E'$  of uniform convergence on the precompact sets of  $E$ , then  $(E', \tau_P)$  is quasi-Suslin, (ii) every quasi-Suslin locally convex space is web-compact, and (iii) as observed above, every web-compact space is trans-separable.

In what follows, motivated by Orihuela's angelicity theorem (see Theorem 1.3 below), we introduce the notion of  $\sigma$ -complete family (cf. Definition 2.1) and show that if  $X$  contains a dense subspace covered by a  $\sigma$ -complete family consisting of bounded sets, then  $C_p(X)$  is angelic (Theorem 2.6). Then we prove that if  $X$  has a dense subspace  $Y$  covered by a  $\sigma$ -complete family consisting of  $Y$ -bounded sets, every compact set in  $C_b(X)$  is metrizable (Theorem 2.8). We shall later on use the two following results.

**Theorem 1.3** (Orihuela). [10], [7, Theorem 4.5] *If  $X$  is web-compact then  $C_p(X)$  is angelic.*

**Theorem 1.4** (Ferrando-Kąkol-López Pellicer). [7, Theorem 6.3] *Compact subsets of a locally convex space  $E$  are metrizable if and only if  $E'$  endowed with the topology of uniform convergence on the compact sets of  $E$  is trans-separable.*

## 2. COMPACT SETS IN $C_b(X)$

If  $X$  is a Hausdorff completely regular space, let us denote by  $\mathcal{D}$  the admissible uniform structure for  $X$  generated by the family of pseudometrics  $\{d_f : f \in C(X)\}$  with  $d_f(x, y) = |f(x) - f(y)|$  for  $x, y \in X$ . According to [6, 15.14 Corollary, (a)], the uniform space  $(X, \mathcal{D})$  is complete if and only if  $X$  is realcompact. Alternatively,  $X$  is realcompact if and only if every net  $\{x_d : d \in \mathcal{D}\}$  in  $X$  converges if  $\lim f(x_d)$  exists for each  $f \in C(X)$  (see [2, Theorem 2]).

Observe that if  $\{\alpha_n\}_{n=1}^\infty$  is a bounded sequence of the metric space  $\mathbb{N}^\mathbb{N}$  then  $\{\alpha_n(i)\}_{n=1}^\infty$  is a bounded sequence in  $\mathbb{N}$  for each  $i \in \mathbb{N}$ , so that  $\gamma(i) = \sup_{n \in \mathbb{N}} \alpha_n(i)$  belongs to  $\mathbb{N}$  and  $\alpha_n \leq \gamma$  for all  $i \in \mathbb{N}$ . Consequently, if a set  $X$  contains a family of sets  $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$  such that  $A_\alpha \subseteq A_\beta$  if  $\alpha \leq \beta$ , for any bounded sequence  $\{\alpha_n\}_{n=1}^\infty$  in  $\mathbb{N}^\mathbb{N}$  and any sequence  $\{x_n\}_{n=1}^\infty$  in  $X$  such that  $x_n \in A_{\alpha_n}$  for each  $n \in \mathbb{N}$  it holds that  $x_n \in A_\gamma$  for all  $n \in \mathbb{N}$ . This fact motivates the following definition.

**Definition 2.1.** An unbounded subspace  $\Sigma$  of  $\mathbb{N}^\mathbb{N}$  is called  $\sigma$ -complete if for each bounded sequence  $\{\alpha_n\}_{n=1}^\infty$  in  $\Sigma$  it holds that  $\sup_{n \in \mathbb{N}} \alpha_n \in \Sigma$ . If  $\Sigma$  is a  $\sigma$ -complete subspace of  $\mathbb{N}^\mathbb{N}$  and  $\mathcal{A} = \{A_\alpha : \alpha \in \Sigma\}$  a family of subsets of a set  $X$  such that  $A_\alpha \subseteq A_\beta$  if  $\alpha \leq \beta$ , we say that  $\mathcal{A}$  is a  $\sigma$ -complete family of sets.

**Example 2.2.** *Some  $\sigma$ -complete subspaces of  $\mathbb{N}^\mathbb{N}$ .* Note that every  $\sigma$ -complete subspace  $\Sigma$  of  $\mathbb{N}^\mathbb{N}$  is a directed subset of the ordered set  $(\mathbb{N}^\mathbb{N}, \leq)$ . The whole space  $\mathbb{N}^\mathbb{N}$  is clearly  $\sigma$ -complete. For every  $\alpha \in \mathbb{N}^\mathbb{N}$  and every  $n > 1$  the set  $\Sigma(\alpha, n) = \{\gamma \in \mathbb{N}^\mathbb{N} : \gamma(i) \leq \alpha(i), i \geq n\}$  is also a  $\sigma$ -complete subspace of  $\mathbb{N}^\mathbb{N}$ . Setting  $\gamma_n = (n, n+1, n+2, \dots)$  then  $\{\gamma_n : n \in \mathbb{N}\}$  is a countable  $\sigma$ -complete subspace of  $\mathbb{N}^\mathbb{N}$ .

**Example 2.3.** *Examples of  $\sigma$ -complete family of sets.* If  $\Sigma$  is a  $\sigma$ -complete subspace of  $\mathbb{N}^\mathbb{N}$  and  $\mathcal{B} = \{B_\alpha : \alpha \in \Sigma\}$  is an arbitrary family of subsets of a set  $X$ , setting

$$A_\alpha := \bigcup \{B_\beta : \beta \in \Sigma, \beta \leq \alpha\}$$

the family  $\mathcal{A} = \{A_\alpha : \alpha \in \Sigma\}$  is clearly  $\sigma$ -complete and  $B_\alpha \subseteq A_\alpha$  for every  $\alpha \in \Sigma$ .

**Lemma 2.4.** *If  $X$  contains a  $\sigma$ -complete family consisting of bounded sets whose union is dense in  $X$ , then the Hewitt realcompactification  $vX$  of  $X$  is web-compact.*

*Proof.* Let  $\{A_\alpha : \alpha \in \Sigma\}$  be a  $\sigma$ -complete family of  $X$ -bounded sets in  $X$  covering a dense subspace  $Y$  of  $X$ . Let us start with some useful remarks. If  $\delta : X \rightarrow L_p(X)$  stands for the canonical homomorphism  $\delta(x) = \delta_x$  that embeds  $X$  into a (closed) subspace  $\delta(X)$  of  $L_p(X)$ , the uniformity  $\mathcal{D}$  for  $X$  is (uniformly) isomorphic to the relative uniformity on  $\delta(X)$  induced by the uniform structure for  $L(X)$  associated to its weak\* topology, i.e. the (unique) translation invariant uniformity for  $L(X)$  defined by the locally convex structure of  $L_p(X)$ . So, if necessary, we can identify  $X$  with  $\delta(X)$  and  $\mathcal{D}$  with the uniformity for  $\delta(X)$  induced by the weak\* topology of

$L(X)$ . On the other hand, note that clearly  $A$  is a (functionally) bounded subset of  $X$  if and only if  $\delta(A)$  is a (linearly) bounded set in  $L_p(X)$ . Hence, given that the weak\* topology of  $L(X)$  is a weak locally convex topology and every weakly bounded set in a locally convex space is precompact [9, 20.9 (3)], it follows that  $A$  is bounded if and only if  $\delta(A)$  is a precompact subset of the locally convex space  $L_p(X)$ . In other words, a subset  $A$  of the topological space  $X$  is  $X$ -bounded if and only if  $A$  is a  $\mathcal{D}$ -precompact set of the uniform space  $(X, \mathcal{D})$ .

Let  $(\tilde{X}, \tilde{\mathcal{D}})$  denote the completion of  $(X, \mathcal{D})$  and  $\tau_{\tilde{\mathcal{D}}}$  be the uniform topology on  $\tilde{X}$  induced by the uniformity  $\tilde{\mathcal{D}}$  for  $\tilde{X}$ . According to [6, 15.13 Theorem, (a)], the topological space  $(\tilde{X}, \tau_{\tilde{\mathcal{D}}})$  coincides with  $vX$ . We claim that  $vX$  is a web-compact space.

If we define the map  $T : \Sigma \rightarrow \mathcal{P}(\tilde{X})$  by  $T(\alpha) = A_\alpha$ , on the one hand we have that  $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = Y$ . Moreover, since  $Y$  is dense in  $X$  and  $X$  is dense in  $(\tilde{X}, \tau_{\tilde{\mathcal{D}}})$ , it follows that that  $\overline{\bigcup\{T(\alpha) : \alpha \in \Sigma\}}^{\tau_{\tilde{\mathcal{D}}}} = \tilde{X}$ . On the other hand, if  $\alpha_n \rightarrow \alpha$  in  $\mathbb{N}^{\mathbb{N}}$  and  $x_n \in T(\alpha_n)$  for each  $n \in \mathbb{N}$ , since  $\Sigma$  is a  $\sigma$ -complete subspace of  $\mathbb{N}^{\mathbb{N}}$  one has that  $\sup \alpha_n \in \Sigma$ . Hence, setting  $\gamma(i) := \sup\{\alpha_n(i) : n \in \mathbb{N}\}$ , then  $\gamma \in \Sigma$  and  $\alpha_n \leq \gamma$  for every  $n \in \mathbb{N}$ . Consequently,  $A_n \subseteq A_\gamma$  and  $x_n \in A_\gamma$  for every  $n \in \mathbb{N}$ . So, the fact that  $A_\gamma$  is  $\mathcal{D}$ -precompact, ensures that the sequence  $\{x_n\}_{n=1}^\infty$  has a  $\mathcal{D}$ -Cauchy subnet in  $X$  which converges in  $\tilde{X}$  under the uniform topology  $\tau_{\tilde{\mathcal{D}}}$ . Hence  $(\tilde{X}, \tau_{\tilde{\mathcal{D}}})$  is web-compact.  $\square$

**Remark 2.5.** *It follows from the previous lemma that every realcompact space  $X$  that contains a  $\sigma$ -complete family consisting of bounded sets covering  $X$  is web-compact. Every non-separable Banach space  $E$  with closed unit ball  $B$  contains a  $\sigma$ -complete family of sets, namely  $\{\alpha(1)B : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  (consisting of linearly bounded but not functionally bounded sets) which covers  $E$ . But  $E$  is not web-compact since it is not trans-separable.*

**Theorem 2.6.** *If  $X$  contains a  $\sigma$ -complete family consisting of bounded sets whose union is dense in  $X$ , then  $C_p(X)$  is angelic.*

*Proof.* Lemma 2.4 and Theorem 1.3 combine to get that the space  $C_p(vX)$  is angelic, which as is well known forces the space  $C_p(X)$  to be angelic as well.  $\square$

The following lemma extends the beautiful main theorem of [11].

**Lemma 2.7.** *Let  $(X, \mathcal{N})$  be a uniform space. If the space  $X$  has a  $\sigma$ -complete covering consisting of precompact sets, then  $(X, \mathcal{N})$  is trans-separable.*

*Proof.* Let  $\{A_\alpha : \alpha \in \Sigma\}$  be a  $\sigma$ -complete covering of  $X$  consisting of precompact sets. If  $(\tilde{X}, \tilde{\mathcal{N}})$  stands for the completion of  $(X, \mathcal{N})$  and  $T : \Sigma \rightarrow \mathcal{P}(\tilde{X})$  is defined in the usual way by  $T(\alpha) = A_\alpha$ , then  $\overline{\bigcup\{T(\alpha) : \alpha \in \Sigma\}^{\tau_{\tilde{\mathcal{N}}}}} = \tilde{X}$ . On the other hand, if  $\alpha_n \rightarrow \alpha$  in  $\Sigma$  and  $x_n \in T(\alpha_n)$  for each  $n \in \mathbb{N}$ , the  $\sigma$ -completeness of  $\Sigma$  yields that  $\sup \alpha_n \in \Sigma$ . Setting  $\gamma(i) := \sup\{\alpha_n(i) : n \in \mathbb{N}\}$  as in the proof of Lemma 2.4, then  $\gamma \in \Sigma$  and  $\alpha_n \leq \gamma$  for all  $n \in \mathbb{N}$ . Thus  $x_n \in A_\gamma$  for all  $n \in \mathbb{N}$  which, by virtue of the precompactness of  $A_\gamma$ , implies that  $\{x_n\}_{n=1}^\infty$  has a convergent subnet in  $(\tilde{X}, \tau_{\tilde{\mathcal{N}}})$ . This shows that  $(\tilde{X}, \tau_{\tilde{\mathcal{N}}})$  is web-compact, so that  $(\tilde{X}, \tilde{\mathcal{N}})$  is trans-separable. Thus  $(X, \mathcal{N})$  is trans-separable too.  $\square$

**Theorem 2.8.** *If there is a dense subspace  $Y$  of  $X$  covered by a  $\sigma$ -complete family consisting of  $Y$ -bounded sets, then  $C_b(X)$  is angelic and every compact set in  $C_b(X)$  is metrizable.*

*Proof.* First note that every  $Y$ -bounded subset of  $Y$  is  $X$ -bounded, so Theorem 2.6 ensures that  $C_p(X)$  is angelic. Since the compact-bounded topology is stronger than the pointwise convergence topology, the angelic lemma [5] guarantees that the space  $C_b(X)$  is angelic.

For the second statement let  $\{A_\alpha : \alpha \in \Sigma\}$  be a  $\sigma$ -complete family of  $Y$ -closed and  $Y$ -bounded sets in  $X$  covering  $Y$ . Let  $\tau_p$  and  $\tau_b$  denote the pointwise convergence and the compact-bounded topology on  $C(Y)$ , respectively. Since  $\Sigma$  is unbounded in  $\mathbb{N}^{\mathbb{N}}$  there is  $k \in \mathbb{N}$  such that  $\sup\{\alpha(k) : \alpha \in \Sigma\} = +\infty$ . Define

$$U_\alpha = \{f \in C(Y) : \sup_{y \in A_\alpha} |f(y)| \leq \alpha(k)^{-1}\}$$

for  $\alpha \in \Sigma$  and set  $\mathcal{U} := \{U_\alpha : \alpha \in \Sigma\}$ . Since  $(\Sigma, \leq)$  is a directed set and  $\bigcup\{A_\alpha : \alpha \in \Sigma\} = Y$ , we can see that  $\mathcal{U}$  is a family of absolutely convex and absorbing sets in  $C(Y)$  that compose a filter base. It can be easily seen that  $\bigcap\{U_\alpha : \alpha \in \Sigma\} = \{0\}$  and that for each  $\alpha \in \Sigma$  and  $\epsilon > 0$  there is  $\gamma \in \Sigma$  with  $U_\gamma \subseteq \epsilon \cdot U_\alpha$ . For instance, for the latter statement choose  $\beta \in \Sigma$  such that  $\beta(k) \geq \alpha(k) \cdot \epsilon^{-1}$  and then select  $\gamma \in \Sigma$  with  $\gamma(i) = \max\{\alpha(i), \beta(i)\}$  for every  $i \in \mathbb{N}$ . If  $f \in U_\gamma$ , since  $A_\alpha \subseteq A_\gamma$  we have

$$\sup_{y \in A_\alpha} |\epsilon^{-1} f(y)| \leq \epsilon^{-1} \cdot \sup_{y \in A_\gamma} |f(y)| \leq \epsilon^{-1} \cdot \gamma(k)^{-1} \leq \epsilon^{-1} \cdot \beta(k)^{-1} \leq \alpha(k)^{-1}$$

which means that  $f \in \epsilon \cdot U_\alpha$ . Hence  $\tau$  is a Hausdorff locally convex topology on  $C(Y)$  for which  $\mathcal{U}$  is a base of neighborhoods of the origin which satisfies that  $\tau_p \leq \tau \leq \tau_b$ .

Setting  $F := (C(Y), \tau)'$ , if  $\alpha \leq \beta$  with  $\alpha, \beta \in \Sigma$  it turns out that  $U_\beta \subseteq U_\alpha$ , which implies that  $U_\alpha^0 \subseteq U_\beta^0$ , the polars being taken in  $F$ . Thus  $\{U_\alpha^0 : \alpha \in \Sigma\}$  is a  $\sigma$ -complete covering of  $F$  formed by equicontinuous sets. Since the topology  $\zeta$  on  $F$  of uniform convergence on the compact

sets of  $(C(Y), \tau)$  agrees with the weak\* topology  $\sigma(F, C(Y))$  of  $F$  on every equicontinuous set, it follows that each  $U_\alpha^0$  is a  $\zeta$ -compact set. This implies that the uniform space  $(F, \mathcal{N})$ , where  $\mathcal{N}$  is the uniform structure for  $F$  associated to the locally convex topology  $\zeta$ , has a  $\sigma$ -complete covering consisting of precompact (in fact, compact) sets, which according to Lemma 2.7 means that  $(F, \mathcal{N})$  is trans-separable. Hence  $(F, \zeta)$  is a trans-separable locally convex space and Theorem 1.4 guarantees that every compact set in  $(C(Y), \tau)$  is metrizable.

Now observe that the restriction map  $S : C_b(X) \rightarrow (C(Y), \tau)$  defined  $S(f) = f|_Y$  is a continuous (linear) injection from  $C_b(X)$  into  $(C(Y), \tau)$ . So if  $K$  is a compact set in  $C_b(X)$  its image  $S(K)$  is a compact set in  $(C(Y), \tau)$ , hence metrizable. Given that  $S$  restricts itself to an homeomorphism on  $K$ , it follows that  $K$  is metrizable in  $C_b(X)$  as required.  $\square$

**Example 2.9.** For  $C_p(X)$  the previous theorem does not hold. If  $X$  is a non-metrizable Talagrand compact space then  $C_p(X)$  is  $K$ -analytic. Hence, according to a classic result of [12], there is a family  $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$  of compact sets covering  $C_p(X)$  such that  $K_\alpha \subseteq K_\beta$  whenever  $\alpha \leq \beta$ . Therefore, although  $C_p(X)$  has a  $\sigma$ -complete covering consisting of compact sets (hence bounded sets), the space  $C_p(C_p(X))$  contains a non-metrizable Fréchet-Urysohn compact set.

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**AUTHOR'S COMMENT ON:  
"ON COMPACT SETS IN  $C_b(X)$ "**

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The name 'compact-bounded' used in [1] for the topology of uniform convergence on bounded subsets of the domain is not standard, this topology is usually known as the 'bounded-open' topology.

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