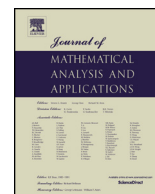




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## Compact covers and function spaces <sup>☆</sup>


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### ABSTRACT

For a Tychonoff space  $X$ , we denote by  $C_p(X)$  and  $C_c(X)$  the space of continuous real-valued functions on  $X$  equipped with the topology of pointwise convergence and the compact-open topology respectively. Providing a characterization of the Lindelöf  $\Sigma$ -property of  $X$  in terms of  $C_p(X)$ , we extend Okunev's results by showing that if there exists a surjection from  $C_p(X)$  onto  $C_p(Y)$  (resp. from  $L_p(X)$  onto  $L_p(Y)$ ) that takes bounded sequences to bounded sequences, then  $\nu Y$  is a Lindelöf  $\Sigma$ -space (respectively  $K$ -analytic) if  $\nu X$  has this property. In the second part, applying Christensen's theorem, we extend Pelant's result by proving that if  $X$  is a separable completely metrizable space and  $Y$  is first countable, and there is a quotient linear map from  $C_c(X)$  onto  $C_c(Y)$ , then  $Y$  is a separable completely metrizable space. We study also a non-separable case, and consider a different approach to the result of J. Baars, J. de Groot, J. Pelant and V. Valov, which is based on the combination of two facts: Complete metrizability is preserved by  $\ell_p$ -equivalence in the class of metric spaces (J. Baars, J. de Groot, J. Pelant). If  $X$  is completely metrizable and  $\ell_p$ -equivalent to a first-countable  $Y$ , then  $Y$  is metrizable (V. Valov). Some additional results are presented.

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## 1. Introduction

All spaces considered in this article are assumed to be completely regular and Hausdorff. We use terminology and notation as in [10]. We say that a set  $A$  in a space  $X$  is *functionally bounded* in  $X$  if every continuous real-valued function on  $X$  is bounded on  $A$ . A space  $X$  is a  $\mu$ -space if every closed functionally bounded subspace of  $X$  is compact. A *Polish space* is a separable completely metrizable space. The symbol  $\omega$  denotes the smallest infinite ordinal (so  $\omega$  is the set of all non-negative integers).

We denote by  $C_p(X)$  and  $C_c(X)$  the spaces of continuous real-valued functions on  $X$  endowed with the topology of pointwise convergence and the compact-open topology respectively.  $L_p(X)$  is the topological dual of  $C_p(X)$  endowed with the weak\* topology. We assume that  $X$  is a subspace of  $L_p(X)$  by virtue of the standard embedding  $x \mapsto \hat{x}$  where  $\hat{x}(f) = f(x)$  for each  $f \in C_p(X)$ .

Recall that Nagata's theorem states that if the topological rings  $C_p(X)$  and  $C_p(Y)$  are topologically isomorphic, then  $X$  and  $Y$  are homeomorphic, see [5]. This suggests the following problem, see [2].

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Two spaces  $X$  and  $Y$  are said to be  $t$ -equivalent ( $\ell_p$ -equivalent) if the spaces  $C_p(X)$  and  $C_p(Y)$  are homeomorphic (linearly homeomorphic). We say that a topological property  $\mathcal{P}$  is preserved by  $t$ -equivalence ( $\ell_p$ -equivalence) if whenever two spaces  $X$  and  $Y$  are  $t$ -equivalent ( $\ell_p$ -equivalent) and  $X$  has the property  $\mathcal{P}$ ,  $Y$  has the property  $\mathcal{P}$  too.

*What topological properties are preserved by the relations of  $t$ -equivalence and  $\ell_p$ -equivalence?*

Clearly, a property  $\mathcal{P}$  is preserved by the relation of  $t$ -equivalence ( $\ell_p$ -equivalence) if and only if there is a dual topological (linear topological) property  $\mathcal{Q}$  such that  $X$  has  $\mathcal{P}$  if and only if  $C_p(X)$  has  $\mathcal{Q}$ ; in different words, if  $\mathcal{P}$  “admits a description in terms of the (linear) topological structure of  $C_p(X)$ ”.

We will say that two spaces  $X$  and  $Y$  are  $\ell_c$ -equivalent if the spaces  $C_c(X)$  and  $C_c(Y)$  are linearly homeomorphic. Note that if  $X$  and  $Y$  are  $\ell_p$ -equivalent and  $X$  is a  $\mu$ -space, then the spaces  $X$  and  $Y$  are also  $\ell_c$ -equivalent, see [4]. It is well known (by a combination of Milyutin’s and Pestov’s results, see [4, Theorem 3]) that  $[0, 1]$  and  $[0, 1] \times [0, 1]$  are  $\ell_c$ -equivalent but not  $\ell_p$ -equivalent. On the other hand, if  $X$  and  $Y$  are  $\ell_p$ -equivalent and  $X$  is Dieudonné complete (in particular, if  $X$  is paracompact or realcompact), then  $X$  and  $Y$  are  $\ell_c$ -equivalent, see [4, Theorem 1].

We say that a space  $X$   $\ell_c$ -covers a space  $Y$  if there is a continuous open linear mapping from  $C_c(X)$  onto  $C_c(Y)$ . Clearly, if  $X$  and  $Y$  are  $\ell_c$ -equivalent, then each of the two  $\ell_c$ -covers the other.

There are many known results about preservation and non-preservation of various topological properties by  $t$ -equivalence and  $\ell_p$ -equivalence; see, e.g., [4–6,20,27]. For example, metrizability, local compactness, the countability of weight, normality and paracompactness are not  $\ell_p$ -invariant. On the other hand, hemicompactness, the property of being an  $\aleph_0$ -space, the Lindelöf  $\Sigma$ -property,  $K$ -analyticity and analyticity are preserved by  $\ell_p$ -equivalence.

We denote by  $\mathbb{P}$  the space  $\omega^\omega$  endowed with the Tychonoff product topology (with all the factors discrete). We equip the space  $\mathbb{P}$  with the natural partial order:  $p \leq q$  if and only if  $p(n) \leq q(n)$  for all  $n \in \omega$ . For an element  $p$  of  $\mathbb{P}$  and a natural number  $k$  we denote by  $p|k$  the finite sequence  $(p(1), \dots, p(k))$ . Given a finite sequence  $\sigma = (\sigma_1, \dots, \sigma_n)$  of natural numbers, we denote by  $W(\sigma)$  the set  $\{p \in \mathbb{P} : p|n = \sigma\}$ . Clearly, for every  $p \in \mathbb{P}$ , the family of sets  $\{W(p|n) : n \in \omega\}$  is a base of open neighborhoods of  $p$  in  $\mathbb{P}$ .

A family of subspaces  $\mathcal{R} = \{A_p : p \in \mathbb{P}\}$  of a space  $X$  is called a resolution of  $X$  if it covers  $X$  and  $A_p \subset A_q$  whenever  $p \leq q$ . We say that a resolution  $\mathcal{R}$  is compact if each element  $A_p$  of  $\mathcal{R}$  is compact. If  $X$  is a topological vector space, we say that a resolution  $\mathcal{R}$  of  $X$  is bounded if each element of  $\mathcal{R}$  is bounded in  $X$  (that is, absorbed by any neighborhood of zero). A resolution  $\mathcal{R}$  swallows compact sets if every compact subspace of  $X$  is contained in some element of  $\mathcal{R}$ .

As usual, a set-valued mapping  $T : X \rightarrow Y$  is called compact valued if the set  $T(x)$  is compact for every  $x \in X$ , and is upper semicontinuous if for every open set  $V$  in  $Y$ , the set  $\{x \in X : T(x) \subset V\}$  is open. We abbreviate “compact valued upper semicontinuous” as “usco”. For a set  $A \subset X$  we denote  $T(A) = \bigcup\{T(x) : x \in A\}$ , and we say that  $T$  is onto  $Y$  if  $T(X) = Y$ . We denote the family of all compact subspaces of a space  $X$  by  $\mathcal{K}(X)$  (so compact-valued mappings to  $X$  are the same as functions to  $\mathcal{K}(X)$ ).

In Section 2 we find a characterization of Lindelöf  $\Sigma$ -property of  $\nu X$  in terms of the linear topological structure of  $C_p(X)$  and use it to show that if  $\nu X$  is a Lindelöf  $\Sigma$ -space or a  $K$ -analytic space and there exists a surjection from  $L_p(X)$  onto  $L_p(Y)$  that takes bounded sequences to bounded sequences, then  $\nu Y$  is a Lindelöf  $\Sigma$ -space (respectively,  $K$ -analytic); we prove a similar statement for the Lindelöf  $\Sigma$ -property of  $\nu X$  and mappings between  $C_p(X)$  and  $C_p(Y)$ . This supplements some earlier results of Okunev [18].

A. Arhangel’skii asks in [3, Problem 20] if a first-countable space  $Y$  which is  $\ell_p$ -equivalent to a metrizable space  $X$  must also be metrizable. In [6, Theorem 3.3] J. Baars, J. de Groot and J. Pelant proved that complete metrizability is preserved by  $\ell_p$ -equivalence in the class of metrizable spaces. They also gave an alternative proof for separable metrizable spaces by using Christensen’s theorem (Theorem 1 below), [6, Theorem 5.1, Theorem 5.3]. Later on, Valov proved, using the results in [25], that the answer to Arhangel’skii’s problem is positive for Čech-complete spaces  $Y$ , see [27, Corollary 4.6]. The combination of the above facts yields: *The property of being a completely metrizable space is preserved by the  $\ell_p$ -equivalence for spaces satisfying the first axiom of countability.*

In Section 3 of this article we discuss the preservation of complete metrizability by  $\ell_c$ -equivalence. We give a different proof of the preservation of complete metrizability in the case of  $\ell_c$ -equivalent spaces  $X$  and  $Y$  where  $X$  is Polish and  $Y$  is first countable, and discuss the non-separable case. Note however that from Valov’s quite technical [27, Corollary 4.6] it follows that if  $X$  is completely metrizable,  $Y$  is paracompact first countable, and there is a continuous surjection from  $C_c(X)$  onto  $C_c(Y)$ , then  $Y$  is completely metrizable. Our approach is different and uses a property of  $C_c(X)$  which is preserved by linear open maps and characterizes spaces  $X$  with a compact resolution swallowing compact sets (Theorem 14, Corollary 15). The importance of this concept stems from the following deep result of J.P.R. Christensen [8, Theorem 3.3].

**Theorem 1.** *A metrizable space  $X$  is Polish if and only if  $X$  admits a compact resolution swallowing compact sets.*

We partially extend Theorem 1 to the non-separable case, see Proposition 16 below, and we apply this extension to show that if a space  $Y$  is  $\ell_c$ -covered by a completely metrizable space of weight  $\kappa$ , then  $Y$  admits a compact cover swallowing compact sets similar to a resolution. In particular, if  $Y$  is of pointwise countable type and  $X$  is Polish, then  $Y$  is separable and completely metrizable (Corollary 24). The separable case will be deduced from Theorem 21 showing that if  $X$  is an  $\aleph_0$ -space and there is an open continuous linear mapping from  $C_c(X)$  onto  $C_c(Y)$ , then  $Y$  is an  $\aleph_0$ -space. Indeed, the latter fact and Theorem 1 apply to prove that if for a separable completely metrizable space  $X$  there exists a quotient linear map from  $C_c(X)$  onto  $C_c(Y)$  and  $Y$  is first countable, then  $Y$  is separable and completely metrizable.

Motivated by the argument providing [Corollary 23](#), we present another proof (quite different from the one in [\[27, Proposition 4.8\]](#)) showing that if  $Y$  is a  $wq$ -space  $\ell_c$ -covered by a locally compact  $\mu$ -space  $X$ , then  $Y$  is a locally compact  $\mu$ -space. Our approach uses the concept of Baire-likeness of topological vector spaces. This extends a result of [\[3\]](#) or [\[4, Theorem 12\]](#) with the same conclusion for  $\ell_p$ -equivalent spaces  $X$  and  $Y$ . McCoy and Ntantu [\[15\]](#) proved a similar result for  $\ell_p$ -equivalent spaces  $X$  and  $Y$  where  $X$  is separable, metrizable and locally compact, and  $Y$  is first countable. The same conclusion was obtained in [\[12\]](#) for locally compact paracompact spaces  $X$ .

Some interesting part of the paper deals with compact  $\mathbb{P}(\kappa)$ -resolutions swallowing compact sets ([Theorem 14](#)) to provide a nice characterization for a large class of locally convex spaces  $C_c(X)$  to have a  $\mathfrak{G}_\kappa$ -basis. This leads to [Proposition 16](#) stating that if  $X$  is a completely metrizable space of weight  $\kappa$ , then  $C_c(X)$  has a  $\mathfrak{G}_\kappa$ -basis. The concept of a  $\mathfrak{G}_\kappa$ -basis seems to be a good tool to study a non-separable version (still an open question) of a remarkable Christensen theorem several times mentioned in the paper, see [Theorem 17](#).

Recall that a space  $Y$  is of *pointwise countable type* [\[5\]](#) if for every  $y \in Y$  there exists a compact set  $K$  that contains  $y$  and has a countable base of neighborhoods in  $Y$ . The class of spaces of pointwise countable type contains, in particular, all first-countable spaces and all Čech-complete spaces.

A space  $Y$  is a *wq-space* [\[27\]](#) if for each  $y \in Y$  there is a sequence  $\{U_n : n \in \omega\}$  of open neighborhoods of  $y$  such that if  $y_n \in U_n$  for each  $n \in \omega$  then  $\{y_n : n \in \omega\}$  is functionally bounded in  $Y$ . It is well known (and is easy to verify) that every space of pointwise countable type is a  $wq$ -space; moreover, every  $wq$ - and  $\mu$ -space is of pointwise countable type.

## 2. $\ell_p$ -equivalence and the Lindelöf $\Sigma$ -property of $\nu X$

A space  $X$  is a *Lindelöf  $\Sigma$ -space* if there is a compact-valued upper semicontinuous mapping from a separable metrizable space onto  $X$ ; since every separable metrizable space is a continuous image of a subspace of  $\mathbb{P}$ , we can characterize Lindelöf  $\Sigma$ -spaces as images underusco mappings of subspaces of  $\mathbb{P}$ .

If there is a compact-valued upper semicontinuous mapping from  $\mathbb{P}$  onto  $X$ , then  $X$  is called *K-analytic*, see [\[22\]](#). A space  $X$  is called *quasi-Souslin* if there exists a set-valued mapping  $T$  from  $\mathbb{P}$  onto  $X$  such that if a sequence  $p_n$  converges to a point  $p$  in  $\mathbb{P}$  and for each  $n \in \omega$ ,  $x_n \in T(p_n)$ , then the sequence  $\{x_n : n \in \omega\}$  has a limit point in  $T(p)$ , see [\[26, I.4.2\]](#). It is easy to verify that a space  $X$  is quasi-Souslin if and only if there is a countably compact-valued upper semicontinuous mapping from  $\mathbb{P}$  onto  $X$ .

A topological vector space  $E$  is called *web-bounded* if there exist an  $S \subset \mathbb{P}$  and a set-valued mapping  $A$  from  $S$  onto  $E$  such that whenever  $p \in S$  and  $x_k \in A(W(p|k))$ , the sequence  $\{x_k : k \in \omega\}$  is bounded in  $E$  (that is, is absorbed by any neighborhood of zero in  $E$ ). Clearly, every topological vector space with a bounded resolution, in particular, every vector quasi-Souslin and every vector Lindelöf  $\Sigma$ -space, is web-bounded. Note that every linear subspace of a web-bounded space is web-bounded, and every image of a web-bounded space under a continuous linear mapping is web-bounded.

The next statement is [Theorem 3.5](#) in [\[19\]](#).

**Proposition 2.** *The space  $\nu X$  is a Lindelöf  $\Sigma$ -space if and only if there exists a Lindelöf  $\Sigma$ -space  $Z$  such that  $C_p(X) \subset Z \subset \mathbb{R}^X$ .*

As usual, we denote by  $E'$  the weak dual space of  $E$ . Clearly,  $E'$  is a subspace of  $\mathbb{R}^E$ .

**Theorem 3.** *Let  $E$  be a locally convex space. If  $E$  is web-bounded, then there is a linear Lindelöf  $\Sigma$ -space  $Z$  such that  $E' \subset Z \subset \mathbb{R}^E$ . In particular,  $E'$  is web-bounded.*

**Proof.** Let  $S \subset \mathbb{P}$  and  $A : S \rightarrow E$  be a set-valued mapping as in the definition of a web-bounded space. Put

$$Z = \{\phi \in \mathbb{R}^E : \text{for each } p \in S \text{ there is an } n \in \omega \text{ such that } \phi \text{ is bounded on } A(W(p|n))\}.$$

Let us verify that  $E' \subset Z$ . Let  $\phi \in E'$ , and suppose  $\phi \notin Z$ . Then for some  $p \in S$  and every  $n \in \omega$  there is a point  $x_n \in A(W(p|n))$  such that  $|\phi(x_n)| > n$ . Then the sequence  $\{x_n : n \in \omega\}$  is unbounded in  $E$ , in contradiction with the definition of a web-bounded space.

Let  $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$  be the natural two-point compactification of  $\mathbb{R}$ . For every finite sequence  $\sigma$  of naturals of length  $n$  put

$$Z_\sigma = \{\phi \in \bar{\mathbb{R}}^E : \phi(A(W(\sigma))) \subset [-n, n]\}.$$

The set  $Z_\sigma$  is closed in the compact space  $\bar{\mathbb{R}}^E$ , hence is compact. Furthermore, the family  $\{Z_\sigma : \sigma \text{ is a finite sequence of naturals}\}$  is countable, and  $Z = \bigcap_{p \in S} \bigcup_{n \in \omega} Z_{p|n}$ . It follows that for every  $\phi \in Z$  and  $\psi \in \bar{\mathbb{R}}^E \setminus Z$  there are  $p \in S$  and  $n \in \omega$  such that  $\phi \in Z_{p|n}$  and  $\psi \notin Z_{p|n}$ . By [Proposition IV.9.2](#) in [\[5\]](#),  $Z$  is a Lindelöf  $\Sigma$ -space.

Obviously,  $Z$  is a linear subspace of  $\mathbb{R}^E$ , so  $Z$  is web-bounded. Since  $E'$  is a linear subspace of  $Z$ ,  $E'$  is web-bounded.  $\square$

We say that a locally convex space  $E$  is *weak* if its topology coincides with the  $*$ -weak topology. It is well known that if a space  $E$  is weak, then  $(E)'$  is linearly homeomorphic to  $E$  (see, e.g., [IV.1.2](#) in [\[23\]](#)).

**Corollary 4.** *Let  $E$  be a weak space. Then  $E$  is web-bounded if and only if  $E'$  is web-bounded.*

In particular,  $C_p(X)$  is a weak space.

**Corollary 5.** *The space  $C_p(X)$  is web-bounded if and only if  $L_p(X)$  is web-bounded.*

The following lemma is a part of [14, Theorem 9.15]; for the sake of completeness we give the proof here.

**Lemma 6.** *The space  $C_p(X)$  is web-bounded if and only if  $\nu X$  is a Lindelöf  $\Sigma$ -space.*

**Proof.** If  $\nu X$  is a Lindelöf  $\Sigma$ -space, then  $L_p(\nu X)$  is a Lindelöf  $\Sigma$ -space (because  $\nu X$  is a Hamel base for  $L_p(\nu X)$ , so  $L_p(\nu X)$  is a countable union of continuous images of products of finite powers of  $\nu X$  with compact spaces; see [5, Proposition 0.5.13]), hence  $L_p(\nu X)$  is web-bounded. It follows by Corollary 5 that the space  $C_p(\nu X)$  is web-bounded. The restriction mapping  $r_X: C_p(\nu X) \rightarrow C_p(X)$  (defined by the rule  $r_X(f) = f|X$  for all  $f \in C_p(\nu X)$ ) is linear, continuous and onto, so the space  $C_p(X)$  is web-bounded.

Conversely, assume that  $C_p(X)$  is web-bounded. By Theorem 3, there is a Lindelöf  $\Sigma$ -space  $Z$  such that  $L_p(X) \subset Z \subset \mathbb{R}^{C(X)}$ .

From the fact that  $X$  is  $C$ -embedded in  $\mathbb{R}^{C(X)}$  it follows that  $\nu X$  is homeomorphic to the closure of  $X$  in  $\mathbb{R}^{C(X)}$ . Since  $X \subset Z \subset \mathbb{R}^{C(X)}$ , and  $Z$  is Lindelöf,  $\nu X$  is homeomorphic to a closed subspace of  $Z$ , and therefore is a Lindelöf  $\Sigma$ -space.  $\square$

**Corollary 7.** *The space  $L_p(X)$  is web-bounded if and only if  $\nu X$  is a Lindelöf  $\Sigma$ -space.*

Indeed, it is clear from the definition that web-boundedness is preserved under mappings that take bounded sequences to bounded sequences.

A set  $B$  in a topological vector space is bounded if and only if every sequence in  $B$  is bounded. It follows that the image of a space with a bounded resolution under a mapping that takes bounded sequences to bounded sequences has a bounded resolution. Now from Lemma 6 we arrive at the following.

**Theorem 8.** *Let  $X$  and  $Y$  be spaces, and assume that there exists a surjective mapping  $h: C_p(X) \rightarrow C_p(Y)$  that takes bounded sequences to bounded sequences. If  $\nu X$  is a Lindelöf  $\Sigma$ -space, then  $\nu Y$  is a Lindelöf  $\Sigma$ -space.*

**Theorem 9.** *Let  $X$  and  $Y$  be spaces, and assume that there exists a surjective mapping  $h: L_p(X) \rightarrow L_p(Y)$  that takes bounded sequences to bounded sequences. If  $\nu X$  is a Lindelöf  $\Sigma$ -space, then  $\nu Y$  is a Lindelöf  $\Sigma$ -space.*

**Theorem 10.**  *$L_p(X)$  has a bounded resolution if and only if  $\nu X$  is a  $K$ -analytic space. In particular, if  $X$  is quasi-Souslin, then  $L_p(X)$  has a bounded resolution.*

**Proof.** Let  $\{A_p: p \in \mathbb{P}\}$  be a bounded resolution in  $L_p(X)$ . The family  $\{A_p \cap X: p \in \mathbb{P}\}$  is a resolution in  $X$  consisting of functionally bounded sets. For each  $p \in \mathbb{P}$  let  $B_p$  be the closure of  $A_p \cap X$  in the space  $\nu X$ . Then the subspace  $Y = \bigcup\{B_p: p \in \mathbb{P}\}$  is quasi-Souslin. It follows that  $\nu Y$  is  $K$ -analytic. Indeed, let  $T$  be a set-valued mapping that witnesses  $Y$  being quasi-Souslin. Each  $T(p)$  is countably compact, so its closure  $\overline{T(p)}$  in  $\nu Y$  is compact. The mapping  $\bar{T}: p \mapsto \overline{T(p)}$  is upper semicontinuous, so  $Z = \bigcup_{p \in \mathbb{P}} \overline{T(p)}$  is  $K$ -analytic.

Since  $Y \subset Z \subset \nu Y$ , we have  $Z = \nu Z = \nu Y$ , so  $\nu Y$  is  $K$ -analytic. Since  $X \subset Y \subset \nu X$ , we conclude that  $\nu X = \nu Y$  is  $K$ -analytic.

Conversely, assume that  $\nu X$  is  $K$ -analytic. Then by [5, Proposition 0.5.13], the space  $L_p(\nu X)$  is  $K$ -analytic; therefore, it has a compact resolution  $\{A_p: p \in \mathbb{P}\}$ . As  $L_p(X)$  is embedded in  $L_p(\nu X)$ , the family  $\{A_p \cap L_p(X): p \in \mathbb{P}\}$  is a bounded resolution of  $L_p(X)$ .  $\square$

**Theorem 11.** *If there exists a mapping from  $L_p(X)$  onto  $L_p(Y)$  that takes bounded sequences to bounded sequences, and  $\nu X$  is  $K$ -analytic, then  $\nu Y$  is  $K$ -analytic.*

**Problem 12.** *Suppose the space  $\nu X$  is  $K$ -analytic and there is a mapping from  $C_p(X)$  onto  $C_p(Y)$  that takes bounded sequences to bounded sequences. Must the space  $\nu Y$  be  $K$ -analytic?*

### 3. Resolutions and completeness

Let  $\kappa$  be an infinite cardinal. We denote  $\mathbb{P}(\kappa) = ([\kappa]^{<\omega})^\omega$  where  $[\kappa]^{<\omega}$  is the family of all finite subsets of  $\kappa$ ; thus, for every  $p \in \mathbb{P}(\kappa)$  and  $n \in \omega$ ,  $p(n)$  is a finite subset of  $\kappa$ . We endow the set  $\mathbb{P}(\kappa)$  with the partial order by putting  $p \leq q$  if for every  $n \in \omega$ ,  $p(n) \subset q(n)$ . We denote by  $c(p)$  the cardinality of the set  $p(0)$ .

We say that a compact-valued mapping  $\phi: \mathbb{P}(\kappa) \rightarrow X$  is a *compact  $\mathbb{P}(\kappa)$ -resolution for  $X$*  if  $X = \phi(\mathbb{P}(\kappa))$  and  $\phi(p) \subset \phi(q)$  whenever  $p \leq q$ . If, moreover, every compact subset of  $X$  is contained in  $\phi(p)$  for some  $p \in \mathbb{P}(\kappa)$ , then we say that  $\phi$  is a *compact  $\mathbb{P}(\kappa)$ -resolution swallowing compact sets*.

Let  $i: \mathbb{P} \rightarrow \mathbb{P}(\omega)$  be the function such that  $i(p)(n) = \{k \in \omega: k \leq p(n)\}$  for all  $p \in \mathbb{P}$  and  $n \in \omega$ ; put  $\mathbb{P}_0 = i(\mathbb{P})$ . Obviously,  $\mathbb{P}_0$  is a cofinal subset of  $\mathbb{P}(\omega)$  order-isomorphic to  $\mathbb{P}$ ; furthermore, for every  $s \in \mathbb{P}(\omega)$  there is a unique minimal element  $s_0$  in the set  $\{t \in \mathbb{P}_0: s \leq t\}$ .

**Proposition 13.** *A space  $X$  has a compact resolution if and only if  $X$  has a compact  $\mathbb{P}(\omega)$ -resolution. Moreover,  $X$  has a compact resolution swallowing compact sets if and only if  $X$  has a compact  $\mathbb{P}(\omega)$ -resolution swallowing compact sets.*

**Proof.** This proposition follows from [7, Proposition 3.3(a)]. For the sake of completeness we give the following direct proof. Suppose  $X$  has a compact resolution  $K: \mathbb{P} \rightarrow \mathcal{K}(X)$ . Define  $\phi: \mathbb{P}(\omega) \rightarrow \mathcal{K}(X)$  by the rule:  $\phi(p) = K(i^{-1}(p_0))$  where  $p_0$  is the minimum element of  $\mathbb{P}_0$  greater or equal to  $p$ . Then  $\phi$  is a compact  $\mathbb{P}(\omega)$ -resolution for  $X$ . Clearly,  $\phi(\mathbb{P}(\omega)) = K(\mathbb{P})$ , so if  $K$  is compact swallowing, then so is  $\phi$ .

Now assume that  $\phi$  is a compact  $\mathbb{P}(\omega)$ -resolution for  $X$ . For each  $s \in \mathbb{P}$  put  $K(s) = \phi(i(s))$ . Since  $\mathbb{P}_0$  is cofinal in  $\mathbb{P}(\omega)$ , the function  $K: \mathbb{P} \rightarrow \mathcal{K}(X)$  is a compact resolution for  $X$ . Obviously, if  $\phi$  is compact swallowing, then so is  $K$ .  $\square$

Following [11], we will say that a base  $\{U_p: p \in \mathbb{P}\}$  of neighborhoods of zero of a locally convex space  $E$  is a  $\mathfrak{G}$ -base if  $U_q \subset U_p$  whenever  $p \leq q$ .

We will say that a base  $\{U_p: p \in \mathbb{P}(\kappa)\}$  of neighborhoods of zero of  $E$  is a  $\mathfrak{G}_\kappa$ -base if  $U_q \subset U_p$  whenever  $p \leq q$ . By an argument similar to that in the proof of Proposition 13, a space  $E$  has a  $\mathfrak{G}$ -base if and only if it has a  $\mathfrak{G}_\omega$ -base.

**Theorem 14.** *Let  $\kappa$  be an infinite cardinal. A space  $X$  has a compact  $\mathbb{P}(\kappa)$ -resolution swallowing compact sets if and only if  $C_c(X)$  has a  $\mathfrak{G}_\kappa$ -base.*

**Proof.** Let  $\phi: \mathbb{P}(\kappa) \rightarrow \mathcal{K}(X)$  be a compact  $\mathbb{P}(\kappa)$ -resolution swallowing compact sets. Assign to each  $p \in \mathbb{P}(\kappa)$  the set  $U_p = \{f \in C_c(X): |f(x)| \leq \frac{1}{c(p)+1} \text{ for all } x \in \phi(p)\}$ . The compactness of each  $\phi(p)$ ,  $p \in \mathbb{P}(\kappa)$ , and the compact-swallowing property imply that the family  $\mathcal{U} = \{U_p: p \in \mathbb{P}(\kappa)\}$  is a base of neighborhoods of zero in  $C_c(X)$ . If  $p \leq q$ , then  $\phi(p) \subset \phi(q)$  and  $p(0) \subset q(0)$ , so  $U_q \subset U_p$ , and  $\mathcal{U}$  is a  $\mathfrak{G}_\kappa$ -base.

Conversely, let  $\{U_p: p \in \mathbb{P}(\kappa)\}$  be a  $\mathfrak{G}_\kappa$ -base of  $C_c(X)$ . For each  $p \in \mathbb{P}(\kappa)$  put

$$K_p = \{x \in X: |f(x)| \leq c(p) \text{ for all } f \in U_p\}.$$

Clearly, the sets  $K_p$  are closed in  $X$ , and  $K_p \subset K_q$  whenever  $p \leq q$ . Let us verify that these sets are compact.

Let  $p \in \mathbb{P}(\kappa)$ . Then the set  $\{f \in C_c(X): f(K_p) \subset (-1, 1)\}$  contains the set  $\frac{1}{c(p)+1}U_p$ , so it is a neighborhood of 0 in  $C_c(X)$ . It follows that  $K_p$  is contained in a compact subset of  $X$ ; since  $K_p$  is closed in  $X$ , it is compact.

Let us now verify that every compact set  $C$  in  $X$  is contained in  $K_p$  for some  $p \in \mathbb{P}(\kappa)$ . Indeed, since the family  $\{U_p: p \in \mathbb{P}(\kappa)\}$  is a base at 0 of  $C_c(X)$ , we have  $U_q \subset \{f \in C_c(X): f(C) \subset (-1, 1)\}$  for some  $q \in \mathbb{P}(\kappa)$ . Find a  $p \in \mathbb{P}(\kappa)$  so that  $q \leq p$  and  $p(0) \neq \emptyset$ . Then for every  $x \in C$  and  $f \in U_p$  we have  $|f(x)| < 1 \leq c(p)$ , so  $x \in K_p$ .

Thus, the mapping  $\phi: \mathbb{P}(\kappa) \rightarrow \mathcal{K}(X)$  such that  $\phi(p) = K_p$  for all  $p \in \mathbb{P}$ , is a compact-swallowing compact  $\mathbb{P}(\kappa)$ -resolution for  $X$ .  $\square$

**Corollary 15.** *If a space  $X$  has a compact  $\mathbb{P}(\kappa)$ -resolution swallowing compact sets, and  $X$   $\ell_c$ -covers  $Y$ , then  $Y$  has a compact  $\mathbb{P}(\kappa)$ -resolution swallowing compact sets.*

If  $X$  and  $Y$  are  $\ell_p$ -equivalent, the conclusion of the last theorem holds. Indeed, if  $T: C_p(X) \rightarrow C_p(Y)$  is a linear homeomorphism, then  $T$  is also a linear homeomorphism between  $C_c(X)$  and  $C_c(Y)$  by [4, Propositions II.1.1, II.1.4].

Note also that Corollary 15 fails if we only assume  $t$ -equivalence of the spaces  $X$  and  $Y$  (see [24, Example 3.4]): the spaces  $C_p(\mathbb{Q})$  and  $C_p(\omega+1)$  are homeomorphic by [9], but the space of rationals  $\mathbb{Q}$  does not admit a compact resolution swallowing compact sets, because otherwise by Theorem 1 it would have to be Čech-complete. Corollary 15 also shows that  $C_p(\mathbb{Q})$  and  $C_p(\omega+1)$  are not linearly homeomorphic. By [24, Corollary 2.2], the property of having a compact resolution is preserved by  $t$ -equivalence.

**Proposition 16.** *Every Čech-complete space  $X$  of weight  $\kappa$  has a  $\mathbb{P}(\kappa)$ -compact resolution swallowing compact sets. Consequently, if  $X$  is a completely metrizable space of weight  $\kappa$ , then  $C_c(X)$  admits a  $\mathfrak{G}_\kappa$ -base.*

Conversely, if  $X$  is a metrizable space and  $C_c(X)$  has a  $\mathfrak{G}$ -base, then  $X$  is Polish.

**Proof.** Let  $\mathfrak{B}$  be a base of cardinality  $\kappa$  for  $X$ . Fix a countable family  $\{W_n: n \in \omega\}$  of open sets in  $\beta X$  so that  $X = \bigcap_{n \in \omega} W_n$ . Let  $\mathfrak{B}_n$  be the family of all elements of  $\mathfrak{B}$  whose closures in  $\beta X$  are contained in the set  $W_n$ . Clearly,  $\mathfrak{B}_n$  is a base of  $X$  for each  $n \in \omega$ . Enumerate each  $\mathfrak{B}_n$  in type  $\kappa$ :  $\mathfrak{B}_n = \{B_{n\alpha}: \alpha \in \kappa\}$ . For every finite set  $A \subset \kappa$  put  $F_n(A) = \bigcup \{\overline{B_{n\alpha}}: \alpha \in A\}$

where the closures are taken in  $\beta X$ . Then  $F_n(A)$  is a compact subset of  $\beta X$  contained in  $W_n$ . Define  $\phi: \mathbb{P}(\kappa) \rightarrow \mathcal{K}(X)$  by the rule:

$$\phi(p) = \bigcap \{F_n(p(n)): n \in \omega\}.$$

The function  $\phi$  is well defined, since the intersection of compact sets  $F_n(p(n))$  is compact, and is contained in  $X$ , because  $F_n(p(n)) \subset W_n$  and  $\bigcap_{n \in \omega} W_n = X$ . It is obvious from the construction that  $\phi(p) \subset \phi(q)$  whenever  $p \leq q$ . Let us verify that for every compact  $K$  in  $X$  there exists  $p \in \mathbb{P}(\kappa)$  such that  $K \subset \phi(p)$ . Indeed, for each  $n \in \omega$  there exists a finite  $A_n \subset \kappa$  such that  $K \subset \bigcup \{B_{n\alpha}: \alpha \in A_n\}$ . Then the element  $p$  of  $\mathbb{P}(\kappa)$  such that  $p(n) = A_n$  for all  $n \in \omega$  is as required.

If  $X$  is metrizable and  $C_c(X)$  has a  $\mathfrak{G}$ -base, then by [Theorem 14](#),  $X$  has a compact resolution swallowing compact sets, so  $X$  is Polish by [Theorem 1](#).  $\square$

It is natural to ask whether every metrizable space of weight  $\kappa$  such that  $C_c(X)$  has a  $\mathfrak{G}_\kappa$ -base must be completely metrizable. The next statement, proved by David Guerrero Sánchez and presented here with his kind permission, shows that the answer generally is negative.

**Theorem 17.** *Let  $X$  be a metrizable space of weight  $\leq \kappa$ . Then  $X$  has a compact-swallowing compact  $\mathbb{P}(\kappa^\omega)$ -resolution.*

**Proof.** Let  $X$  be a metric space of weight  $\leq \kappa$ . Then  $X$  has at most  $\lambda = \kappa^\omega$  compact subsets.

Therefore, we can write  $\mathcal{K}(X) = \{K_\alpha: \alpha \in \lambda\}$  (we do not assume that the enumeration is injective).

Define  $\phi(p) = \bigcup \{K_\alpha: \alpha \in p(0)\}$  for each  $p \in \mathbb{P}(\lambda)$ . Clearly,  $\phi$  is a compact-swallowing compact  $\mathbb{P}(\lambda)$ -resolution for  $X$ .  $\square$

In particular, every metric space  $X$  of weight  $\mathfrak{c}$  has a compact-swallowing compact  $\mathbb{P}(\mathfrak{c})$ -resolution, so by [Theorem 14](#),  $C_c(X)$  has a  $\mathfrak{G}_\mathfrak{c}$ -base. Of course, a metrizable space of weight  $\mathfrak{c}$  need not be completely metrizable.

#### 4. Metrizability and local compactness

It is well known that metrizability and local compactness are not preserved by the relation of  $\ell_p$ -equivalence: this is shown by the very first known example of non-homeomorphic  $\ell_p$ -equivalent spaces, the countable sum of convergent sequences and the countable Fréchet fan (see, e.g., [\[5\]](#)).

On the other hand, there are several results that show that a space  $\ell_p$ -equivalent or  $\ell_c$ -equivalent to a metrizable or a locally compact space must be metrizable or locally compact if we assume that it has some additional property, such as first countability or pointwise countable type.

In this section we obtain some extensions and versions of results of this type.

We need the following few auxiliary results. Recall that a space  $X$  is *submetrizable* if there is a continuous bijection from  $X$  onto a metrizable space.

**Proposition 18.** *Let  $X$  be a submetrizable space. If there exists a continuous map from  $C_c(X)$  onto  $C_c(Y)$ , then  $C_c(C_c(Y))$  is submetrizable. In particular, if  $Y$  is of pointwise countable type, then  $Y$  is submetrizable and first countable.*

**Proof.** If  $X$  is submetrizable, then by [\[15, Theorem 5.6.2\]](#),  $C_c(X)$  has a dense  $\sigma$ -compact subspace. Hence,  $C_c(Y)$  has a dense  $\sigma$ -compact subspace, and by [\[15, Corollary 4.3.2\]](#),  $C_c(C_c(Y))$  is submetrizable.

Recall that every space of pointwise countable type is a  $k$ -space, see [\[10, Chapter 2.3\]](#), so  $Y$  is homeomorphic to a subspace of  $C_c(C_c(Y))$ . Hence,  $Y$  is submetrizable, so all compact sets in  $Y$  are metrizable. To see that  $Y$  is first countable we apply the *transitivity of character for compact sets* (the fact that if  $F_1 \subset F_2$  are compact sets in a space  $Y$ ,  $F_1$  has countable character in  $F_2$ , and  $F_2$  has countable character in  $Y$ , then  $F_1$  has countable character in  $Y$ , see [\[1, Proposition 3.3\]](#)); we apply this to any singleton  $F_1 = \{y_0\}$  and a compact set  $F_2$  of countable character in  $Y$  that contains  $y_0$  to prove the countability of character of  $Y$  at  $y_0$ . Thus,  $Y$  is first countable.  $\square$

Recall that a space  $X$  is an  $\aleph_0$ -space if  $X$  has a countable  $k$ -network, that is, a countable family of sets  $\mathcal{N}$  such that for every compact set  $K$  and every open set  $U$  in  $X$  such that  $K \subset U$ , there is an  $N \in \mathcal{N}$  with  $K \subset N \subset U$ , see [\[16\]](#). It is well known [\[16\]](#) that every image of an  $\aleph_0$ -space under a perfect mapping is an  $\aleph_0$ -space and every closed subspace of an  $\aleph_0$ -space is an  $\aleph_0$ -space.

A function  $f: X \rightarrow Y$  is called *k-continuous* if its restriction to every compact subspace of  $X$  is continuous. For a space  $X$ , let  $\Delta_X: X \rightarrow C_c(C_c(X))$  be the map such that  $\Delta_X(x)(f) = f(x)$  for all  $x \in X$  and  $f \in C_c(X)$ . It is well known that the mapping  $\Delta_X$  is injective, and that it is an embedding if and only if it is continuous.

**Lemma 19.** *The mapping  $\Delta_X$  is k-continuous.*

**Proof.** Let  $K$  be a compact subset of  $X$  and  $F = C_c(C_c(X)) \setminus [C, V]$  where  $C$  is a compact subset of  $C_c(X)$  and  $V$  is an open subset of  $\mathbb{R}$ . Clearly,  $\Delta_X^{-1}(F) = \{x \in X: f(x) \notin V \text{ for some } f \in C\}$ .

Let  $x$  be the limit of a convergent net  $(x_i: i \in I)$ , with each  $x_i \in K \cap (\Delta_X)^{-1}(F)$ . To prove the lemma it is enough to show that  $x \in \Delta_X^{-1}(F)$ .

For each  $i \in I$  fix an  $f_i \in C$  with  $f_i(x_i) \notin V$ . The compactness of  $C$  implies that the net  $(f_i: i \in I)$  has a subnet that converges uniformly on  $K$  to some  $f \in C$ . Then from  $f_i(x_i) \notin V$ , for each  $i \in I$ , follows that  $f(x) \notin V$ , implying that  $x \in \Delta_X^{-1}(F)$ .  $\square$

Denote by  $\tilde{X}$  the image of  $X$  under the mapping  $\Delta_X: X \rightarrow C_c(C_c(X))$  and by  $\hat{X}$  the image of  $X$  under the mapping  $\Delta_X: X \rightarrow C_p(C_c(X))$ . It is well known (see e.g. [15]) that  $\Delta_X$  is a homeomorphism from  $X$  to  $\tilde{X}$ , and  $\hat{X}$  is closed in  $C_p(C_c(X))$ ; it follows that  $\tilde{X}$  is a closed subspace of  $C_c(C_c(X))$  and that the restriction to  $\tilde{X}$  of the identity mapping  $C_c(C_c(X)) \rightarrow C_p(C_c(X))$  is a continuous bijection onto  $\hat{X}$  whose inverse is  $k$ -continuous.

**Lemma 20.** *Let  $X$  be a space and  $f: X \rightarrow Y$  a continuous bijection whose inverse is  $k$ -continuous. If  $X$  is an  $\aleph_0$ -space, then so is  $Y$ .*

**Proof.** Note that a set  $K$  in  $Y$  is compact if and only if  $f^{-1}(K)$  is compact. Now it is immediate that if  $\mathcal{N}$  is a countable  $k$ -network in  $X$ , then  $\{f(N): N \in \mathcal{N}\}$  is a countable  $k$ -network in  $Y$ .  $\square$

The following theorem extends Arhangel'skii's result from [3] with the same conclusion as below but for  $\ell_p$ -equivalent spaces  $X$  and  $Y$ , see also [4, Theorem 12]. Note that in general the  $\aleph_0$ -space property is not preserved by open maps.

**Theorem 21.** *If  $X$  is an  $\aleph_0$ -space and there is an open continuous linear mapping from  $C_c(X)$  onto  $C_c(Y)$ , then  $Y$  is an  $\aleph_0$ -space.*

**Proof.** Let  $h: C_c(X) \rightarrow C_c(Y)$  be an open continuous linear mapping. Let  $h_c^*: C_c(C_c(Y)) \rightarrow C_c(C_c(X))$  and  $h_p^*: C_p(C_c(Y)) \rightarrow C_p(C_c(X))$  be the dual mappings; the mapping  $h_c^*$  is continuous, and the mapping  $h_p^*$  is an embedding, see e.g. [15, Corollary 2.2.8].

The mapping  $h_p^*$  is a closed embedding, because  $h$  is quotient. Denote

$$Z_p = h_p^*(\hat{Y}) \text{ and } Z_c = h_c^*(\tilde{Y}).$$

It follows that  $Z_p$  is closed in  $C_p(C_c(X))$ , and hence  $Z_c$  is closed in  $C_c(C_c(X))$ . Moreover,  $h_p^*$  maps homeomorphically  $\hat{Y}$  onto  $Z_p$ , because  $h_p^*$  is an embedding.

Let  $j: Z_c \rightarrow Z_p$  be the restriction to  $Z_c$  of the identity mapping  $C_c(C_c(X)) \rightarrow C_p(C_c(X))$ , and let  $i: \tilde{Y} \rightarrow \hat{Y}$  be the restriction to  $\tilde{Y}$  of the identity mapping  $C_c(C_c(Y)) \rightarrow C_p(C_c(Y))$ . Clearly,  $i$  and  $j$  are continuous bijections, and  $i^{-1}$  is  $k$ -continuous by Lemma 19.

We have  $j \circ (h_c^*|\tilde{Y}) = (h_p^*|\hat{Y}) \circ i$ , so  $j^{-1} = (h_c^*|\tilde{Y}) \circ i^{-1} \circ (h_p^*|\hat{Y})^{-1}$ , whence  $j^{-1}$  is  $k$ -continuous.

By [16] the space  $C_c(C_c(X))$  is an  $\aleph_0$ -space, so its closed subspace  $Z_c$  is an  $\aleph_0$ -space. From Lemma 20 now follows that  $Z_p$  is an  $\aleph_0$ -space. Since  $Z_p$  is homeomorphic to  $Y$ ,  $Y$  is an  $\aleph_0$ -space.  $\square$

Since every first-countable  $\aleph_0$ -space is second countable, see [16,17], Theorem 21 applies to prove the following extension of Arhangel'skii's [3, Theorem 16] (see also [4]).

**Corollary 22.** *Let  $X$  be a second-countable space and  $Y$  be first-countable. If there is an open continuous linear mapping from  $C_c(X)$  onto  $C_c(Y)$ , then  $Y$  is second countable.*

The next corollary extends Pelant's result [4, Theorem 3.27].

**Corollary 23.** *Let  $X$  be a separable completely metrizable space and  $Y$  a first-countable space. If there is an open continuous linear mapping from  $C_c(X)$  onto  $C_c(Y)$ , then  $Y$  is a separable completely metrizable space.*

**Proof.** By Theorem 21,  $Y$  is second countable. Since  $X$  is Polish, it has a compact-swallowing compact resolution, so  $C_c(X)$  has a  $\mathfrak{G}$ -base. It follows that  $C_c(Y)$  has a  $\mathfrak{G}$ -base, so  $Y$  has a compact-swallowing compact resolution, and hence is completely metrizable by Christensen's theorem.  $\square$

Proposition 18 and Corollary 23 give a relatively simple proof for the next statement, which can also be obtained from a combination of results of V. Valov, J. de Groot and J. Pelant.

**Corollary 24.** *If  $X$  is a Polish space,  $Y$  is a space of pointwise countable type, and  $X$   $\ell_c$ -covers  $Y$ , then  $Y$  is Polish.*

It is well known that  $C_c(X)$  is metrizable if and only if  $X$  is hemicompact, i.e. has a compact-swallowing sequence of compact sets, see [15, Theorem 4.4.2].

**Proposition 25.** *Let  $X$  be a locally compact space. Then  $X$  is hemicompact if and only if  $X$  is a  $\mu$ -space with a compact resolution swallowing compact sets.*

**Proof.** Clearly, every hemicompact space is a  $\mu$ -space with a compact resolution swallowing compact sets.

For the converse assume that  $X$  is a  $\mu$ -space admitting a compact resolution swallowing compact sets. The  $\mu$ -property of  $X$  implies that  $C_c(X)$  is barrelled (see Theorem 10.1.20 in [21]). Since  $X$  is locally compact, the space  $C_c(X)$  is Baire-like, that is, for every increasing countable cover  $\{A_n: n \in \omega\}$  of  $C_c(X)$  by symmetric convex closed sets there exists an  $m \in \omega$  such that  $A_m$  is a neighborhood of zero, see [13, Lemma 2.1] or the proof of Theorem 27 below. Next, by Theorem 14 (with  $\kappa = \omega$ ), the space  $C_c(X)$  has a  $\mathfrak{G}$ -base, say,  $\{U_p: p \in \mathbb{P}\}$ . We may assume without loss of generality that all sets  $U_p$  are convex, symmetric and closed in  $C_c(X)$ . For each finite sequence of naturals  $\sigma$  of length  $n$ , put  $C_\sigma = \bigcap \{U_p: p|n = \sigma\}$ . Then  $C_{p|n} \subset U_p$ . Moreover, the sequence  $\{C_{p|n}: n \in \omega\}$  is increasing, and for each bounded set  $B \subset C_c(X)$ ,  $B \subset kC_{p|k}$  for some  $k \in \omega$ . Indeed, assume that there is a bounded set  $B$  in  $C_c(X)$  such that  $B \not\subset kC_{p|k}$  for all  $k \in \omega$ . For each  $k \in \omega$  choose  $x_k \in B \setminus kC_{p|k}$ . Choose  $p_k \in \mathbb{P}$  so that  $p_k|k = p|k$  and  $x_k \notin kU_{p_k}$ . Let  $q(n) = \max\{p_k(n): k \in \omega\}$ ; then  $q \in \mathbb{P}$  (note that  $p_k(n) = p(n)$  for  $k \geq n$ , so  $q$  is well defined). Clearly,  $p_k \leq q$  for all  $k \in \omega$ . Therefore,  $U_q \subset U_{p_k}$  for all  $k \in \omega$ , and  $x_k \notin kU_q$ . Thus,  $\{x_k: k \in \omega\}$  is an unbounded sequence in  $B$ , a contradiction with  $B$  being bounded.

The Baire-likeness of  $C_c(X)$  now implies that there exists a  $k \in \omega$  such that  $C_{p|k}$  is a neighborhood of zero in  $C_c(X)$ . Put

$$\mathfrak{D} = \{C_\sigma: \sigma \text{ is a finite sequence of naturals and } C_\sigma \text{ is a neighborhood of zero in } C_c(X)\}.$$

Then  $\mathfrak{D}$  is a countable family of neighborhoods of zero in  $C_c(X)$ . By the above argument, for any neighborhood of zero  $U$  in  $C_c(X)$  there are a  $p \in \mathbb{P}$  and  $k \in \omega$  such that  $C_{p|k} \in \mathfrak{D}$  and  $C_{p|k} \subset U$ . Thus,  $\mathfrak{D}$  is a countable base of neighborhoods of zero in  $C_c(X)$ . It follows that  $C_c(X)$  is metrizable. Hence,  $X$  is hemicompact.  $\square$

**Example 26.** The assumption of local compactness of  $X$  in Proposition 25 cannot be omitted: there exists a  $\sigma$ -compact Čech-complete not hemicompact space that has a compact resolution swallowing compact sets.

Put  $X = [0, 1] \setminus \{\frac{1}{n+1}: n \in \omega\}$ . Then  $X$  is a  $G_\delta$ -set in  $[0, 1]$ , so  $X$  is a Polish space; therefore  $X$  has a compact resolution swallowing compact sets. Clearly,  $X$  is  $\sigma$ -compact and not locally compact; since all hemicompact first-countable spaces are locally compact,  $X$  is not hemicompact.

In [27, Proposition 4.8] Valov proved that if  $X$  and  $Y$  are  $\mu$ -spaces,  $X$  is locally compact,  $Y$  is a  $wq$ -space, and there exists a continuous linear surjection from  $C_c(X)$  onto  $C_c(Y)$ , then  $Y$  is locally compact. The above proof of Proposition 25 motivated us to present an apparently simpler approach to Valov's [27, Proposition 4.8]. We use an argument similar to one in [13].

**Theorem 27.** *Let  $X$  be a locally compact  $\mu$ -space and let  $Y$  be a  $wq$ -space which is  $\ell_c$ -equivalent to  $X$ . Then  $Y$  is a locally compact  $\mu$ -space.*

**Proof.** Let us first verify that if  $X$  is a locally compact  $\mu$ -space, then  $C_c(X)$  is barrelled and Baire-like. To prove the claim it is sufficient to prove that for any decreasing sequence  $\{A_n: n \in \omega\}$  of closed non-compact subsets of  $X$  there is a function  $f \in C(X)$  that is unbounded on each  $A_n$ , see [13, Proposition 1.2].

Let  $\{A_n: n \in \omega\}$  be a sequence as above. For each  $n \in \omega$  choose an  $f_n \in C(X)$  unbounded on  $A_n$ . If there exists a number  $m \in \omega$  such that  $f_m$  is unbounded on each  $A_n$ , we are done. Therefore, assume that for each  $n \in \omega$  there is a  $k_n > n$  such that  $f_n$  is bounded on  $A_{k_n}$ . Since the sequence  $\{A_n: n \in \omega\}$  is decreasing, we may assume (taking a suitable subsequence if necessary) that  $f_n$  is bounded on  $A_{n+1}$  for each  $n \in \omega$ . If  $A = \bigcap_n A_n$  is non-compact, the proof is complete. So assume that  $A$  is compact. Then there exists an open neighborhood  $H_0$  of  $A$  whose closure  $W_0$  is compact. Since  $f_1$  is bounded on  $A_2$  and  $W_0$  and unbounded on  $A_1$ , there exists  $x_1 \in A_1 \setminus (A_2 \cup W_0)$ . Then there exists an open neighborhood  $H_1$  of  $x_1$  with compact closure  $W_1$  and  $W_1 \subset X \setminus (A_2 \cup W_0)$ . This procedure yields a sequence  $D = \{x_n: n \in \omega\}$  in  $X$  with a pairwise disjoint sequence  $\{H_n: n \in \omega\}$  of its open neighborhoods whose closures are compact. It is easy to see that the set  $Z = D \cup A$  is closed and non-compact. Since  $X$  is a  $\mu$ -space, there exists a function  $f \in C(X)$  unbounded on  $Z$ . Then  $f$  is unbounded on each  $A_n$ . This proves that  $C_c(X)$  is Baire-like. By the assumption,  $C_c(Y)$  is Baire-like (because the Baire-likeness is inherited by Hausdorff locally convex quotients).

Finally, assume that  $Y$  is a  $wq$ -space. We already know that  $C_c(Y)$  is Baire-like. Then  $Y$  is locally compact. Indeed, given a point  $y_0 \in Y$  fix a decreasing sequence  $\{U_n: n \in \omega\}$  of its neighborhoods as in the definition of a  $wq$ -space. Put  $W_n = \{f \in C(Y): \sup_{y \in U_n} |f(y)| \leq n\}$  for each  $n \in \omega$ . Then each  $W_n$  is a closed absolutely convex subset of  $C_c(Y)$ . Note that the sequence  $\{W_n: n \in \omega\}$  covers  $C_c(Y)$ . Indeed, if some function  $f_0$  is not covered, then for each  $n \in \omega$  there is a point  $z_n \in U_n$  such that  $|f_0(z_n)| > n$ . It follows that the sequence  $\{z_n: n \in \omega\}$  is not functionally bounded, a contradiction with the choice of the sets  $U_n$ .



By the Baire-likeness of  $C_c(Y)$ , there exist  $n \in \omega$ ,  $\epsilon > 0$ , and a compact subset  $S$  of  $Y$  such that  $\{f \in C(Y) : \sup_{y \in S} |f(y)| < \epsilon\} \subset W_n$ . Then  $U_n \subset S$ , and  $S$  is a compact neighborhood of  $y_0$  in  $Y$ .  $\square$

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