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Chapter 4
The Density Character of the Space $C_p(X)$

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Abstract The main purpose of this survey is to introduce to the reader the adequate framework and motivation for the recent results obtained relating the density character and the space of the continuous functions, [16]. The interest in this cardinal function has been continuous over the years. We will offer a vision of the process along the time and we will point out different general results. Specially, we are interested in those in which the space of the continuous functions appears as well as those in which duality plays an important role. Of course, precise classes of spaces are considered in each case to apply the results, which will take us forward to expose a parallel development and description of a specific class, in fact it will be the development of a different cardinal function, the number of Nagami, which measures the specific property of the space what makes things work well.

Keywords Density character · Lindelöf Σ-space · Locally convex space · Space of the continuous functions · Weakly compact generated

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4.1 Introduction

Following Hödel [21, p. 10] a *cardinal function* Ψ is a function from the class of all topological spaces (or some precisely defined subclass) into the class of all cardinals such that $\Psi(X) = \Psi(Y)$ whenever X and Y are homeomorphic. Cardinal functions have been used to establish general topological properties. In some sense cardinal functions measure the cardinality of "good" topological properties.

The *density character* is a cardinal function whose definition can be found in early papers, see for example [18, p. 329].

Definition 4.1 Let X be a topological space. The *density character of* X, denoted by $d(X)$, is the least infinite cardinal number of a dense subset of X.

When the density character is countable we say that X is a *separable space*.

The interest in this property has been continuous over the years. Recently, in [16] a general result involving the space of the continuous functions with the pointwise convergence topology and the density character has been proved. Throughout this survey we will offer a general framework for the density character along the time. We will point out different general results, specially we are interested in those in which the space of the continuous functions appears as well as those in which duality plays an important role. Of course, precise classes of spaces are considered in each case to apply that results, which will take us forward to expose a parallel development of a description of a specific class, in fact it will be the development of a different cardinal function which measures the specific property of the space what makes things work well.

Proofs are included, in other case we indicate the reference where they can be found.

As we have already mentioned, classic results refer to the density character. B. Pospíšil pointed out in [33] that for a topological space X the cardinality of X, $|X|$, is bounded by $2^{2^{d(X)}}$.

About the stability properties we must mentioned that the density character is not stable respect to subspaces. A cardinal function ϕ is *monotone* if $\phi(Y) \leq \phi(X)$ for every subspace Y of X. After this definition we have that the density character is not monotone. An easy example is the following one, take $X = \beta\mathbb{N}$ (the Stone-Čech compactification of $\mathbb{N}$) which is separable while $X \backslash \mathbb{N}$ is not. Respect to the product we have the following theorem:

Theorem 4.1 ([19, 32]) *Let* $\mathfrak{m}$ *be an infinite cardinal number. Let* $X = \prod_{i \in I} X_i$ *where* X_i *is a topological space with* $d(X_i) \leq \mathfrak{m}$. *If* $|I| \leq 2^{\mathfrak{m}}$, *then* $d(X) \leq \mathfrak{m}$.

Throughout this chapter *topological space* means completely regular Hausdorff topological space. We use X, Y to denote topological spaces. Notation and terminology are standard, our basic references are [13, 15, 23–25, 39] and it will be provided as needed. The set of the continuous functions from X to Y is denoted by $C(X, Y)$, when $Y = \mathbb{R}$, the set is simply denoted by $C(X)$. $C_c(X)$ and $C_p(X)$ denote the space of the continuous functions endowed with the compact-open topology and pointwise convergence topology, respectively. The *weight* of X, $w(X)$, is the least cardinality of an open base of X. The *weak weight* of X, $iw(X)$, is the smallest infinite cardinal number $\mathfrak{m}$ such that there exists a continuous bijection from X onto a space Y with $w(Y) \leq \mathfrak{m}$. The density character of the space of the continuous functions was considered by N. Noble in [30]. In the following theorem the density character of the space of the continuous functions is related with the weak weight of X.

Theorem 4.2 ([30, Theorem 1]) *Let X be a topological space and let $C(X)$ be the set of the continuous functions on X endowed with any topology between the pointwise convergence topology and the compact-open topology. Then*

$$d(C(X)) = iw(X). \tag{4.1}$$

This result can also be found in [27, Theorem 4.2.1]. The proof that follows is obtained from this last reference.

Proof ([27, p. 53]) To prove $d(C(X)) = iw(X)$, it is enough to prove that

$$d(C_c(X)) \leq iw(X) \leq d(C_p(X)).$$

Let $\phi : X \to Y$ be a continuous bijection where $w(Y) = iw(X)$. Then the induced function $\phi^* : C_c(Y) \to C_c(X)$ defined as $\phi^*(f) := f \circ \phi$ gives us the following chain of inequalities

$$d(C_c(X)) \leq d(C_c(Y)) \leq nw(C_c(Y)) \leq w(Y) = iw(X),$$

where has been used that the density character is less than or equal to the network weight, see Definition 4.8.3. To see that $iw(X) \leq d(C_p(X))$ consider an infinite dense subset D of $C_p(X)$ having cardinality $d(C_p(X))$. Define $\psi : X \to \mathbb{R}^D$ by $\pi_f \circ \psi(x) = f(x)$ for each $x \in X$ and $f \in D$. Now ψ is a bijective continuous map since D separates points of X. Thus, $iw(X) \leq w(\psi(X)) \leq |D| = d(C_p(X))$. □

The next corollary follows immediately from Theorem 4.2.

Corollary 4.1 ([27, p. 54]) *The following statements are equivalent:*

1. *$C_c(X)$ is separable.*
2. *$C_p(X)$ is separable.*
3. *X has a coarser separable metrizable topology.*

4.2 The *Nice* Lindelöf-Σ Class

In order to get the adequate framework, in this section we point out the importance of a "nice" class of spaces that will give us the key to develop a general cardinal function that will be used in the results that involve, on one hand, the density character and, on the other hand, the space of the continuous functions.

Let us recall the definition of Lindelöf-Σ-space. Following Arkhangelskiĭ, [1, p. 6] the class of Lindelöf Σ-spaces is defined as

> *the smallest class of spaces containing all compacta, all spaces with a countable base, and closed under the following three operations: taking the product of two spaces, transition to a closed subspace, and transition to a continuous image.*

The class of Lindelöf-Σ-spaces has become to be very popular not only in Topology but also in Functional Analysis in which it is called the class of countably K-determined spaces. The class of Σ-spaces was introduced by Nagami in 1969, [29]. Later, the class of Σ-spaces with the Lindelöf property was considered. In the frame of Functional Analysis the class of K-analytic spaces was developed, see [10, 11, 17]. In fact, the class of K-analytic spaces was proved as an important class since M. Talagrand proved that each weakly compactly generated Banach space is K-analytic in the weak topology. M. Talagrand introduced K-analyticity in Banach spaces, see [35, 36], in order to find a *good* class of Banach spaces with the property of being Lindelöf for the weak topology. The more general class of countably K-determined spaces was developed, [40]. The related literature is full of references to the Lindelöf Σ-class (or equivalently, countably K-determined class), mainly because their good properties. The title of the survey of V. Tkachuck: *Lindelöf Σ-spaces: an omnipresent class* [37] is quite representative of this fact. Other reason for the presence of the Lindelöf Σ-class is that there are tools to use it in different ways. This means that the definition with which we began this section is descriptive but not too useful to work with it. We will show different characterizations of the notion of Lindelöf Σ-class.

In [10] a topological space is called *K-analytic* if it is the continuous image of a $K_{\sigma\delta}$ of a compact space. In fact, the notion of countably K-determined space was described as follows:

Definition 4.2 ([40, Definition 1]) Let Y be a topological space, $X \subset Y$ is *countably determined inY* if and only there are a countable family of compact sets $\{A_n : n \in \mathbb{N}\}$ in Y such that for any $x \in X$ there exists $J \subset \mathbb{N}$ such that

$$x \in \bigcap_{n \in J} A_n \subset X.$$

Nevertheless, M. Talagrand showed up the importance of using the notion of upper semicontinuous maps, [35].

Definition 4.3 Let X and Y be topological spaces. A multivalued map $\phi : X \to 2^Y$ is said to be *upper semicontinuous* in $x_0 \in X$ if $\phi(x_0)$ is not empty and for each open set V in Y with $\phi(x_0) \subset V$ there exists an open set U of x_0 in X such that $\phi(U) \subset V$. A multivalued map ϕ is said to be upper semicontinuous if it is upper semicontinuous for each point in X. We will say that a multivalued map $\phi : X \to 2^Y$ is *usco* if ϕ is upper semicontinuous and the set $\phi(x)$ is compact for each $x \in X$.

More about usco maps can be found in [7] and the references therein.

Now we can introduce the definition of K-analyticity in different terms.

Definition 4.4 A topological space is *K-analytic* if there exists an usco map from $\mathbb{N}^{\mathbb{N}}$ into 2^X covering X.

Using the notion of usco map, an equivalent definition for the class of countably K-determined can also be obtained.

Definition 4.5 ([35, Proposition 1.1]) A topological space is called *countably* K-determined is there exists a subspace $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and an usco map $\phi : \Sigma \to 2^X$ covering X.

And the notion of Lindelöf Σ-space can also be given in terms of usco maps.

Definition 4.6 The following conditions are equivalent for any topological space X:

1. X is a Lindelöf Σ-space.
2. There exist a second countable space M and an usco map $\phi : M \to 2^X$ such that $X = \bigcup\{\phi(x) : x \in M\}$.
   The previous definitions are equivalent to the following one.
3. There exist a compact cover $\mathscr{C}$ of the space X and a countable family $\mathscr{N}$ of subsets of X such that for any $C \in \mathscr{C}$ and any open set U such that $C \subset U$ there exists $N \in \mathscr{N}$ such that $C \subset N \subset U$.

In fact, there are many more equivalent definitions to the notion of Lindelöf Σ-space. In [37, Theorem 1] a summary of the different equivalences can be found.

The proposition that follows give us the key to see that the class of countably K-determined spaces and the class of Lindelöf Σ-spaces are the same class.

Proposition 4.1 *Let M be a metric space and $\mathfrak{m}$ an infinite cardinal number such that the weight of M is $\mathfrak{m}$, $w(M) = \mathfrak{m}$. Then there exist a set I with $|I| = \mathfrak{m}$, a subspace $\Sigma \subset I^{\mathbb{N}}$, where I is endowed with the discrete topology and a continuous onto map $f : \Sigma \to (M, d)$.*

Now, since the different equivalent definitions, the following inclusions are obvious.

K-analytic space $\Rightarrow$ Lindelöf Σ-space $\Rightarrow$ Lindelöf space.

We close this section with a well-known result of M. Talagrand that relates the density character and the weight (of a topology coarser than the original one) for a Lindelöf Σ-space, see [35, Theorem 2.4]. This result will be of great interest for their applications, as we will see later.

Theorem 4.3 ([35]) *Let (X, τ) be a Lindelöf Σ-space and τ' a topology coarser than τ, then*

$$d(X, \tau) \leq w(X, \tau').$$

4.3 Generalizing the Notion of Lindelöf Σ-Spaces

In the light of the results obtained with the class of Lindelöf Σ-spaces, the following step is trying to generalize this notion. In this sense, there are some cardinal functions which have been introduced with the idea of generalizing the notion of Lindelöf Σ-space. Some of them are the following:

Definition 4.7 Let X be a topological space.

1. The strong Σ-degree, $\Sigma(X)$, [20], is the smallest infinite cardinal number $\mathfrak{m}$ such that X has a strong Σ-net $\{\mathscr{F}_\alpha : \alpha \in A\}$ with $|A| = \mathfrak{m}$. A family $\{\mathscr{F}_\alpha : \alpha \in A\}$ of locally finite closed covers of a space X is a strong Σ-net for X if the following hold for each $x \in X$:
   a. $C(x) = \bigcap\{C(x, \mathscr{F}_\alpha) : \alpha \in A\}$ is compact;
   b. $\{C(x, \mathscr{F}_\alpha) : \alpha \in A\}$ is a base for $C(x)$ in the sense that given any open set U containing $C(x)$, there exists α in A such that $C(x, \mathscr{F}_\alpha) \subset U$,

   where $C(x, \mathscr{F}_\alpha) := \bigcap\{F \in \mathscr{F}_\alpha : x \in F\}$.
2. The $\ell\Sigma$-number of X, $\ell\Sigma(X)$, [6], is the smallest infinite cardinal number such that there exist a metric space M and an usco map $\phi : M \to 2^X$ covering X.
3. The Nagami number of X, $Nag(X)$, [6, 29], is the smallest infinite cardinal number $\mathfrak{m}$ such that there exist a topological space Y of weight $\mathfrak{m}$ and an usco map $\phi : Y \to 2^X$ covering X.
4. The index of $\mathscr{K}$-analyticity, $\ell K(X)$, [3] is the least infinite cardinal $\mathfrak{m}$ for which there exists a complete metric space M of weight $\mathfrak{m}$ and an usco map $\phi : M \to 2^X$ covering X.

5. The index of compact generation of X, $CG(X)$, [3] is the least infinite cardinal number $\mathfrak{m}$ such that there exists a family $\{K_\lambda : \lambda < \mathfrak{m}\}$ of compact subsets of X whose union is a dense subset of X.

Analyzing the previous definitions it is obvious that all of them are related in some sense. The following results show the relationships between them. Recall that for a topological space X, the *Lindelöf number*, $\ell(X)$, is the smallest infinite cardinal $\mathfrak{m}$ such that for any open cover of Y there exists a subcover of cardinality less than or equal to $\mathfrak{m}$.

Proposition 4.2 ([3, 6]) *Let X be a topological space. Then, the following inequalities hold*

$$\ell(X) \leq Nag(X) \leq \ell\Sigma(X) \leq \ell K(X) \leq CG(X).$$

Proposition 4.3 ([6]) *Let X be a topological space. Then,*

$$Nag(X) = \max\{\ell(X), \Sigma(X)\}.$$

4.4 The Cardinal Function *Nag*(*X*)

The cardinal functions $Nag(X)$ and $\ell\Sigma(X)$ generalize the notion of Lindelöf Σ-space, in the sense that if $Nag(X)$ or $\ell\Sigma(X)$ are countable then X is a Lindelöf Σ-space. Indeed, both cardinal functions (with some differences) measure the minimal number of compact sets we need to get a continuous cover of a topological space X. The cardinal function $Nag(X)$ is less than or equal to $\ell\Sigma(X)$, although they are in general different. In [6] an example in which both cardinal functions are not the same is showed. We know that the Lindelöf Σ-class has a good behavior with respect to the stability properties. The following natural question is if the generalization of this notion, in our case, the number of Nagami, verifies the same stability properties. We have that the answer is positive and the same occurs for the general case. The following proposition, that can be found in [6], give us the mentioned properties.

Proposition 4.4 1. *For any family* $(X_i)_{i\in I}$ *we have that*

$$Nag\left(\prod_{i\in I} X_i\right) \le \max\left\{|I|, \sup_{i\in I} Nag(X_i)\right\}.$$

2. *Let* X *be a topological space and* $Z \subset X$ *a closed subspace. Then* $Nag(Z) \le Nag(X)$.
3. *Let* X *and* Y *be topological spaces and* $\phi : X \to 2^Y$ *an usco map such that* $Y = \bigcup_{x\in X} \phi(x)$. *Then*
$$Nag(Y) \le Nag(X).$$
In particular, the statement is true for onto continuous functions.
4. *Let* $(X_i)_{i\in I}$ *a family of subspaces of* X, *then*

$$Nag\left(\bigcup_{i\in I} X_i\right) \le \max\left\{|I|, \sup_{i\in I} Nag(X_i)\right\}.$$

We must recall that we are interested in applying these cardinal functions to state some topological cardinal inequalities involving $C_p(X)$, that is, the space of the real-valued continuous functions endowed with the pointwise convergence topology. For that we need also other cardinal functions. For a topological space X, we consider the following ones.

Definition 4.8 Let X be a topological space.

1. The *tightness* of X, $t(X)$, is the smallest infinite cardinal $\mathfrak{m}$ such that for any set $A \subset X$ and any point $y \in \overline{A}$ there is a set $B \subset A$ for which $|B| \le \mathfrak{m}$ and $y \in \overline{B}$.
2. The *hereditarily density* of X, $hd(X)$, is the maximal cardinality of $\{d(Y) : Y \subset X\}$.
3. The *network weight* of X, $nw(X)$ is the minimal cardinality of a network in X (a network in a space X is a family S of subsets of the set X such that for any point $x \in X$ and any open set O_x such that $x \in O_x$ there is $P \in S$ such that $x \in P \subset O_x$.
4. The *Hewitt-Nachbin number* of X, $q(X)$, is the minimal cardinality $\mathfrak{m}$ such that X is $\mathfrak{m}$-placed in βX, that is for each $x \in \beta X \backslash X$ there is a set P of type $G_{\mathfrak{m}}$ (intersection of a family of open sets of cardinality less than or equal to $\mathfrak{m}$) such that $x \in P \subset \beta X \setminus X$. A Hewitt $\mathfrak{m}$-extension of X is the subspace $\upsilon_{\mathfrak{m}} X$ of βX consisting of all $x \in \beta X$ for which any set of type $G_{\mathfrak{m}}$ in βX containing x intersects X. If $\mathfrak{m} = \omega_0$ the set $\upsilon_{\mathfrak{m}} X$ is just the (Hewitt) realcompactification of X. When $q(X)$ is countable, X is called a realcompact space.

The following proposition generalizes the Theorem 4.3 of M. Talagrand.

Proposition 4.5 ([6]) *Let* (X, τ) *be a topological space and* τ' *a topology in* Y *coarser than* τ. *Then*

$$d(X, \tau) \le \max\{Nag(X, \tau), nw(X, \tau')\}. \tag{4.2}$$

The following result establishes the relationship between the network and the weak weight of a topological space through the number of Nagami.

Proposition 4.6 ([28]) *Let X and Y be topological spaces and $f : X \to Y$ a continuous bijection, then*

$$nw(X) \leq \max\{Nag(X), w(Y)\}. \tag{4.3}$$

Respect to the relationship between the number of Nagami and the number of Hewitt-Nachbin we have that the inclusions $X \subset \upsilon_\gamma X \subset \upsilon_\lambda X$ for $\lambda \leq \gamma$ are obvious, in fact, $\upsilon_\gamma X$ is C -embedded in $\upsilon_\lambda X$.

In particular, X is C-embedded in $\upsilon_\lambda X$ for every infinite cardinal number λ. We recall that a subspace Y of a space X is *C-embedded in X* if every real-valued continuous function on Y can be extended to a real-valued continuous function on X.

Observe also that if Y is a subspace of X which is C-embedded in X, then the restriction map, $\pi : C_p(X) \to C_p(Y)$, is a continuous onto map, hence

$$Nag(C_p(Y)) \leq Nag(C_p(X)).$$

The following proposition establishes different bounds which will be useful later, see [38, Theorem 1].

Proposition 4.7 *Let X be a topological space. The following inequalities hold.*

1. $q(X) \leq d(C_p(X)) \leq hd(C_p(X))$.
2. $q(X) \leq Nag(X)$.

4.5 The Density Character and the Duality

In previous sections we have shown some general results about the density character and its relationship with other cardinal functions. In this section all our efforts are directed in attempting to motivate the recent results for the space of the continuous functions given in [16]. For that, we will show some results that involve not only the density character but also the duality. In particular, we are interested in those results in which we can obtain information about the density character of a space and the density character of the dual.

Additional terminology is required. All topological vector spaces and all locally convex spaces are assumed over the real field. If E is a topological vector space, the space E' denotes the set of all continuous linear functionals on E. Let F be a subset of E', we write $\sigma(E, F)$ to denote the locally convex topology on X of pointwise convergence on F. The topology $\sigma(E, E')$ is the weak topology of E and $\sigma(E', E)$ is the weak* topology of E'. The locally convex topology on E of uniform convergence of all $\sigma(E', E)$-compact convex balanced sets in E' is called the Mackey topology on E and denoted by $\mu(E, E')$.

4.5.1 The Density Character for WCG

D. Amir and J. Lindenstrauss [2, 26] studied the structure of the class of weakly compact sets in Banach spaces. In this section we consider a well-known result for the class of weakly compactly generated spaces.

Definition 4.9 A locally convex space E is *weakly compactly generated* (WCG) if there exists an absolutely convex weakly compact total subset of E.

Recall that a subset of a locally convex space E is *total* if its closed linear span is equal to E. Subspaces of WCG spaces behave badly. Rosenthal [34], has given an example of a closed linear subspace of a WCG Banach space which is not WCG.

The following result can be found in [26, Proposition 2.2] in a slightly different terminology. The proof follows the steps described in the book [15, Theorem 11.3].

Proposition 4.8 ([26, pp. 239–240]) *Let E be a weakly compactly generated* (*WCG*) *Banach space then,*

$$d(E) \leq d(E', \sigma(E', E)).$$

Proof [15] Let K be a weakly compact convex subset of E that generates E, and let D be a set of cardinality $d(E', \sigma(E', E))$ that is $\sigma(E', E)$-dense in E'.

Define an operator $T : E' \to C(K)$ by $T(f) : K \to \mathbb{R}$ given by $T(f)(k) = f(k)$ for each $k \in K$.

Since $T(D)$ separates points of K, the algebra generated by $T(D)$ and the constant functions contains a dense set N of cardinality $d(E', \sigma(E', E)) = |T(D)|$ that is, by the Stone-Weierstrass Theorem, dense in $C(K)$.

For every $g \in N$, choose $k_g \in K$ such that $g(k_g) = \sup_{k \in K} g(k)$. We claim that $S = \{k_g : g \in N\}$ is $\sigma(E, E')$ dense in K. Indeed, we have that $\sup_K(\tilde{g}) = \sup_{g \in N}(\tilde{g}(k_g))$ for every $\tilde{g} \in C(K)$ since otherwise for $g \in N$ sufficiently close in $C(K)$ to $\tilde{g}$ we can not have that $\sup_{k \in K}(g) = g(k_g)$.

Therefore, S is weakly dense in K by Urysohn's Theorem, so *span*(S) is $\sigma(E, E')$-dense in E. Since *span*(S) is weakly and thus norm dense in E, we have that $d(E) \leq d(E', \sigma(E', E))$. □

The inequality $d(E) \geq d(E', \sigma(E', E))$ also holds. Let K be a weakly compact set generating E, and let D be a set of cardinality $d(E', \sigma(E', E))$ that is $\sigma(E', E)$-dense in E'. The topology τ of pointwise convergence on D coincides with the weak topology of K because K is $\sigma(E, E')$-compact. The topology τ has a base of cardinality $|D|$ and thus there is a set S of cardinality $|D|$ which is $\sigma(E, E')$ dense in K. Therefore, according to [12, Theorem 13, p. 422], $d(E, \sigma(E, E')) \leq |D|$.

4.5.2 The Density Character for Weakly Compactly Generated Locally Convex Spaces

R. J. Hunter and J. Lloyd considered in [22] the class of *weakly compactly locally convex spaces*, denoted cWCG. Let's see the definition.

Definition 4.10 A locally convex space E is *countably weakly compactly generated* if there exists a sequence of absolutely convex weakly compact subsets of E whose union is total (or equivalently, dense) in E.

Every separable locally convex space is cWCG. If E is a Fréchet space, then $(E', \sigma(E', E))$ is cWCG. In general, WCG and cWCG spaces are different. Each cWCG Fréchet space is a WCG space, [22, Proposition 1.1], but $\mathbb{R}^{(\mathbb{N})}$, which denotes the linear subspace of $\mathbb{R}^{\mathbb{N}}$ consisting of all maps taking all but finitely many elements of $\mathbb{N}$ equal to zero is cWCG but not WCG, see [22].

The following proposition [22, Proposition 1.2], give us the key to prove a result in which the density character for this class is involved.

Proposition 4.9 *Let E be a locally convex space. Then E is cWCG if and only if there exists a metrizable locally convex topology τ on E', which is coarser than $\mu(E', E)$.*

Now, a theorem for this class involving the density character and duality follows, extending the result of J. Lindenstrauss to the class of cWCG locally convex spaces.

Theorem 4.4 *Let E be a cWCG locally convex space. Then*

$$d(E) \leq d(E', \sigma(E', E)).$$

Proof Let E be a cWCG locally convex space. By Proposition 4.9, there exists a metrizable locally convex topology τ on E' coarser than $\mu(E', E)$. The topological dual G of (E', τ) is contained in E and $\sigma(E', G)$ is a Hausdorff topology on E' which is coarser than $\sigma(E', E)$. Let D be a $\sigma(E', E)$-dense subset of E' of cardinality equal to $\mathfrak{m}$. Since (E', τ) is metrizable, G has a $\sigma(G, E')$-dense subset of cardinality equal to $\mathfrak{m}$. But G is $\sigma(E, E')$-dense in E because G separates points of E'. Thus E has a $\sigma(E, E')$-dense subset of cardinality $\mathfrak{m}$ and the proof is over. □

4.5.3 The Density Character for Weakly Countably K-Determined Spaces

A Banach space E is said to be *weakly countably K-determined* if E endowed with the weak topology is countably K-determined (or equivalently a Lindelöf Σ-space). We have used in the title of this subsection the notation *countably K-determined* instead of *Lindelöf Σ-class* in order to be coherent with the original notation since the following result was established in these terms by L. Vašák [40, Corollary 2].

Theorem 4.5 *Let E be a weakly countably K-determined space, then*

$$d(E) = d(E', \sigma(E', E)).$$

The same theorem was proved by M. Talagrand [35, Théorème 6.1] in a different way, as an application of Theorem 4.3, with τ the weak topology and τ' the poinwise convergence topology on a weak*-dense set of the dual.

4.5.4 The Density Character for K-Analytic Locally Convex Spaces

As we have mentioned, M. Talagrand introduced K-analyticity in Banach spaces, see [35, 36], in order to find a *good* class of Banach spaces with the property of being Lindelöf for the weak topology and countably K-determined.

M. Talagrand has proved that a WCG Banach space is weakly K-analytic and has constructed an example of type $C(K)$ to show that both cases are not the same (this is a consequence of the fact that closed subspaces of WCG Banach spaces can fail to be WCG, which was proved previously by Rosenthal [34]).

Now we can introduce a theorem that involves the density character for this class of spaces.

Theorem 4.6 *Let E be a weakly K-analytic locally convex space. Then the following statements follow:*

1. $d(E) \leq d(E', \sigma(E', E))$.
2. *If E is metrizable,* $d(E) = d(E', \sigma(E', E))$.

Proof 1. Let D be an infinite $\sigma(E', E)$-dense subset of E' of cardinality $\mathfrak{m}$. Now the topology $\sigma(E, D)$ is a Hausdorff topology on E coarser than $\sigma(E, E')$, which admits a fundamental system of zero neighborhoods of cardinality $\mathfrak{m}$. Since $(E, \sigma(E', E))$ is K-analytic, $(E, \sigma(E, E'))$ is Lindelöf, hence $(E, \sigma(E, D))$ is Lindelöf too and it admits a base of open sets of cardinality $\mathfrak{m}$. Now using Theorem 4.3, we have that $d(E, \sigma(E, E')) \leq \mathfrak{m}$.

2. Let E be a dense subset of cardinality $\mathfrak{m}$ and K be a $\sigma(E', E)$-compact subset of E', then K admits a base of uniformity of cardinality $\mathfrak{m}$ for the $\sigma(E', E)$ topology, and therefore a base of open sets of cardinality $\mathfrak{m}$. Then, K has a dense subset of cardinality $\mathfrak{m}$. Now the result follows because E' is a countable union of $\sigma(E', E)$-compact subsets. □

4.5.5 The Density Character for the Class $\mathfrak{G}$

B. Cascales and J. Orihuela have introduced in [8] a large class of locally convex spaces that it is called the class $\mathfrak{G}$. The definition of this class is as follows.

Definition 4.11 A locally convex space E is said to be in the *class* $\mathfrak{G}$ if there exists a family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of E' (called a $\mathfrak{G}$-representation of E) such that

1. $E' = \bigcup\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}$.
2. $A_\alpha \subset A_\beta$ if $\alpha \leq \beta$.
3. In each A_α all sequences are equicontinuous.

Condition 3 implies that every A_α is $\sigma(E', E)$-relatively countably compact. Therefore, if E is in the class $\mathfrak{G}$, the space $(E', \sigma(E', E))$ has a relatively countably compact resolution. The class $\mathfrak{G}$ contains the metrizable and dual metric space and it is stable by taking subspaces, separated quotients, completions, countable direct sums and countable products. A good reference in which a completed description and more results about the class $\mathfrak{G}$ can be found is [24].

Theorem 4.7 ([8, Theorem 13]) *Let E be a locally convex space of the class $\mathfrak{G}$ which is weakly K-countably determined. Then,*

$$d(E, \sigma(E, E')) = d(E', \sigma(E', E)).$$

Proof Let D be a dense subset in $(E, \sigma(E, E'))$. By [31, Theorem 6] the space $(E', \sigma(E', E))$ is angelic. Now, the space $(E', \sigma(E', E))$ is K-analytic using the $\mathfrak{G}$-representation of E in E'. Let $\sigma(E', D)$ be the topology on E' of pointwise convergence on D. Applying Theorem 4.3, we obtain that $d(E', \sigma(E', E))$ is less than or equal to $|D|$.

Conversely, if D is a dense subset of $(E', \sigma(E', E))$, using the topology $\sigma(E, D)$ on E and since $(E, \sigma(E, E'))$ is countably K-determined, again using Theorem 4.3, we obtain that $d(E, \sigma(E, E'))$ is less than or equal to the cardinality of D and we are done. □

4.6 The Density Character for $C_p(X)$

A Banach space $C(K)$ of continuous functions on a compact space K is WCG in and only if K is an *Eberlein compact*, (i.e. if and only if K is homeomorphic to a weakly compact set in some $c_0(\Gamma)$), [2]. On the other hand, every closed linear subspace Y of a WCG Banach space is a Lindelöf Σ-space in the weak topology of Y, [14, Proposition 7.1.6]. Recall that a compact space X is said to be a *Gul'ko compact* if $C_p(X)$ is a Lindelöf Σ-space. According to Theorem [14, 7.1.8] a compact space is Gul'ko compact if and only if the Banach space $C(X)$ is weakly Lindelöf Σ-space.

Theorem 4.8 *If X and $C_p(X)$ are Lindelöf Σ-spaces, then $d(X) = d(C_p(X))$.*

Proof If $L_p(X)$ denotes the weak* dual of $C_p(X)$, Theorem 4.3 asserts that

$$d(C_p(X)) \leq d(L_p(X)).$$

If D is a dense subspace of X, the rational span H of D is dense in $L_p(X)$. Since $|H| = |D|$, it follows that $d(L_p(X)) \leq d(X)$. On the other hand, if $\mathscr{F}$ is a family of real-valued functions dense in $C_p(X)$, let η be the weakest topology on X that makes continuous each $f \in \mathscr{F}$. Since η is coarser than the original topology of X, Theorem 4.3 ensures that $d(X) \leq w(X, \eta) = |\mathscr{F}|$. So $d(X) \leq d(C_p(X))$ and we are done.

In particular, if K is a Gul'ko compact space, then

$$d(K) = d(C_p(K)) = iw(C_p(K)),$$

by Theorem 4.2. This implies in particular that there exists on $C(K)$ a weaker completely regular topology τ' such that $w(C(K), \tau') = d(C_p(K)) = d(K)$.

All these results give us reasons to think about establishing a general result involving the density character and the space of the continuous functions. Using the number of Nagami, a version of the previous result (relating duality and density character) is provided in [16] for spaces $C_p(X)$. The main theorem is as follows.

Theorem 4.9 ([16]) *Let X be a topological space and $L \subset C_p(X)$. Then there exist a space Y and completely regular Hausdorff topologies $\tau' \leq \tau$ for Y such that:*

1. $Nag(Y, \tau) \leq Nag(X)$, $\ell\Sigma(Y, \tau) \leq \ell\Sigma(X)$, $d(Y, \tau) \leq d(X)$.
2. $w(Y, \tau') \leq d(L)$.
3. $nw(C_p(Y, \tau)) = nw(Y, \tau) \leq \max\{Nag(X), d(L)\}$, *hence*

$$d(Y, \tau) \leq \max\{Nag(X), d(L)\},$$

$$d(C_p(Y, \tau)) \leq \max\{Nag(X), d(L)\}.$$

4. *L is embedded into $C_p(Y, \tau)$.*
5. $d(L) \leq \max\{Nag(L), d(Y, \tau)\}$.

Proof Let D be a dense subset of L, such that $|D| = d(L)$. Let $\mathscr{T}_D$ and $\mathscr{T}_L$ be the weakest topologies on X that make continuous all those real-valued functions that belong to D or L, respectively. By density, $f(x) = f(y)$ for each $f \in D$ implies $f(x) = f(y)$ for each $f \in L$. Let $(\widehat{X}, \widehat{\mathscr{T}_D})$ and $(\widehat{X}, \widehat{\mathscr{T}_L})$ be the topological quotients of $(X, \mathscr{T}_D)$ and $(X, \mathscr{T}_L)$ with respect to the relation $x \sim y$ if and only if $f(x) = f(y)$ for all f of D and L, respectively. If we define the map $F : (X, \mathscr{T}_D) \to \mathbb{R}^D$ by $F(z) = \rho_z$, where $\rho_z(f) = f(z)$ for all $f \in D$, then clearly F is continuous and $x \sim y$ if and only if $F(x) = F(y)$. Hence, $(\widehat{X}, \widehat{\mathscr{T}_D})$ is homeomorphic to a subspace of $\mathbb{R}^D$ and consequently

$$w(\widehat{X}, \widehat{\mathscr{T}_D}) \leq w(\mathbb{R}^D) \leq |D| = d(L).$$

On the other hand, since $(\widehat{X}, \widehat{\mathscr{T}_L})$ is a continuous image of X, we note that

$$Nag(\widehat{X}, \widehat{\mathscr{T}_L}) \leq Nag(X),\ \ \ell\Sigma(\widehat{X}, \widehat{\mathscr{T}_L}) \leq \ell\Sigma(X),\ \ d(\widehat{X}, \widehat{\mathscr{T}_L}) \leq d(X),$$

see [6, Proposition 7 (iv), Remark 8] and [13, Theorem 1.4.10]. Now, applying Proposition 4.6 we have that

$$nw(\widehat{X}, \widehat{\mathscr{T}_L}) \leq \max\{Nag(\widehat{X}, \widehat{\mathscr{T}_L}), w(\widehat{X}, \widehat{\mathscr{T}_D})\} \leq \max\{Nag(X), d(L)\}.$$

On the other hand, by [1, Theorem I.1.3] we have

$$nw(C_p(\widehat{X}, \widehat{\mathscr{T}_L})) = nw(\widehat{X}, \widehat{\mathscr{T}_L}).$$

In particular, as the density is less than or equal to the network weight we have that

$$d(\widehat{X}, \widehat{\mathscr{T}_L}) \leq \max\{Nag(X), d(L)\},$$
$$d(C_p(\widehat{X}, \widehat{\mathscr{T}_L})) \leq \max\{Nag(X), d(L)\}.$$

Therefore $(Y, \tau) := (\widehat{X}, \widehat{\mathscr{T}_L})$ is a completely regular Hausdorff space such that

$$d(Y, \tau) \leq \max\{Nag(X), d(L)\}$$

and

$$d(C_p(Y, \tau)) \leq \max\{Nag(X), d(L)\}.$$

Now set $(Y, \tau') := (\widehat{X}, \widehat{\mathscr{T}_D})$. Clearly $\tau' \leq \tau$ on Y.

Recall that

$$\widehat{x} := \{y \in X : f(y) = f(x) \text{ for all } f \in L\},$$

define $T : L \to C_p(Y, \tau)$ by $T(f) = \widehat{f}$, where $\widehat{f}(\widehat{x}) := f(x)$. Note that if $y \in \widehat{x}$, then

$$\widehat{f}(\widehat{y}) = f(y) = f(x) = \widehat{f}(\widehat{x}),$$

since $f \in L$, so that $\widehat{f}$ is well-defined. On the other hand, $\widehat{f} \in C_p(Y, \tau)$. Indeed, if $\widehat{x_d} \to \widehat{x}$ in (Y, τ), then

$$\widehat{f}(\widehat{x_d}) = f(x_d) \to f(x) = \widehat{f}(\widehat{x}).$$

If $\widehat{f}(\widehat{x}) = \widehat{g}(\widehat{x})$ for all $x \in X$ then $f(x) = g(x)$ for all $x \in X$, which implies that $f = g$ and consequently that $\widehat{f} = \widehat{g}$, so that T is injective. Moreover, if $f_p \to f$ in L then $f_p(x) \to f(x)$ for every $x \in X$, which implies that $\widehat{f_p}(\widehat{x}) \to \widehat{f}(\widehat{x})$. Hence, T is continuous. Finally, if $\widehat{f_p} \to \widehat{f}$ in $T(L)$ under the pointwise convergence topology, then $\widehat{f_p}(\widehat{x}) \to \widehat{f}(\widehat{x})$ and hence $f_p(x) \to f(x)$ for all $x \in X$. This shows that T is a homeomorphism from L into $C_p(Y, \tau)$.

Lastly, let B be a dense subset of (Y, τ) of cardinality $d(Y, \tau)$. The restriction map

$$j : C_p(Y, \tau) \to C_p(B)$$

is bijective and continuous. Since $w(C_p(B)) = |B|$, we deduce that $C_p(Y, \tau)$ admits a weaker Hausdorff topology ξ such that $w(C(Y, \tau), \xi) \le |B|$. Hence, L admits a weaker topology ξ_L such that $w(L, \xi_L) \le |B|$. By Proposition 4.5 we get that

$$d(L) \le \max\{Nag(L), w(L, \xi_L)\} \le \max\{Nag(L), |B|\}. \qquad \square$$

From statement 5 of Theorem 4.9, if L is a Lindelöf Σ-space and $Nag(X) \le d(L)$ (this holds for example if X is a Lindelöf Σ-space), then

$$d(L) = d(Y, \tau).$$

Consequently, if X is a Lindelöf Σ-space and $Nag(C_p(X)) \le d(X)$, since X is embedded into $C_p\left(C_p(X)\right)$ there exists a space Y admitting a weaker completely regular Hausdorff topology τ' such that X embeds into $C_p(Y)$ and

$$d(X) = d(Y) \ge w(Y, \tau').$$

The following corollary, [16, Corollary 1], provides bounds considering extensions of the space X.

Corollary 4.2 ([16]) *Let X be a topological space and $\mathfrak{m}$ an infinite cardinal number with $\aleph_0 \le \mathfrak{m} \le q(X)$. Let $L \subset C_p(X)$ be a subspace of $C_p(X)$ such that $d(L) = \mathfrak{m}$. Then there exist a space Y and completely regular Hausdorff topologies τ and τ' on Y such that $\tau' \le \tau$ and:*

1. $Nag(Y, \tau) \le \min\{Nag(\upsilon_\lambda X) : \lambda \ge \mathfrak{m}\} \le Nag(\upsilon_{\mathfrak{m}} X)$,
   $\ell\Sigma(Y, \tau) \le \min\{\ell\Sigma(\upsilon_\lambda X) : \lambda \ge \mathfrak{m}\} \le \ell\Sigma(\upsilon_{\mathfrak{m}} X)$.
2. $w(Y, \tau') \le d(L)$.
3. $d(Y, \tau) \le \min\{Nag(\upsilon_\lambda X) : \lambda \ge \mathfrak{m}\} \le Nag(\upsilon_{\mathfrak{m}} X)$.
4. *L is embedded into $C_p(Y, \tau)$.*

Particular results are obtained simply applying the results to the countable case, see [16]. In order to establish a clear relationship with the previous results we point out some applications to the case of locally convex spaces. The following applications can be found in [16].

Corollary 4.3 ([9]) *A $\sigma(E, E')$-compact set Y in a lcs E from the class $\mathfrak{G}$ is $\sigma(E, E')$-metrizable if and only if Y is contained in a $\sigma(E, E')$-separable subset of E.*

As we have been noted in [16], Theorem 4.9 applies to show Theorem 4.7.

Corollary 4.4 *Let E be a lcs from the class $\mathfrak{G}$ such that $(E, \sigma(E, E'))$ is a Lindelöf Σ-space, then $d(E, \sigma(E, E')) = d(E', \sigma(E', E))$.*

Proof Since E belongs to $\mathfrak{G}$ and $L := (E, \sigma(E, E'))$ is a Lindelöf Σ-space, the space $X := (E', \sigma(E', E))$ is K-analytic by applying [31, Theorem 21] and [5, Corollary 1.1]. Clearly, $L \subset C_p(X)$, so by Theorem 4.9 (5,1) there exists Y such that $d(L) = d(Y) \leq d(X)$. Note that $d(X) \leq d(L)$ also holds. Indeed, since $X \subset C_p(C_p(X))$, by Theorem 4.9 (5,1) there exists Z such that

$$d(X) \leq \max\{Nag(X), d(Z)\} = d(Z) \text{ and } d(Z) \leq d(C_p(X)).$$

On the other hand, $d(C_p(X)) = iw(X)$, Theorem 4.2. If B is a dense subset of L, then $\sigma(E', B) \leq \sigma(E', E)$, so $iw(X) \leq d(L)$. Hence, $d(X) \leq d(L)$. □

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