Journal of Convex Analysis
Volume 21 (2014), No. 3, 833–849

Measures of Weak Noncompactness in Non-Archimedean Banach Spaces*

C. Angosto
Depto. de Matemática Aplicada y Estadistica,
Universidad Politécnica de Cartagena, 30203 Cartagena, Spain
carlos.angosto@upct.es

J. Kąkol
Faculty of Mathematics and Informatics,
A. Mickiewicz University, 61-614 Poznań, Poland
kakol@amu.edu.pl

A. Kubzdela
Institute of Civil Engineering, Poznań University of Technology,
Ul. Piotrowo 5, 61-138 Poznań, Poland
albert.kubzdela@put.poznan.pl

Received: January 4, 2013
Revised manuscript received: July 20, 2013
Accepted: July 21, 2013

Let E be a non-Archimedean Banach space over a non-Archimedean locally compact non-trivially valued field $\mathbb{K} := (\mathbb{K}, |.|)$. Let E'' be its bidual and M a bounded set in E. We say that M is $\varepsilon-$weakly relatively compact if $\overline{M}^{\sigma(E'',E')} \subset E+B_{E'',\varepsilon}$, where $B_{E'',\varepsilon}$ is the closed ball in E'' with the radius $\varepsilon \geq 0$. In this paper we describe measures of noncompactness γ, k and De Blasi measure ω. We show that $\gamma(M) \leq k(M) \leq \omega(M) = \omega(\operatorname{aco} M) \leq \frac{1}{|\rho|}\gamma(M)$, where ρ ($|\rho| < 1$) is an uniformizing element in $\mathbb{K}$, and $\omega(M) = \sup\{\overline{\lim}_m \operatorname{dist}(x_m, [x_1, \ldots, x_{m-1}]) : (x_m) \subset M\}$; the latter equality is purely non-Archimedean. In particular, assuming $|\mathbb{K}| = \{||x|| : x \in E\}$, we prove that the absolutely convex hull $\operatorname{aco} M$ of a $\varepsilon-$weakly relatively compact subset M in E is $\varepsilon-$weakly relatively compact. In fact we show that in this case for a bounded set M in E we have $\gamma(M) = \gamma(\operatorname{aco} M) = k(M) = k(\operatorname{aco} M) = \omega(M)$. Note that the above equalities fail in general for real Banach spaces by results of Granero ([9]) and Astalla and Tilly ([5]). Most of proofs are strictly non-Archimedean. A non-Archimedean variant of another quantitative Krein's theorem due to Fabian, Hajek, Montesinos and Zizler is also provided, see Corollary 3.9.

Keywords: Krein's theorem, Compactness, Measures of weak noncompactness

2001 Mathematics Subject Classification: 46S10, 46A50, 54C35

*The research was supported for the first named author by the project MTM2008-05396 of the Spanish Ministry of Science and Innovation, for the second named author by National Center of Science, Poland, grant no. N N201 605340 and also by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2013/058.

ISSN 0944-6532 / $ 2.50

1. Introduction

Several theorems related with compactness both in functional analysis and topology can be detected from suitable inequalities about distances to spaces of continuous functions. This motivated specialists to study quantitative counterparts of some classical results. The reader is addressed to works [1], [4], [6], [7], [10], [11] also as a good source of references. These papers provided some tools which have been used for new quantitative versions of Gantmacher's theorem about weak compactness of adjoint operators in Banach spaces, Eberlein–Grothendieck's theorem, Grothendieck's characterization of weak compactness in real Banach spaces $C(X, \mathbb{R})$, and the classical Krein-Smulyan's theorem.

The classical Krein's theorem states [8, Theorem 7.1] that in a real or complex Banach space E the convex hull $\operatorname{co} M$ is weakly relatively compact for a weakly relatively compact set $M \subset E$. A quantitative version of this theorem has been proved by Fabian, Hajek, Montesinos and Zizler [7, Theorems 2 and 13]. They asked:

(*) Assume that M is a bounded set in a Banach space E and let $B_{E''}$ be the closed unit ball in the bidual E'' of E. Assume that M is a ε-weakly relatively compact set, i.e. $\overline{M}^{\sigma(E'',E')} \subset E + \varepsilon B_{E''}$ for some $\varepsilon \geq 0$. Does the same hold for its convex envelope $\operatorname{co} M$?

The case $\varepsilon = 0$ is the classical Krein's theorem. In [7, Theorem 2] it was proved that whenever M is ε-weakly relatively compact for some $\varepsilon > 0$, then $\operatorname{co} M$ is 2ε-weakly relatively compact. Moreover if $B_{E'}$ is $\sigma(E', E)$-angelic, then $\operatorname{co} M$ is ε-weakly relatively compact.

The Krein's theorem apparently plays a very important role both in functional analysis and topology, so it is natural to ask if its non-Archimedean version holds also for non-Archimedean Banach spaces.

In [12, Proposition 2.2] we proved that the absolutely convex hull of a weakly relatively compact subset of a non-Archimedean Banach space (over a non-Archimedean locally compact non-trivially valued field) is weakly relatively compact (see also [17, Theorem 1.5 and Proposition 1.6]). The situation for function spaces $C_p(X, \mathbb{K})$ with the pointwise topology is totally different. In contrast to real spaces, we showed also in [12, Example 2.3] that for an infinite zero-dimensional compact space X the absolutely convex hull $\operatorname{aco} H$ of a τ_p-relatively compact and uniformly bounded set H in $C(X, \mathbb{K})$ need not be τ_p-relatively compact for a locally compact non-Archimedean $\mathbb{K}$. Nevertheless, the main result of [12] stated that if $H \subset C(X, \mathbb{K})$ is uniformly bounded, then $\operatorname{aco} H$ is τ_p-relatively compact if and only if H is weakly relatively compact (see [12, Theorem 2.8]). This line of research is continued in the present paper for non-Archimedean Banach spaces E.

We provide a non-Archimedean version of Fabian, Hajek, Montesinos and Zizler result (Theorem 3.3) as well as quantitative versions of Krein's theorem (Proposition 3.5, (3) and Corollary 3.9). Working with measures of weak noncompactness we show (Theorem 3.10) the following formula for a bounded set $M \subset E$:

$$\gamma(M) \leq k(M) \leq k(\operatorname{aco} M) \leq \omega(M) = \omega(\operatorname{aco} M) \leq |\rho|^{-1}\gamma(M),$$

where $\gamma(M)$ is the worst distance between iterated limits for sequences in M and sequences in the dual unit ball $B_{E'}$, $k(M)$ is the worst distance to E of points of the $\sigma(E'', E')$ −closure of M in the bidual space E'', and ω denotes the De Blasi measure of weak noncompactness. This proves also that γ, k and ω are equivalent, which for the real case fails in general (see [5, p. 372]). Nevertheless, for the case $|\mathbb{K}| = \{||x|| : x \in E\}$ we show the chain of equalities

$$\gamma(M) = \gamma(\operatorname{aco} M) = k(M) = k(\operatorname{aco} M) = \omega(M) = \omega(\operatorname{aco} M).$$

In this case we note that the absolutely convex hull $\operatorname{aco} M$ of a ε-weakly relatively compact subset M of a non-Archimedean Banach space over $\mathbb{K}$ is ε-weakly relatively compact. Note that such a result fails in general for real Banach spaces [9, Theorem 7].

Note also that Proposition 3.5 describes the De Blasi measure purely non-Archimedean as $\omega(M) = \sup\{\overline{\lim}_m \operatorname{dist}(x_m, [x_1, \ldots, x_{m-1}]) : (x_m) \subset M\}$.

2. Definitions and Notations

Let $\mathbb{K}$ be a non-Archimedean non-trivially valued field, i.e. equipped with the non-Archimedean valuation $|.| : \mathbb{K} \to [0, \infty)$, satisfying the strong triangle inequality: $|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$ $(\lambda, \mu \in \mathbb{K})$. Note that every complete valued field is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or is non-Archimedean, [13, Theorem 1.2.18]. Recall that if $\mathbb{K}$ is locally compact, then $|\mathbb{K}| := \{|\lambda| : \lambda \in \mathbb{K}\}$ is discretely valued, i.e. 0 is the only accumulation point of $|\mathbb{K}|$ ([13, Theorem 1.2.8]) and $|\mathbb{K}| = \{|\rho|^n : n \in \mathbb{Z}\} \cup \{0\}$ for some uniformizing element $\rho \in \mathbb{K}$, $|\rho| < 1$ (see [13, Chapter 1.2]).

We say that a normed space E over $\mathbb{K}$ is non-Archimedean if $||x + y|| \leq \max\{||x||, ||y||\}$ for all $x, y \in E$.

Let $B_{E,r} := \{x \in E : ||x|| \leq r\}$ and $B^-_{E,r} := \{x \in E : ||x|| < r\}$; we will write B_E (B^-_E) instead of $B_{E,1}$ ($B^-_{E,1}$). In particular, $B_{\mathbb{K}} = \{\lambda \in \mathbb{K} : |\lambda| \leq 1\}$. It follows directly from the strong triangle inequality that

$$B_{E,r_1} + B_{E,r_2} = B_{E,\max\{r_1,r_2\}}. \tag{1}$$

For a set $A \subset E$, we define the absolutely convex hull of A as

$$\operatorname{aco} A = \left\{\sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in B_{\mathbb{K}}, a_1, \ldots, a_n \in A\right\};$$

we say that A is absolutely convex if $A = \operatorname{aco} A$. A subset X of E, $0 \notin X$, is called orthogonal if for each $x \in X$ we have $\operatorname{dist}(x, [X \setminus \{x\}]) = ||x||$, where $[X \setminus \{x\}]$ means the linear span of the set $X \setminus \{x\}$. A projection $P : E \to E$ is called an orthoprojection if $||x+y|| = \max\{||x||, ||y||\}$ for all $x \in P(E)$, $y \in \operatorname{Ker} P$. For more background on normed spaces over non-Archimedean valued fields we refer the reader to the monographs [13] and [15].

By E' we denote a topological dual of E. We define the natural map $j_E : E \to E''$ by the formula $j_E(x)(z^*) := z^*(x)$ for all $x \in E, z^* \in E'$. j_E is an isometry; using the natural identification, we will identify E with $j_E(E) \subset E''$ and for $x \in E$ we will write $x \in E''$ instead of $j_E(x) \in E''$.

Let $M \subset E$ be a bounded set. Then M is relatively weakly compact if and only if $\overline{M}^{\sigma(E'',E')} \subset E$. We say that M is $\varepsilon-$weakly relatively compact if $\overline{M}^{\sigma(E'',E')} \subset E + B_{E'',\varepsilon}$. We shall say that M $\varepsilon-$interchanges limits with $B_{E'}$ if for any two sequences $(x_n) \subset M$ and $(z_n^*) \subset B_{E'}$, assuming that the limits $\lim_m \lim_n z_m^*(x_n)$ and $\lim_n \lim_m z_m^*(x_n)$ exist, we have

$$|\lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n)| \leq \varepsilon.$$

Define the following three functions - measures of weak noncompactness, mentioned above.

$$\omega(M) := \inf\{\varepsilon > 0 : M \subset K_\varepsilon + B_{E,\varepsilon};\ K_\varepsilon \text{ is } \sigma(E,E') - \text{compact}\},$$

$$k(M) := \sup_{x^{**} \in \overline{M}^{\sigma(E'',E')}} \operatorname{dist}(x^{**}, E)$$

$$\gamma(M) := \sup\{|\lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n)| : (z_m^*) \subset B_{E'}, (x_n) \subset M\}.$$

assuming the involved limits exist. By [14, Theorem 5.2] (see also [16, Theorem 2.3] and [18, Propositions 18, 19 and 20]) every compact (closed) set of E over locally compact $\mathbb{K}$ is weakly compact (weakly closed), thus the measure of weak noncompactness ω introduced by De Blasi compares with the Hausdorff measure of noncompactness.

Clearly, for each bounded set $M \subset E$ $k(M) = 0$ if and only if $\overline{M}^{\sigma(E'',E')} \subset E$ that is equivalent to the fact that M is relatively weakly compact. Since 0 is the only accumulation point of $|\mathbb{K}|$, we observe that for each bounded $M \subset E$ we have $\gamma(M) \in |\mathbb{K}|$. If $||E|| \neq |\mathbb{K}|$ the defined functions ω, k and γ may have different sets of values, as Example 3.12 shows.

Define

$$\phi_{\mathbb{K}}(\varepsilon) := \max\{|\lambda| : \lambda \in \mathbb{K}, |\lambda| \leq \varepsilon\};$$

clearly, $\phi_{\mathbb{K}}(\varepsilon) \in |\mathbb{K}|$ and $\phi_{\mathbb{K}}(\varepsilon) = \varepsilon$ if $\varepsilon \in |\mathbb{K}|$. We say that $t > 0$ is an upper accumulation point of $||E||$ if there exists $(x_n) \subset E$ such that $||x_1|| < ||x_2|| < ... < t$ and $\lim_n ||x_n|| = t$.

3. Results and proofs

Throughout by $\mathbb{K}$ we will denote a non-Archimedean locally compact and non-trivially valued field. By ρ we denote an uniformizing element of $\mathbb{K}$ with $|\rho| < 1$;

by E we will denote a non-Archimedean Banach space over $\mathbb{K}$; we assume $|\mathbb{K}| \subset ||E||$, where $||E|| := \{||x|| : x \in E\}$.

We start with some facts which will be used in the sequel.

Proposition 3.1.

(*a*) $\overline{j_{E'}\left(B_{E'}^{-}\right)}^{\sigma(E''',E'')} = B_{E'''}^{-}$.

(*b*) *If* 1 *is not an upper accumulation point of* $||E||$, *then* $\overline{j_{E'}\left(B_{E'}\right)}^{\sigma(E''',E'')} = B_{E'''}$.

Proof. (*a*) follows from [13, Corollary 7.4.8]. (*b*) Since $j_{E'}$ is an isometry and $B_{E'''}$ is $\sigma\left(E''', E''\right)$-closed, we have $\overline{j_{E'}\left(B_{E'}\right)}^{\sigma(E''',E'')} \subset B_{E'''}$. Assume that there exists

$$f \in B_{E'''} \backslash \overline{j_{E'}\left(B_{E'}\right)}^{\sigma(E''',E'')}.$$

It follows from [13, Theorem 7.4.6] that there exists $x^{**} \in E''$ such that $|f\left(x^{**}\right)| \geq \frac{1}{|\rho|}$ and $|x^{**}\left(z^{*}\right)| \leq 1$ for all $z^{*} \in B_{E'}$; thus, we can easily deduce that $||x^{**}|| \leq \frac{1}{|\rho|}$. Since

$$1 \geq ||f|| \geq \frac{|f\left(x^{**}\right)|}{||x^{**}||},$$

we obtain

$$\frac{1}{|\rho|} \geq ||x^{**}|| \geq |f\left(x^{**}\right)| \geq \frac{1}{|\rho|} \Longrightarrow ||x^{**}|| = |f\left(x^{**}\right)| = \frac{1}{|\rho|}. \tag{2}$$

On the other hand,

$$||x^{**}|| = \sup_{z^{*} \in E' \backslash \{0\}} \frac{|x^{**}\left(z^{*}\right)|}{||z^{*}||}.$$

Hence, we can select $\left(z_n^{*}\right) \subset B_{E'}$, assuming that $||z_n^{*}|| > |\rho|$ for all $n \in \mathbb{N}$, such that $||x^{**}|| = \lim_n \frac{|x^{**}(z_n^{*})|}{||z_n^{*}||}$. Suppose that there is $n_0 \in \mathbb{N}$ for which

$$||x^{**}|| = \frac{\left|x^{**}\left(z_{n_0}^{*}\right)\right|}{\left|\left|z_{n_0}^{*}\right|\right|}.$$

Then, by (2), we get

$$\left|\left|z_{n_0}^{*}\right|\right| = \frac{\left|x^{**}\left(z_{n_0}^{*}\right)\right|}{||x^{**}||} = \left|x^{**}\left(z_{n_0}^{*}\right)\right| \cdot |\rho| \leq |\rho|,$$

which contradicts the assumption that $||z_n^{*}|| > |\rho|$ for all $n \in \mathbb{N}$. So, we can assume, selecting a subsequence, if necessary, that

$$1 < \frac{|x^{**}\left(z_n^{*}\right)|}{||z_n^{*}||} < \frac{\left|x^{**}\left(z_{n+1}^{*}\right)\right|}{\left|\left|z_{n+1}^{*}\right|\right|} < ||x^{**}|| = \frac{1}{|\rho|}$$

for every $n \in \mathbb{N}$. Then, for every $k \in \mathbb{N}$ we can choose $n_k \in \mathbb{N}$ such that

$$\frac{\left|x^{**}\left(z_{n_k}^*\right)\right|}{\left|\left|z_{n_k}^*\right|\right|} > \frac{1}{|\rho|} - \frac{1}{k}.$$

Thus,

$$\frac{k}{k-|\rho|}\,|\rho| > \frac{\left|\left|z_{n_k}^*\right|\right|}{\left|x^{**}\left(z_{n_k}^*\right)\right|} \geq \left|\left|z_{n_k}^*\right|\right| > |\rho|\,.$$

But then, for every $k \in \mathbb{N}$ we can choose $x_k \in E$ satisfying

$$\frac{k}{k-|\rho|}\,|\rho| > \frac{\left|z_{n_k}^*\left(x_k\right)\right|}{||x_k||} > |\rho|\,.$$

Without loss of generality, we can assume that $\left|z_{n_k}^*\left(x_k\right)\right| = |\rho|$ for all $k \in \mathbb{N}$. Then, we obtain

$$\frac{k-|\rho|}{k} < ||x_k|| < 1.$$

Hence, $\lim_k ||x_k|| = 1$ and we deduce that 1 is an upper accumulation point of $||E||$, a contradiction. □

Lemma 3.2. *Let $x^{**} \in E''$ and assume that $d = \operatorname{dist}(x^{**}, E) > 0$. For every $x_1, \ldots, x_n \in E$, a non-zero $\lambda \in \mathbb{K}$ and $\varepsilon > 0$ with $0 < \varepsilon < |\lambda| < d$ there exist $z^* \in B_{E'}$ and $\lambda_0 \in \mathbb{K}$, $|\lambda_0| < \varepsilon$, such that $x^{**}(z^*) = \lambda + \lambda_0$ and $|x_i(z^*)| < \varepsilon$ for each $i \in \{1, \ldots, n\}$. If 1 is not an upper accumulation point of $||E||$ then we can even assume that $|\lambda| \leq d$.*

Proof. First, define a linear functional $f_0 : E + [x^{**}] \to \mathbb{K}$ for which $f_0(x^{**}) = \lambda$ and $f_0(y) = 0$ for all $y \in E$. Then, $||f_0|| = \frac{|\lambda|}{d} < 1$ if we assume that $|\lambda| < d$ ($||f_0|| \leq 1$ if we assume that $|\lambda| \leq d$). Applying Ingleton's theorem ([13, Corollary 4.1.2]), define $f \in E'''$ such that $||f|| = ||f_0||$ and $f|_{E+[x^{**}]} = f_0$. Let

$$V = \{g \in E''' : |(g-f)(x^{**})| < \varepsilon,\ |(g-f)(x_i)| < \varepsilon \text{ for all } i = 1, \ldots, n\}.$$

Then, since $j_{E'}\left(B_{E'}^-\right)$ is $\sigma(E''', E'')$-dense in $B_{E'''}^-$ (and $j_{E'}(B_{E'})$ is $\sigma(E''', E'')$-dense in $B_{E'''}$ if 1 is not an upper accumulation point of $||E||$) by Proposition 3.1, there exists $z^* \in V \cap B_{E'}$. Let $\lambda_0 = (z^* - f)(x^{**})$. Then

$$x^{**}(z^*) = f(x^{**}) - f(x^{**}) + z^*(x^{**}) = f(x^{**}) + (z^* - f)(x^{**}) = \lambda + \lambda_0.$$

Since f is zero on E, we finally obtain

$$|x_i(z^*)| = |z^*(x_i) - f(x_i)| = |(z^* - f)(x_i)| < \varepsilon$$

for $i = 1, \ldots, n$. □

We are ready to prove a non-Archimedean version of Fabian, Hajek, Montesinos and Zizler result.

Theorem 3.3. *Let $M \subset E$ be a bounded set and let $\varepsilon > 0$. Then,*

(a) *if M is $\varepsilon-$weakly relatively compact, then M $\phi_{\mathbb{K}}(\varepsilon)-$interchanges limits with $B_{E'}$;*

(b) *if M $\varepsilon-$interchanges limits with $B_{E'}$, then there exists $\delta_\varepsilon \le \frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$such that M is $\delta_\varepsilon-$weakly relatively compact. If 1 is not an upper accumulation point of $||E||$,then we can select such δ_ε with $\delta_\varepsilon < \frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$.*

Proof. (a) Assume that M is $\varepsilon-$weakly relatively compact. Let $(x_n) \subset M$ and $(z_n^*) \subset B_{E'}$ be sequences such that

$$\lim_n \lim_m x_n(z_m^*), \ \lim_m \lim_n x_n(z_m^*)$$

exist. We prove that

$$\left|\lim_n \lim_m x_n(z_m^*) - \lim_m \lim_n x_n(z_m^*)\right| \le \phi_{\mathbb{K}}(\varepsilon).$$

Let $x^{**} \in \overline{M}^{\sigma(E'',E')}$ be a $\sigma(E'',E')-$cluster point of the sequence (x_n). By the definition of $\varepsilon-$weakly relatively compactness, $\mathrm{dist}(x^{**},E) \le \varepsilon$; fix $\delta > 0$ and choose $x \in E$ for which

$$||x - x^{**}|| \le \mathrm{dist}(x^{**},E) + \delta.$$

Next, take $z^* \in E'$, a $\sigma(E',E)-$cluster point of (z_m^*). Since x and $x_1, x_2, ...$ are in E, $x(z^*)$ and $x_n(z^*)$ $(n = 1, 2, ...)$ are cluster points of $x(z_m^*)$ and $x_n(z_m^*)$, respectively. Thus, we can select a subsequence of (z_m^*), denoted again by (z_m^*), such that $\lim_m x(z_m^*)$ exists. Hence, we obtain

$$\lim_m x(z_m^*) = x(z^*), \tag{3}$$

and

$$\lim_m x_n(z_m^*) = x_n(z^*)$$

for every $n \in \mathbb{N}$. Clearly,

$$\lim_n x_n(z_m^*) = x^{**}(z_m^*) \tag{4}$$

for every $m \in \mathbb{N}$ and

$$\lim_n \lim_m x_n(z_m^*) = \lim_n x_n(z^*) = x^{**}(z^*). \tag{5}$$

Thus, by (4), (3) and (5) we have

$$\begin{aligned} &\left|\lim_n \lim_m x_n(z_m^*) - \lim_m \lim_n x_n(z_m^*)\right| = \left|x^{**}(z^*) - \lim_m x^{**}(z_m^*)\right| \\ &= \left|x^{**}(z^*) - \lim_m x(z_m^*) + \lim_m x(z_m^*) - \lim_m x^{**}(z_m^*)\right| \\ &= \left|x^{**}(z^*) - x(z^*) + \lim_m (x - x^{**})(z_m^*)\right| \\ &\le \max\left\{|(x^{**} - x)(z^*)|, \left|\lim_m (x - x^{**})(z_m^*)\right|\right\} \le ||x^{**} - x||. \end{aligned}$$

Since

$$\left|\lim_m (x - x^{**})(z_m^*)\right|, |(x - x^{**})(z^*)| \in |\mathbb{K}|,$$

$||x^{**} - x|| \leq \varepsilon + \delta$ and $\delta > 0$ is arbitrary, we conclude that

$$\max\left\{\left|\lim_m (x - x^{**})(z_m^*)\right|, |(x - x^{**})(z^*)|\right\} \leq \phi_{\mathbb{K}}(\varepsilon).$$

It finishes the proof of part (a).

(b) Now, suppose that M $\varepsilon-$interchanges limits with $B_{E'}$; i.e. for any two sequences $(x_n) \subset M$ and $(z_n^*) \subset B_{E'}$ we have

$$|\lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n)| \leq \varepsilon, \tag{6}$$

assuming that the involved limits exist. Since the value of the left side of (6) is an element of $|\mathbb{K}|$, we deduce

$$|\lim_m \lim_n z_m^*(x_n) - \lim_n \lim_m z_m^*(x_n)| \leq \phi_{\mathbb{K}}(\varepsilon). \tag{7}$$

Take $x^{**} \in \overline{M}^{\sigma(E'',E')}$ and suppose that $d_0 = \operatorname{dist}(x^{**}, E) > 0$. Set $x_1 \in M$ and $\lambda_0 \in \mathbb{K}$ such that $|\lambda_0| = |\rho| \cdot d_0$ if $d_0 \in |\mathbb{K}|$ and 1 is an upper accumulation point of $||E||$, and $|\lambda_0| = \phi_{\mathbb{K}}(d_0)$, otherwise. Applying Lemma 3.2, we select $\lambda_1 \in \mathbb{K}$, $|\lambda_1| < \frac{|\lambda_0|}{2}$, and $z_1^* \in B_{E'}$ for which $x^{**}(z_1^*) = \lambda_0 + \lambda_1$ and $|x_1(z_1^*)| < \left|\frac{\lambda_0}{2}\right|$. Let

$$V = \left\{u \in E'' : |(x^{**} - u)(z_1^*)| < \frac{|\lambda_0|}{3}\right\}.$$

Taking $x_2 \in M \cap V$, and applying again Lemma 3.2, we choose $\lambda_2 \in \mathbb{K}$ with $|\lambda_2| < \frac{|\lambda_0|}{3}$ and $z_2^* \in B_{E'}$ for which $x^{**}(z_2^*) = \lambda_0 + \lambda_1 + \lambda_2$ and $|x_i(z_2^*)| < \frac{|\lambda_0|}{3}$ for $i = 1, 2$. Continuing on this direction in the n-th step we choose $x_n \in M$ for which

$$|(x^{**} - x_n)(z_i^*)| < \frac{|\lambda_0|}{n+1}, \quad i = 1, \ldots, n-1. \tag{8}$$

Next, using Lemma 3.2, we select $\lambda_n \in \mathbb{K}$ with $|\lambda_n| < \frac{|\lambda_0|}{n+1}$ and $z_n^* \in B_{E'}$ for which $x^{**}(z_n^*) = \lambda_0 + \lambda_1 + ... + \lambda_n$ and

$$|x_i(z_n^*)| < \frac{|\lambda_0|}{n+1} \tag{9}$$

for $i = 1, \ldots, n$. This procedure enables us to form sequences $(x_n) \subset M$, $(\lambda_n) \subset \mathbb{K}$ and $(z_n^*) \subset B_{E'}$ such that for every $n \in \mathbb{N}$

$$x^{**}(z_n^*) = \lambda_0 + \lambda_1 + ... + \lambda_n, \quad |x^{**}(z_n^*)| = |\lambda_0|$$
$$|x_i(z_n^*)| < \frac{|\lambda_0|}{n+1} \quad \text{for } i \in \{1, \ldots, n\}.$$

Clearly, by (8), for every $m \in \mathbb{N}$ we have $x_n(z_m^*) \to x^{**}(z_m^*)$ if $n \to \infty$; hence,

$$\lim_m \lim_n x_n(z_m^*) = \lim_m x^{**}(z_m^*) = \sum_{i=0}^{\infty} \lambda_i.$$

On the other hand, it follows from (9) that for every $n \in \mathbb{N}$ one has $|x_n(z_m^*)| \to 0$ if $m \to \infty$; thus, $\lim_n \lim_m x_n(z_m^*) = 0$. Hence, we conclude that

$$\left|\lim_n \lim_m x_n(z_m^*) - \lim_m \lim_n x_n(z_m^*)\right| = \left|\lim_m \lim_n x_n(z_m^*)\right| = |\lambda_0|,$$

thus, $|\lambda_0| \leq \phi_{\mathbb{K}}(\varepsilon)$ by (7).

Assume that $d_0 \in |\mathbb{K}|$ ($d_0 = \operatorname{dist}(x^{**}, E)$) and 1 is an upper accumulation point of $||E||$; recall that in this case $|\lambda_0| = |\rho| \cdot d_0$, so $d_0 \leq \frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$. Suppose now that $d_0 \notin |\mathbb{K}|$. Then, $\phi_{\mathbb{K}}(d_0) = |\lambda_0|$ and $|\lambda_0| < d_0$. Hence, $d_0 \in (|\lambda_0|, \frac{1}{|\rho|}|\lambda_0|)$ and $d_0 < \frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$. Setting $\delta_\varepsilon := \sup_{x^{**} \in \overline{M}^{\sigma(E'',E')}} d(x^{**}, E)$, we obtain $\delta_\varepsilon \leq \frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$.

Assume now that 1 is not an upper accumulation point of $||E||$. Then $\frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$ is not an accumulation point of $||E||$, either. Thus, we can choose $r > 0$ such that $\operatorname{dist}(x^{**}, E) < \frac{1}{|\rho|}\phi_{\mathbb{K}} - r$ for every $x^{**} \in \overline{M}^{\sigma(E'',E')}$. Defining $\delta_\varepsilon := \sup_{x^{**} \in \overline{M}^{\sigma(E'',E')}} d(x^{**}, E)$ similarly as above, we get promised $\delta_\varepsilon < \frac{1}{|\rho|}\phi_{\mathbb{K}}(\varepsilon)$. □

Corollary 3.4. *If $M \subset E$ is bounded, then M is weakly relatively compact if and only if $\gamma(M) = 0$.*

The next proposition gives some properties of the measure ω.

Proposition 3.5. *Let $M \subset E$ be a bounded set. Then*

(*1*) *for every $\varepsilon > \omega(M)$ there exist $y_1, \ldots, y_k \in E$ such that*

$$M \subset \{y_1, \ldots, y_k\} + B_{E,\varepsilon} \subset \operatorname{aco}\{y_1, \ldots, y_k\} + B_{E,\varepsilon} \subset [y_1, \ldots, y_k] + B_{E,\varepsilon};$$

(*2*) $\omega(M) = \inf\{\varepsilon > 0 : M \subset [F_\varepsilon] + B_{E,\varepsilon}$ *where* $F_\varepsilon \subset E$ *is finite*$\}$;

(*3*) $\omega(M) = \omega(\operatorname{aco} M)$;

(*4*) $\omega(M) = \sup\{\overline{\lim}_m \operatorname{dist}(x_m, [x_1, \ldots, x_{m-1}]) : (x_m) \subset M\}$.

(*5*) $k(M) \leq \omega(M)$.

Proof. (*1*) Let $\varepsilon > 0$. If $\varepsilon > \omega(M)$, then, by definition, there exists a weakly compact set K_ε (in fact compact by [14, Theorem 5.2]), for which $M \subset K_\varepsilon + B_{E,\varepsilon}$. By compactness of K_ε we can select $y_1, \ldots, y_k \in E$ such that $K_\varepsilon \subset \bigcup_{i=1}^k U_i$, where

$$U_i = \{x \in E : ||x - y_i|| \leq \varepsilon\} = \{y_i\} + B_{E,\varepsilon}, \quad i = 1, \ldots, k.$$

Since $B_{E,\varepsilon} + B_{E,\varepsilon} = B_{E,\varepsilon}$ (see (1)), we get

$$M \subset \bigcup_{i=1}^{k} (\{y_i\} + B_{E,\varepsilon}) + B_{E,\varepsilon} \subset \{y_1, \ldots, y_k\} + B_{E,\varepsilon}.$$

The other inclusions in (*1*) are obvious.

(*2*) Denote

$$\omega_0 := \inf\{\varepsilon > 0 : M \subset [F_\varepsilon] + B_{E,\varepsilon} \text{ where } F_\varepsilon \subset E \text{ is finite}\}.$$

To prove $\omega_0 \geq \omega(M)$ take $\varepsilon > 0$, and assume that there exists a finite set $F_\varepsilon \subset E$ such that $M \subset [F_\varepsilon] + B_{E,\varepsilon}$. Since M is bounded, there exists $r > \varepsilon > 0$ for which $M \subset B_{E,r}$. Then, $K'_\varepsilon = [F_\varepsilon] \cap B_{E,r}$ is compact. Set $x \in M$. Then, $x = x_F + x_\varepsilon$, where $x_F \in [F_\varepsilon]$ and $x_\varepsilon \in B_{E,\varepsilon}$. Clearly,

$$x_F \in [F_\varepsilon] \cap (M + B_{E,\varepsilon}) \subset [F_\varepsilon] \cap (B_{E,r} + B_{E,\varepsilon}) = [F_\varepsilon] \cap B_{E,r}$$

by (1). Thus, $x \in K'_\varepsilon + B_{E,\varepsilon}$ and we imply $M \subset K'_\varepsilon + B_{E,\varepsilon}$. Hence, $\omega(M) \leq \omega_0$. The inequality $\omega_0 \leq \omega(M)$ follows directly from (*1*).

(*3*) Clearly $\omega(M) \leq \omega(\operatorname{aco} M)$. Assume that $M \subset F + B_{E,\varepsilon}$ for some finite-dimensional subspace $F \subset E$ and $\varepsilon > 0$. Take $z \in \operatorname{aco} M$. Then $z = \sum_{i=1}^n \lambda_i x_i$ for some $\lambda_i \in B_{\mathbb{K}}$ and $x_i \in M$, $i = 1, \dots, n$. Since $x_i \in M$ for every $i \in \{1, \dots, n\}$, we can choose $x'_i \in F$ and $x_i^\varepsilon \in B_{E,\varepsilon}$ such that $x_i = x'_i + x_i^\varepsilon$. Then we have

$$z = \sum_{i=1}^n \lambda_i (x'_i + x_i^\varepsilon) = \sum_{i=1}^n \lambda_i x'_i + \sum_{i=1}^n \lambda_i x_i^\varepsilon,$$

and conclude that $z \in F + B_{E,\varepsilon}$, since $\sum_{i=1}^n \lambda_i x'_i \in F$ and $\sum_{i=1}^n \lambda_i x_i^\varepsilon \in B_{E,\varepsilon}$. Hence, $\operatorname{aco} M \subset F + B_{E,\varepsilon}$ and $\omega(M) \geq \omega(\operatorname{aco} M)$.

(*4*) Denote

$$\omega_{NA} = \sup\{\overline{\lim_m} \operatorname{dist}(x_m, [x_1, \dots, x_{m-1}]) : (x_m) \subset M\}.$$

Let $\varepsilon_0 = \omega(M)$. Fix $\varepsilon > \varepsilon_0$ and assume that there exists a sequence $(x_n) \subset M$ for which

$$\overline{\lim_n} \operatorname{dist}(x_n, [x_1, \dots, x_{n-1}]) > \varepsilon.$$

Then we can choose a subsequence (x_{n_k}) of (x_n) for which $\lim_k \operatorname{dist}(x_{n_k}, [x_1, \dots, x_{n_k - 1}]) > \varepsilon$ and even, removing finitely many elements, such that

$$\operatorname{dist}(x_{n_k}, [x_1, \dots, x_{n_k - 1}]) > \varepsilon \tag{10}$$

for all $k \in \mathbb{N}$. Clearly, $||x_{n_k}|| > \varepsilon$ for all $k \in \mathbb{N}$. By (*1*), we can select $y_1, \dots, y_p \in E$ such that $M \subset \{y_1, \dots, y_p\} + B_{E,\varepsilon}$; we can assume that $||y_i - y_j|| > \varepsilon$ for all $i, j \in \{1, \dots, p\}$ with $i \neq j$. Since $x_{n_1} \in M$, we find $j_1 \in \{1, \dots, p\}$ for which

$$||x_{n_1} - y_{j_1}|| \leq \varepsilon. \tag{11}$$

By (10), $\operatorname{dist}(x_{n_2}, [x_1, \dots, x_{n_2 - 1}]) > \varepsilon$, hence, we have

$$||x_{n_2} - x_{n_1}|| > \varepsilon.$$

Applying (11), we obtain

$$||x_{n_2} - y_{j_1}|| = ||x_{n_2} - x_{n_1} + x_{n_1} - y_{j_1}|| = ||x_{n_2} - x_{n_1}|| > \varepsilon.$$

Thus, we can choose $j_2 \in \{1, \dots, p\} \setminus \{j_1\}$ for which $||x_{n_2} - y_{j_2}|| \le \varepsilon$. Continuing on this direction, we show that $||x_{n_i} - y_{j_i}|| \le \varepsilon$ for each $i = 1, ..., p$, where $\{j_1, ..., j_p\} = \{1, \dots, p\}$. Hence, $M \subset \{x_{n_1}, \dots, x_{n_p}\} + B_{E,\varepsilon}$. Then, $\left|\left|x_{n_{p+1}} - x_{n_i}\right|\right| \le \varepsilon$ for some $i \in \{1, ..., p\}$. But, by (10)

$$\operatorname{dist}\left(x_{n_{p+1}}, \left[x_1, \dots, x_{n_p}\right]\right) > \varepsilon,$$

providing a contradiction. Thus $\omega_{NA} \le \varepsilon$, and we conclude $\omega_{NA} \le \omega(M)$.

In order to show $\omega_{NA} \ge \omega(M)$ take $\varepsilon < \omega(M)$. Since, by (*1*), $M \nsubseteq F + B_{E,\varepsilon}$ for every finite-dimensional subspace $F \subset E$, setting $x_1 \in M$, we deduce

$$M \nsubseteq [x_1] + B_{E,\varepsilon}.$$

Hence there exists $x_2 \in M$ such that $\operatorname{dist}(x_2, [x_1]) > \varepsilon$. Inductively, we select a sequence $(x_n) \subset M$ for which $\operatorname{dist}(x_n, [x_1, \dots, x_{n-1}]) > \varepsilon$. Thus

$$\overline{\lim_n} \left(\operatorname{dist}(x_n, [x_1, \dots, x_{n-1}])\right) \ge \varepsilon,$$

and the proof of this part is completed.

(*5*) Observe that for $\varepsilon > 0$ and a weakly compact set $K_\varepsilon \subset E$ such that $M \subset K_\varepsilon + B_{E,\varepsilon}$ we have

$$\overline{M}^{\sigma(E'', E')} \subset K_\varepsilon + B_{E'',\varepsilon} \subset E + B_{E'',\varepsilon}.$$

Hence $k(M) \le \omega(M)$. □

Let I be any set. Recall that $l^\infty_{\mathbb{K}}(I)$ is the set of all bounded maps $I \to \mathbb{K}$. It is a Banach space under pointwise operations and norm

$$||.||_\infty : \left(\lambda^i\right)_{i \in I} \to \sup_{i \in I} \left|\lambda^i\right|.$$

By $c^0_{\mathbb{K}}(I) \subset l^\infty_{\mathbb{K}}(I)$ we will denote the closed subspace of all $(\lambda^i)_{i \in I} \in l^\infty_{\mathbb{K}}(I)$ such that for every $\varepsilon > 0$ there exists a finite set $J \subset I$ with $|\lambda^i| < \varepsilon$ for all $i \in I \backslash J$. Clearly, $||c^0_{\mathbb{K}}(I)|| = |\mathbb{K}|$; thus $\omega(M) \in |\mathbb{K}|$ for any bounded set $M \subset c^0_{\mathbb{K}}(I)$. For the case E being the space $c^0_{\mathbb{K}}(I)$ we have the following

Lemma 3.6. *Let $\varepsilon > 0$. If $(w_n) \subset c^0_{\mathbb{K}}(I)$ is a bounded sequence for which there exists an infinite subset $J \subset I$ such that $max_n |w^i_n| = \varepsilon$ for all $i \in J$ then*

- *there exists $(u_n) \in \operatorname{aco}\{w_1, w_2, ...\}$ and $\{k_1, k_2, ...\} \subset J$ such that for every $n \in \mathbb{N}$ $\left|u^{k_n}_n\right| = \varepsilon$, $u^{k_m}_n = 0$ if $m \in \{1, ..., n-1\}$ and $\left|u^{k_m}_n\right| < \varepsilon$ for all $m > n$,*
- *$\omega(\{w_1, w_2, ...\}) \ge \varepsilon$.*

Proof. Take $n_1 \in \mathbb{N}$ and $k_1 \in J$ for which $\left|w_{n_1}^{k_1}\right| = \varepsilon$. Note that $J_1 = \{i \in I : \left|w_{n_1}^{i}\right| \geq \varepsilon\}$ is finite, since w_{n_1} is an element of $c_{\mathbb{K}}^{0}(I)$. Thus, we can find $n_2 > n_1$ and $k_2 \in J \backslash J_1$ such that $\left|w_{n_2}^{k_2}\right| = \varepsilon$; then, clearly $\left|w_{n_1}^{k_2}\right| < \varepsilon$. Next, we find $n_3 > n_2$ and $k_3 \in J \backslash (J_1 \cup J_2)$, where $J_2 = \left\{i \in I : \left|w_{n_2}^{i}\right| \geq \varepsilon\right\}$, such that $\left|w_{n_3}^{k_3}\right| = \varepsilon$. Continuing on this direction we select sequences $(k_j) \subset J$ and $(n_j) \subset \mathbb{N}$ such that $|w_{n_j}^{k_j}| = \varepsilon$ and $|w_{n_i}^{k_j}| < \varepsilon$ for each $i \in \{1, ..., j-1\}$.

Define $u_1 := w_{n_1}$. Suppose that we have defined $u_1, u_2, \ldots, u_{m-1}$ satisfying the required properties. Then we define

$$u_{m,1} := w_{n_m} - \frac{w_{n_m}^{k_1}}{u_1^{k_1}} u_1,$$

$$u_{m,n} := u_{m,n-1} - \frac{u_{m,n-1}^{k_{n-1}}}{u_n^{k_n}} u_n \quad \text{for } n = 2, 3, \ldots, m-1.$$

Next we set $u_m := u_{m,m-1}$. We can easily verify that $(u_m) \subset \operatorname{aco}\{w_1, w_2, ...\}$ and (u_m) satisfies the required properties, i.e. for every $i \in \mathbb{N}$ $\left|u_i^{k_i}\right| = \varepsilon$, $u_i^{k_j} = 0$ for each $j < i$, and $|u_i^{k_j}| < \varepsilon$ for all $j > i$. Let $P : c_{\mathbb{K}}^{0}(I) \to c_{\mathbb{K}}^{0}(J_0)$, where $J_0 = \{k_1, k_2, ...\}$, be the natural orthoprojection. Clearly, $||P(u_n)|| = \varepsilon$ for every $n \in \mathbb{N}$. Then, applying [13, Theorem 2.2.9], we can easily deduce that $(P(u_n))$ is orthogonal. Hence for fixed $m \in \mathbb{N}$, we have

$$\operatorname{dist}(P(u_m), [P(u_1), \ldots, P(u_{m-1})]) = \varepsilon.$$

But,

$$\operatorname{dist}(u_m, [u_1, \ldots, u_{m-1}]) \geq \operatorname{dist}(P(u_m), [P(u_1), \ldots, P(u_{m-1})]),$$

since P is orthoprojection; hence, using Proposition 3.5, (*3*) and (*4*), we finally obtain

$$\omega(\{w_1, w_2, ...\}) = \omega(\operatorname{aco}\{w_1, w_2, ...\}) \geq \varepsilon. \qquad \square$$

Proposition 3.7. *Let $E = c_{\mathbb{K}}^{0}(I)$, $\varepsilon > 0$ and $M \subset E$ be a bounded and infinite set. Then,*

(*1*) *$\omega(M) = \varepsilon$ if and only if there exists $x = (x^i)_{i \in I} \in l_{\mathbb{K}}^{\infty}(I)$ such that the following conditions hold:*
   (*a*) *$|w^i| \leq |x^i|$ for every $w = (w^j)_{j \in I} \in M$ and $i \in I$, and $\{i \in I : |x^i| \neq \varepsilon\}$ is finite;*
   (*b*) *there exists $(w_n) \subset M$ and infinite $J = \{k_1, k_2, ...\} \subset I$ such that $\left|x^{k_n}\right| = \left|w_n^{k_n}\right|$ for every $n \in \mathbb{N}$.*

(*2*) *$\gamma(M) = \omega(M)$.*

(*3*) *M is weakly relatively compact if and only if there exists $x = (x^i)_{i \in I} \in c_{\mathbb{K}}^{0}(I)$ such that $|w^i| \leq |x^i|$ for every $w = (w^j)_{j \in I} \in M$ and $i \in I$.*

Proof. (*1*) Suppose that $x = (x^i)_{i\in I} \in l_{\mathbb{K}}^{\infty}(I)$ is such that (a) and (b) are satisfied. Let

$$M_0 = \left\{x^i e_i : i \in I\right\} \subset c_{\mathbb{K}}^0(I).$$

Then M_0 is an orthogonal set. Using Proposition 3.5, (*4*) we deduce that $\omega(M_0) = \varepsilon$. Clearly, $M \subset \overline{\operatorname{aco} M_0}$ since $|w^i| \leq |x^i|$ for every $w = (w^j)_{j\in I} \in M$ and $i \in I$. Hence, by Proposition 3.5, (*3*) we note

$$\omega(M) \leq \omega(\operatorname{aco} M_0) = \omega(M_0) = \varepsilon.$$

On the other hand, taking a sequence $(w_n) \in M$ defined as in (b), Lemma 3.6 implies $\omega(M) \geq \varepsilon$, so we conclude that $\omega(M) = \varepsilon$.

Now, suppose that $\omega(M) = \varepsilon$. Since $\mathbb{K}$ is discretely valued and M is bounded, for every $i \in I$ we can choose $\lambda_i \in \mathbb{K}$ such that

$$|\lambda_i| = \max\left\{\left|w^i\right| : w = \left(w^j\right)_{j\in I} \in M\right\}.$$

Take $\lambda_0 \in \mathbb{K}$ for which $|\lambda_0| = \varepsilon$. Next, define $x = (x^i)_{i\in I} \in l_{\mathbb{K}}^{\infty}(I)$, setting $x^i = \lambda_i$ if $|\lambda_i| \geq \varepsilon$ and $x^i = \lambda_0$, otherwise. Assume that we can select an infinite set $\{n_1, n_2, ...\} \subset I$ such that $|x^{n_j}| > \varepsilon$, $j \in \mathbb{N}$. But then, for every $j \in \mathbb{N}$ we can find $w_j \in M$ for which $\left|w_j^{n_j}\right| = |x^{n_j}|$. Choosing a subsequence (j_k) such that $\left|w_{j_k}^{n_{j_k}}\right| = |x^{n_{j_k}}| = \varepsilon_0$ for some $\varepsilon_0 > \varepsilon$ and applying Lemma 3.6, we deduce that $\omega(\{w_1, w_2, ...\}) \geq \varepsilon_0 > \varepsilon$. This yields $\omega(M) > \varepsilon$, a contradiction. Hence, the set $J_0 = \{i : |x^i| > \varepsilon,\ i \in I\}$ is finite and (a) is established. To prove (b) it is enough to show that the set

$$J_1 := \{i : |\lambda_i| = \varepsilon,\ i \in I\}$$

is infinite. Having this one can easily form a required sequence $(w_n) \subset M$. Indeed, assume that J_1 is finite. Then, $|w^i| < \varepsilon$ for every $w = (w^j)_{j\in I} \in M$ and $i \in I\backslash(J_0 \cup J_1)$. But then, we can easily deduce that

$$M \subset [\{e_i : i \in J_0 \cup J_1\}] + B_{E,\varepsilon|\rho|},$$

a contradiction.

(*2*) $\gamma(M) \leq \omega(M)$ by Theorem 3.3 and Proposition 3.5, (*5*). Let $\varepsilon = \omega(M)$. Applying (*1*), we can select a finite $I_0 \subset I$, $(w_n) \subset M$ and infinite $J = \{k_1, k_2, ...\} \subset I\backslash I_0$ such that $|w_n^j| \leq \varepsilon$ for every $n \in \mathbb{N}$ and every $j \in I\backslash I_0$, and $\left|w_n^{k_n}\right| = \varepsilon$ for all $n \in \mathbb{N}$. Additionally, since for every $n \in \mathbb{N}$ $|w_n^j| = \varepsilon$ only for finitely many $j \in I\backslash I_0$, passing to a subsequence, if necessary, we can assume that $\left|w_m^{k_n}\right| < \varepsilon$ for all $n \in \mathbb{N}$ and each $m < n$. Let $T : c_{\mathbb{K}}^0(I) \to c_{\mathbb{K}}^0(I\backslash I_0)$ be a natural orthoprojection. Denote $v_n = T(w_n)$, $n \in \mathbb{N}$. Then, $||v_n|| = \varepsilon$ for all $n \in \mathbb{N}$. We prove that (v_n) is orthogonal. Take any $\{p_1, ..., p_l\} \subset \mathbb{N}$, $p_1 < ... < p_l$, and $a_1, ..., a_l \in \mathbb{K}$ with $|a_i| = 1$ for each $i \in \{1, ..., l\}$. Then,

$$\left\|\sum_{i=1}^{l} a_i v_{p_i}\right\| \geq \left|\sum_{i=1}^{l} a_i v_{p_i}^{k_{p_l}}\right| = \left|v_{p_n}^{k_{p_l}}\right| = \varepsilon = \max_{i=1,...,l} ||a_i v_{p_i}||$$

since, by assumption, $|v_{p_i}^{k_{p_l}}| < \varepsilon$ for each $i < l$; thus, (v_n) is orthogonal. Fix $\lambda_0 \in \mathbb{K}$ with $|\lambda_0| = \varepsilon$. Let $v_n^* \ (n \in \mathbb{N})$ denotes a linear functional spanned on $[v_1, v_2, ...]$ given by $v_n^*(v_m) := 0$ if $n \neq m$ and $v_n^*(v_n) := \lambda_0$; since (v_n) is orthogonal, $||v_n^*|| = 1$ for all $n \in \mathbb{N}$. Using Ingleton's theorem ([13, Corollary 4.1.2]), for every $n \in \mathbb{N}$ we find a preserving norm extension of v_n^* on the whole of $c_{\mathbb{K}}^0(I \backslash I_0)$, denoted again by v_n^*, and define $z_n^* = \sum_{i=1}^n v_i^* \circ T$, a linear functional on $c_{\mathbb{K}}^0(I)$. Clearly, $||z_n^*|| \leq 1 \ (n \in \mathbb{N})$. Observe that $z_m^*(w_n) = 0$ if $n > m$ and $z_m^*(w_n) = \lambda_0$ if $n \leq m$. Hence, $\lim_m z_m^*(w_n) = \lambda_0$ for any $n \in \mathbb{N}$ and $\lim_n z_m^*(w_n) = 0$ for every $m \in \mathbb{N}$. Thus,

$$\left|\lim_n \lim_m z_m^*(w_n) - \lim_m \lim_n z_m^*(w_n)\right| = |\lambda_0| = \varepsilon$$

and we conclude $\gamma(M) \geq \omega(M)$.

(3) Suppose that M is weakly relatively compact. For every $i \in I$ choose $a_i \in \mathbb{K}$ such that

$$|a_i| = \max\left\{\left|w^i\right| : w = \left(w^j\right)_{j \in I} \in M\right\},$$

and define $M_0 = \{a_i e_i : i \in I\}$. Assume that there exists $\varepsilon > 0$ and an infinite $J \subset I$ such that $|a_i| > \varepsilon$ for all $i \in J$. Then, we can select $(w_n) \subset M$ and $\{n_1, n_2, ...\} \subset J$ for which $\left|w_j^{n_j}\right| = \left|a_{n_j}\right| = \varepsilon_0$ for some $\varepsilon_0 > \varepsilon$. But applying Lemma 3.6, we conclude that $\omega(M) > \varepsilon$, a contradiction. Hence, setting $y^i := a_i$, $i \in I$, we obtain $(y^i)_{i \in I} \in c_{\mathbb{K}}^0(I)$. Now assume that there exists $x = (x^i)_{i \in I} \in c_{\mathbb{K}}^0(I)$ such that $|w^i| \leq |x^i|$ for every $w = (w^i)_{i \in I} \in M$ and $i \in I$. Define

$$M_0 = \left\{x^i e_i : i \in I\right\} \subset c_{\mathbb{K}}^0(I).$$

Using Proposition 3.5, *(4)* we deduce that $\omega(M_0) = 0$. Since $M \subset \overline{\operatorname{aco} M_0}$, we imply $\omega(M) \leq \omega(\operatorname{aco} M_0) = \omega(M_0) = 0$, thus, M is weakly relatively compact. □

Corollary 3.8. *Let M be a bounded set of E. Then, $\gamma(M) \geq |\rho| \cdot \omega(M)$.*

Proof. By [13, Theorem 2.5.4] and [15, Lemma 4.13] there exist a set I and a linear homeomorphism $T : E \to c_0(I)$ such that

$$|\rho| \cdot ||Tx|| < ||x|| \leq ||Tx||.$$

Hence we have $\omega(M) \leq \omega(T(M))$. Observe that for $z^* \in B_{c_0(I)'}$ we derive

$$||z^* \circ T|| = \sup_{x \in E} \frac{|(z^* \circ T)(x)|}{||x||} \leq \frac{1}{|\rho|} \sup_{x \in E} \frac{|z^*(T(x))|}{||T(x)||} \leq \frac{1}{|\rho|}.$$

Hence $(\rho z^* \circ T) \in B_{E'}$, and then $\gamma(M) \geq |\rho| \cdot \gamma(T(M))$. Applying Proposition 3.7, *(2)* we finally note that

$$\frac{1}{|\rho|} \cdot \gamma(M) \geq \gamma(T(M)) \geq \omega(T(M)) \geq \omega(M).$$ □

Now we present the following quantitative version of Krein's theorem.

Corollary 3.9. *For a bounded set $M \subset E$ we have*

$$\gamma(M) \leq \gamma(\operatorname{aco} M) \leq \frac{1}{|\rho|}\gamma(M).$$

If $|\mathbb{K}| = ||E||$ then

$$\gamma(M) = \gamma(\operatorname{aco} M).$$

Proof. Clearly, $\gamma(M) \leq \gamma(\operatorname{aco} M)$. To complete the proof, observe that

$$\gamma(M) \leq \gamma(\operatorname{aco} M) \leq k(\operatorname{aco} M) \leq \omega(\operatorname{aco} M) = \omega(M) \leq \frac{1}{|\rho|}\gamma(M) \tag{12}$$

by Theorem 3.3, Proposition 3.5, (*5*), (*3*) and Corollary 3.8. If $|\mathbb{K}| = ||E||$, E is isometrically isomorphic to $c_0(I)$ for some I (see [13, Theorem 2.5.4]). Thus, $\omega(M) = \gamma(M)$ by Proposition 3.7, (*2*), and $\gamma(M) = \gamma(\operatorname{aco} M)$ by (12). □

We complete with the following

Theorem 3.10. *If $M \subset E$ is bounded, then*

$$\gamma(M) \leq k(M) \leq k(\operatorname{aco} M) \leq \omega(M) = \omega(\operatorname{aco} M) \leq \frac{1}{|\rho|}\gamma(M). \tag{13}$$

If additionally $|\mathbb{K}| = ||E||$, then

$$\gamma(M) = \gamma(\operatorname{aco} M) = k(M) = k(\operatorname{aco} M) = \omega(M) = \omega(\operatorname{aco} M). \tag{14}$$

Proof. Clearly, $k(M) \leq k(\operatorname{aco} M)$. The rest of (13) follows directly from Theorem 3.3, Proposition 3.5, (*5*), (*3*) and Proposition 3.7, (*2*). Now, assume that $|\mathbb{K}| = ||E||$. Since, by [13, Theorem 2.5.4], E is isometrically isomorphic to $c_{\mathbb{K}}^0(I)$ for some I, we can apply Proposition 3.7, (*2*) obtaining $\gamma(M) = \omega(M)$. Thus, using (13) and Corollary 3.9 we reach (14). □

Remark 3.11. 1. Note that in the corresponding real case, for any bounded set M of a real Banach space we have $k(M) \leq \gamma(M) \leq 2k(M)$, see [4, Theorem 2.3], and the equality $k(M) = k(\operatorname{co} M)$ fails in general, see [9, Theorem 7].

2. It follows from Theorem 3.10 that γ and ω are equivalent for every non-Archimedean Banach space E. This is in contrast to the classical case, where there exist real Banach spaces for which γ and ω are not equivalent (see [4, Remark 3.3 and Corollary 3.4] and [5, p. 372]).

In general, if $|\mathbb{K}| \neq ||E||$, the equality (14) does not hold, as the following example shows.

Example 3.12. Set a real r_0 such that $|\rho| < r_0 < 1$. Let $E = \left(c^0_{\mathbb{K}}(\mathbb{N}), ||.||'\right)$, where the norm $||.||'$ is defined by the formula

$$||\left(x^1, x^2, ...\right)||' = \max\left\{\left|x^1\right|, \left|x^2\right| \cdot r_0, \left|x^3\right| \cdot r_0, ...\right\}$$

Then, $M = \{e_2, e_3, ...\}$ is a bounded subset of E. We prove that $\gamma(M) = |\rho|$. First, note that for every $x \in M$ $||x|| = r_0$, thus, for every $z^* \in B_{E'}$ we get $|z^*(e_i)| \leq |\rho|$, $i = 2, 3, ...$; otherwise, assuming that $|z^*(e_j)| > |\rho|$ for some $j \in \{2, 3, ...\}$ and $z^* \in B_{E'}$ we get $|z^*(e_j)| \geq 1$, since $|\mathbb{K}| \cap (|\rho|, \infty) = \left\{1, |\rho|^{-1}, |\rho|^{-2}, ...\right\}$. Thus, $||x^*|| \geq \frac{|x^*(e_j)|}{||e_j||} \geq \frac{1}{r_0} > 1$, a contradiction. Hence, we get $\gamma(M) \leq |\rho|$.

Now, let (e^*_n) denotes the sequence of functionals such that $e^*_n(e_m) = \rho$ if $n = m$ and $e^*_n(e_m) = 0$ if $n \neq m$. For every $n \in \mathbb{N}$ define $z^*_n = e^*_1 + ... + e^*_n$; clearly, $(z^*_n) \in B_{E'}$. We obtain $\lim_m z^*_n(e_m) = 0$ for every $n \in \mathbb{N}$, and $\lim_n z^*_n(e_m) = \rho$ for every $m \in \{2, 3, ...\}$. Hence

$$\left|\lim_n \lim_m z^*_n(e_m) - \lim_m \lim_n z^*_n(e_m)\right| = |\rho|,$$

and we conclude that $\gamma(M) = |\rho|$.

On the other hand, $||x - y|| = r_0$ for any $x, y \in M$, $x \neq y$. This easily yields the following formula $\omega(M) = r_0$.

References

[1] C. Angosto, B. Cascales: A new look at compactness via distances to function spaces, in: Advanced Courses of Mathematical Analysis (La Rábida, 2007), J. M. Delgado Sánchez et al. (ed.), World Scientific, Hackensack (2008) 49–66.

[2] C. Angosto, B. Cascales: The quantitative difference between countable compactness and compactness, J. Math. Anal. Appl. 343 (2008) 479–491.

[3] C. Angosto, B. Cascales, I. Namioka: Distances to spaces of Baire one functions, Math. Z. 263 (2009) 103–124.

[4] C. Angosto, B. Cascales: Measures of weak noncompactness in Banach spaces, Topology Appl. 156 (2009) 1412–1421.

[5] K. Astala, H. O. Tylli: Seminorms related to weak compactness and to Tauberian operators, Math. Proc. Camb. Philos. Soc. 107(2) (1990) 367–375.

[6] B. Cascales, W. Marciszeski, M. Raja: Distance to spaces of continuous functions, Topology Appl. 153 (2006) 2303–2319.

[7] M. Fabian, P. Hájek, V. Montesinos, V. Zizler: A quantitative version of Krein's theorem, Rev. Mat. Iberoam. 21 (2005) 237–248.

[8] K. Floret: Weakly Compact Sets, Lecture Notes in Math. 801, Springer, Berlin (1980).

[9] A. S. Granero, P. Hájek, V. Montesinos: Convexity and w^*-compactness in Banach spaces, Math. Ann. 328 (2004) 625–631.

[10] A. S. Granero: An extension of the Krein-Smulian theorem, Rev. Mat. Iberoam. 22 (2006) 93–100.

[11] A. S. Granero, M. Sánchez: The class of universally Krein-Šmulian Banach spaces, Bull. Lond. Math. Soc. 39 (2007) 529–540.

[12] J. Kąkol, A. Kubzdela: Non-Archimedean quantitative Grothendieck and Krein's theorems, J. Convex Analysis 20 (2013) 233–242.

[13] C. Perez-Garcia, W. Shikhof: Locally Convex Spaces over Non-Archimedean Valued Fields, Cambridge Univ. Press, Cambridge (2010).

[14] A. C. M. van Rooij: Notes on p-adic Banach spaces, Report 7725, University of Nijmegen (1977).

[15] A. C. M. van Rooij: Non-Archimedean Functional Analysis, Marcel Dekker, New York (1978).

[16] W. Schikhof: On weakly precompact sets in non-Archimedean Banach spaces, Report 8645, University of Nijmegen (1986).

[17] W. Schikhof: The closed convex hull of a compact set in a non-Archimedean locally convex space, Report 8646, University of Nijmegen (1986).

[18] W. Schikhof: Compact-like sets in non-Archimedean functional analysis, in: p-adic Analysis (Houthalen, 1986), N. de Grande-de Kimpe et al. (ed.), Vrije Universiteit Brussel, Brussel (1986) 137–147.