Chapter 5
Compactness and Distances to Spaces of Continuous Functions and Fréchet Spaces

Carlos Angosto and Manuel López-Pellicer

Abstract In recent years, several *quantitative* counterparts for several classical such as Krein-Šmulyan, Eberlein-Šmulyan, Grothendieck, etc. have been proved by different authors. These new versions strengthen the original theorems and lead to new problems and applications in topology and analysis. In this survey, we present several of these quantitative versions of theorems about compactness in Banach spaces with the weak topology, Fréchet spaces with the weak topology and spaces of continuous functions with the pointwise convergence topology. For example if H is a subset of a Banach space E, and w^* is the weak* topology in E'', the index $k(H) := \sup\{d(x^{**}, E), x^{**} \in \overline{H}^{w^*}\}$ is zero if and only if H is relatively compact in (E, w). Then $k(H)$ measures how far is H from being relatively compact in (E, w). The following inequalty $k(co(H)) \leq 2k(H)$ is a quantitative version of the Krein-Šmulian theorem about the w-relative compactness of the convex hull of a weakly compact set.

Keywords Compactness · Fréchet space · Space of continuous functions · Webcompact · Distances

MSC 2010 Classification: 46A50, 54C35

C. Angosto (✉)
Departamento de Matemática Aplicada y Estadistica, Universidad Politécnica de Cartagena, 30203 Cartagena, Spain
e-mail: carlos.angosto@upct.es

M. López-Pellicer
Departamento de Matemática Aplicada and IMPA, Universitat Politècnica de València, 46022 Valencia, Spain
e-mail: mlopezpe@mat.upv.es

J. C. Ferrando and M. López-Pellicer (eds.), *Descriptive Topology and Functional Analysis*, Springer Proceedings in Mathematics & Statistics 80, DOI: 10.1007/978-3-319-05224-3_5,

5.1 Preliminary Results

Recall that a subset H of a topological space X is countably compact (resp. relatively countably compact) if every sequence in H has a cluster point in H (resp. in X) and H is sequentially compact (resp. relatively sequentially compact) if every sequence in H has a convergent subsequence with limit in H (resp. in X). Clearly a (relatively) compact set is (relatively) countably compact and a (relatively) sequentially compact set is (relatively) countably compact, but in general other implications do not hold.

Šmulian showed in 1940 [18] that if H is a relatively compact set in the weak topology of a Banach space E, then H is also relatively sequentially compact and he also proved that the converse implication is also true if the weak* dual is separable. Eberlein [8] proved in 1947 that the converse to Šmulian's theorem is true, i.e., for the weak topology of a Banach space a subset is relatively compact if and only if it is relatively countably compact. Grothendieck proved in 1952 [12] that this result also holds in any locally convex space that is quasicomplete for its Mackey topology. Grothendieck's result is based upon a similar result on spaces $C_p(K)$ of continuous functions on a compact set with the pointwise convergence topology. Kelley and Namioka [14] pointed out that a cluster point of a relatively countably compact subset of $C_p(K)$ is the limit of a sequence in H. J. D. Pryce [17] extended the Eberlein-Šmulian theorem to spaces $C_p(X)$ where X has a dense subset that is the countable union of compact sets. A very useful tool that was used in the previous results was the following notion that was introduced by Grothendieck.

Definition 5.1 Let Z be a topological space, let X be a set and $H \subset Z^X$. It is said that H *interchanges limits with* X if for each sequences $(f_n) \subset H$ and $(x_m) \subset X$

$$\lim_n \lim_m f_n(x_m) = \lim_m \lim_n f_n(x_m)$$

whenever all the limits involved exist.

If $H \subset C(X, Z)$ where X is a countably compact space and Z a compact metric space, then H is relatively compact in the pointwise convergence topology if and only if H interchanges limits with X (X can be replaced by a dense subset of X). The interchange of limits property is also very useful in locally convex spaces. For example a bounded subset of a Banach space is relatively compact in the weak topology of a Banach space if it interchanges limits with the unit dual ball.

Fremlin's notion of angelic space and some of its consequences can be used for proving the above results in a clever and clear way, see the book by Floret [10].

Definition 5.2 A topological space is called *angelic* if for every relatively countably compact set $H \subset X$ the following holds:

- H is relatively compact;
- for each $x \in \overline{H}$ there is a sequence in H which converges to x.

In angelic spaces, a subset is (relatively) compact if and only if it is (relatively) countably compact if and only if it is (relatively) sequentially compact. Orihuela [15] introduced the following class of topological spaces:

Definition 5.3 A topological space X is a *web-compact space* if there is a subset Σ of $\mathbb{N}^{\mathbb{N}}$, a family $\{A_\alpha : \alpha \in \Sigma\}$ of subsets of X whose union D is dense in X and if we denote

$$C_{n_1,\ldots,n_k} := \bigcup\{A_\beta : \beta = (m_k) \in \Sigma, m_j = n_j, j = 1, \ldots, k\}$$

for $\alpha = (n_k) \in \Sigma$, then if $\alpha = (n_k) \in \Sigma$ and $x_k \in C_{n_1,n_2,\ldots,n_k}$, the sequence $(x_k)_k$ has a cluster point in X. If X is web-compact with $D = X$, we call X *strongly web-compact*.

This definition is very technical but includes a big class of spaces, for example spaces with a dense union of compact sets (in particular separable spaces), countably determined topological spaces or quasi-Suslin spaces are web-compact spaces.

Theorem 5.1 (Orihulela [15]) *Let X be a web-compact space. The space $C_p(X)$ is angelic.*

The main tool that Orihule used to prove the previous theorem was the following result.

Theorem 5.2 (Orihulela [15]) *Let (Z, d) be a compact metric space, X a set and H a subset of the space (Z^X, τ_p). We assume that:*

(i) There is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha : \alpha \in \Sigma\}$ of non-void subsets of the set X such that $X = \bigcup\{A_\alpha : \alpha \in \Sigma\}$.
(ii) For every $\alpha = (a_1, a_2, \ldots) \in \Sigma$ the set H interchanges limits in Z with every sequence $(x_n)_n$ in X that is eventually in each set

$$C_{\alpha|m} = \bigcup\{A_\beta : \beta \in \Sigma, b = (b_m) \text{ and } b_j = a_j \text{ for } j = 1, 2, \ldots, m\},$$

for $m \in \mathbb{N}$.

Then for any $f \in \overline{H}^{Z^X}$ there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in H that converges pointwise to f on X.

Proof It follows from Lemma 5.4 □

In recent years, several *quantitative* counterparts for the previous and other classical results have been proved by different authors. These new versions strengthen the original theorems and lead to new problems and applications in topology and analysis. In this survey, we present several of that quantitative versions.

5.2 Distance to Banach Spaces

If H is a bounded subset of a Banach space E, then H is relatively compact in the weak topology if and only if when we consider H as a subset of the bidual space E'' with the weak* topology, the weak* closure $\overline{H}^{w*}$ is a subset of E. Then we can think that the index

$$k(H) := \hat{d}(\overline{H}^{w^*}, E) = \sup\{d(x^{**}, E), x^{**} \in \overline{H}^{w^*}\}$$

measures how far is H from being w-relatively compact. Fabian, Hajek, Montesinos and Zizler used this notion and proved the following quantitative version of the Krein theorem. The second part of the theorem was obtained by Granero.

Theorem 5.3 [9, 11] *Let H be a bounded subset of a Banach space E, then*

$$k(co(H)) \leq 2k(H).$$

If H is a bounded subset of the bidual space E'' then

$$k(co(H)) \leq 5k(H).$$

Observe that if H is w-relatively compact then $k(H) = 0$ and then by the previous theorem $k(co(H)) = 0$ which means that the convex hull of H is also relatively compact in (E, w). One tool that have been used in the proof of previous theorem is the ε-*interchange of limits* (quantitative version of Grothendieck notion when $\varepsilon = 0$). But this notion was previous used in [5] with the following index:

$$\gamma(H) := \sup\{|\lim_m \lim_n f_m(x_n) - \lim_n \lim_m f_m(x_n)| : (f_m)_m \subset B_{E'}, (x_n)_n \subset H\}$$

assuming that the iterated limit exist. Observe that $\gamma(H) = 0$ if and only if H interchanges limits with the unit ball. In the definition of the index used in [5], the supremum is taked for $(x_n)_n \subset co(H)$ instead of H. Fabian et al. [9] also proved that the definition that we are using here is the same that the definition that appears in [5], i.e. $\gamma(H) = \gamma(co(H))$. This last result combined with the following inequalities

$$k(H) \leq \gamma(H) \leq 2k(H).$$

implies first part of Theorem 5.3.

We shall also consider the following index

$$ck(H) := \sup\{d(clust_{E''}(\phi), E), \phi \in H^{\mathbb{N}}\}$$

where $clust_{E''}(\phi)$ denotes the set of cluseter points of ϕ in E'' and $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. It is clear that if H is relatively weakly countably compact then $ck(H) = 0$. The converse implication also holds by the following result.

Theorem 5.4 [2] *If H is a bounded subset of a Banach space E then*

$$ck(H) \leq k(H) \leq \gamma(H) \leq 2ck(H).$$

For any $x^{**} \in \overline{H}^{w*}$ there is a sequence (x_n) in H such that

$$\|x^{**} - y^{**}\| \leq \gamma(H)$$

for any cluster point y^{**} of (x_n) in E''.

Corollary 5.1 *If E is a Banach space then (E, w) is angelic.*

Proof Let H be a w-relatively countably compact subset of E. By the very definition every sequence in H has a w-cluster point in E and therefore $ck(H) = 0$. Then by Theorem 5.4 we have H is w-relatively compact in E. On the other hand, let us pick $x \in \overline{H}^{w}$. Note that then $\gamma(H) = 0$ and thus if we use Theorem 5.4 we obtain the existence of a sequence (x_n) in H such that every w-cluster point $y \in E$ of (x_n) satisfies that $0 \leq \|y - x\| \leq \gamma(H) = 0$. Since H is w-relatively compact and (x_n) in H and x is the unique w-cluster point of (x_n) we conclude that (x_n) weakly converges to x and the proof is over. □

Corollary 5.2 [2] *If H is a bounded subset of a Banach space E, the following conditions are equivalent:*

1. *$ck(H) = 0$.*
2. *$k(H) = 0$.*
3. *H is weakly relatively countably compact.*
4. *H is weakly relatively compact.*

In Theorem 5.4 the involved constants are sharp but sometimes we can obtain some equalities. Recall that a Banach space E is said to have Corson property $\mathscr{C}$ if each collection of closed convex subsets of E with empty intersection has a countable subcollection with empty intersection. If (E, w) is Lindelöf, then E has property $\mathscr{C}$. There are Banach spaces with Corson property $\mathscr{C}$ which are not weakly Lindelöf, [16, p. 146]. It is shown in [16] that a Banach space E has the property $\mathscr{C}$ if and only if, whenever $A \subset E'$ and $x^* \in \overline{A}^{w^*}$, there is a countable subset C of A such that $x^* \in \overline{convC}^{w^*}$. In particular Banach spaces with w^* angelic dual unit balls have Corson property $\mathscr{C}$.

Proposition 5.1 [2] *If E is a Banach space with Corson property $\mathscr{C}$, then for every bounded set $H \subset E$ we have $ck(H) = k(H)$.*

Several of the results of this section can be deduced from similar results in spaces of continuous functions, see Sect. 5.4. Next theorem gives the tool to export results obtained in the context of distances to spaces of continuous functions on a compact set to the context of Banach spaces.

Theorem 5.5 [7] *Let E be a Banach space and let $B_{E'}$ be the closed unit ball in the dual space E' endowed with the w^*-topology. Let $i : E \to E''$ and $j : E'' \to \ell_\infty(B_E)$ the canonical embeddings. Then for every $z \in E''$ we have*

$$d(z, i(E)) = d(j(z), C(B_{E'})).$$

If we consider $\ell_\infty(B_{E'})$ as a subspace of $(\mathbb{R}^{B_{E'}}, \tau_p)$, then the natural embedding $j : (E'', w^*) \to (\ell_\infty(B_{E'}), \tau_p)$ is continuous. For a bounded set $H \subset E''$, the closure $\overline{H}^{w^*}$ is w^*-compact and therefore, the continuity of j gives that $\overline{j(H)}^{\tau_p} = j(\overline{H}^{\tau_p})$. Then

$$d(\overline{j(H)}^{\tau_p}, C(B_{E'}), w^*)) = d((\overline{H}^{w^*}, i(E))$$

and

$$\hat{d}(\overline{j(H)}^{\tau_p}, C(B_{E'}), w^*)) = \hat{d}((\overline{H}^{w^*}, i(E)).$$

5.3 Distance to Fréchet Spaces

In this section we present generalizations of the results given in Sect. 5.2 when we deal con Fréchet spaces. These results can be found in [3, 4]. First introduce some notation. Let E be a Fréchet space and let $(U_n)_n$ be a decreasing basis of absolutely convex neighborhoods of zero. By $\left(E', \beta\left(E', E\right)\right)$ and $\left(E'', \beta\left(E'', E'\right)\right)$ we mean the strong dual of E and $\left(E', \beta\left(E', E\right)\right)$, respectively. In $\left(E'', \beta\left(E'', E'\right)\right)$ the sequence of bipolars $(U_n^{00})_n$ is a decreasing basis of absolutely convex neighborhoods of zero. By $\|h\|_n = \sup\left\{|h(u)| : u \in U_n^0\right\}$ we denote the seminorm in E'' associated with U_n^0 and d_n means the pseudometric defined by $\|.\|_n$. The restriction of $\|.\|_n$ to E, also denoted by $\|.\|_n$, is the seminorm defined by U_n. The topology of E can be defined by the F-norm $d(x, y) := \sum_n 2^{-n}\|x - y\|_n(1 + \|x - y\|_n)^{-1}$ for $x, y \in E$. The topology of the space $(E'', \beta(E'', E'))$ is defined by the F-norm $d(x^{**}, y^{**}) := \sum_n 2^{-n}\|x^{**} - y^{**}\|_n(1 + \|x^{**} - y^{**}\|_n)^{-1}$ for all $x^{**}, y^{**} \in E''$. Additionally without loss of generality, we assume in this paper that $2U_{n+1} \subset U_n$ for $n \in \mathbb{N}$; and this clearly implies that $2\|x^{**}\|_n \leq \|x^{**}\|_{n+1}$ for $n \in \mathbb{N}$ and each $x^{**} \in E''$.

If H is a bounded subset of E then H^0 is a neighborhood of zero in $\left(E', \beta\left(E', E\right)\right)$ and the bypolar H^{00} is a compact subset of $\left(E'', \sigma\left(E'', E'\right)\right)$. Therefore an E-bounded subset H is weakly relatively compact if and only if $\overline{H}^{\sigma(E'',E')}$ is contained in E.

Next concepts are the natural extensions of the given above:

$$\gamma_n(H) := \sup\left\{\left|\lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m)\right| : (u_p) \subset U_n^0, (h_m) \subset H\right\}$$

assuming the involved limits exist. Let

$$ck_n(H) := \sup\left\{d_n\left(clust_{E''}(\varphi), E\right) : \varphi \in H^{\mathbb{N}}\right\}$$

and

$$ck(H) := \sup\left\{d\left(clust_{E''}(\varphi), E\right) : \varphi \in H^{\mathbb{N}}\right\}$$

where $clust_{E''}(\varphi) := \bigcap_p \overline{\{\varphi(m) : m > p\}}^{\sigma(E'',E')}$ is the set of all cluster points in E'' of the sequence $\varphi \in H^{\mathbb{N}}$ and $d_n(A, B) = \inf\{d_n(a, b) : a \in A, b \in B\}$. Also define

$$k_n(H) := \sup\left\{d_n(h, E) : h \in \overline{H}^{\sigma(E'',E')}\right\},$$

and

$$k(H) := \sup\left\{d(h, E) : h \in \overline{H}^{\sigma(E'',E')}\right\}.$$

We say that H ε-interchanges limits with a subset B of E' if

$$\sup\left\{\left|\lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m)\right| : \left(u_p\right) \subset B, (h_m) \subset H\right\} \leq \varepsilon$$

where $\varepsilon \geq 0$ and the involved limits exist. $\gamma_n(H) \leq \varepsilon$ ($\gamma_n(H) = 0$) means: H ε-interchanges (interchanges) limits with U_n^0. Note that

$$2\gamma_n(H) \leq \gamma_{n+1}(H), \quad 2ck_n(H) \leq ck_{n+1}(H), \quad 2k_n(H) \leq k_{n+1}(H).$$

Hence $\sup_n \gamma_n(H) < \infty$, $\sup_n ck_n(H) < \infty$, $\sup_n k_n(H) < \infty$ iff $\gamma_n(H) = 0$, $ck_n(H) = 0$, $k_n(H) = 0$, $n \in \mathbb{N}$, respectively.

Lemma 5.1 [3] *Let H be a bounded subset of a Fréchet space E and let $h \in \overline{H}^{\sigma(E'',E')}$. Then for each $n \in \mathbb{N}$ there exists a net $(u_\beta)_\beta$ in U_n^0 that $\sigma(E', E)$-converges to 0 and such that for each net $(h_\alpha)_\alpha$ in H that $\sigma(E'', E')$-converges to h we have $d_n(h, E) = \lim_\beta \lim_\alpha u_\beta(h_\alpha)$. Consequently, there exist sequences $(h_m)_m$ in H and $(u_p)_p$ in U_n^0 such that $d_n(h, E) = \lim_p \lim_m u_p(h_m)$ and $\lim_m \lim_p u_p(h_m) = 0$. Hence $k_n(H) \leq \gamma_n(H)$.*

Proof The linear functional u defined on the linear hull of E and h by $u(e + \lambda h) = \lambda d_n(h, E)$ for $e \in E$ verifies $|u(e + \lambda h)| = |\lambda|\, d_n(h, E) = d_n(\lambda h, E) = d_n(e + \lambda h, E) \leq \|e + \lambda h\|_n$. By the Hahn-Banach theorem u admits a linear extension to E'', also named u, such that

$$\left|u(x^{**})\right| \leq \left\|x^{**}\right\|_n$$

for each $x^{**} \in E''$. Clearly $u \in \left(U_n^{00}\right)^0 = (U_n^0)^{00}$ and we obtain a net $(u_\beta)_\beta$ in U_n^0 such that

$$u(x^{**}) = \lim_\beta u_\beta(x^{**})$$

for each $x^{**} \in E''$. In particular

$$d_n(h, E) = u(h) = \lim_\beta u_\beta(h), \;\; 0 = d_n(e, E) = u(e) = \lim_\beta u_\beta(e)$$

for each $e \in E$ so $(u_\beta)_\beta$ $\sigma(E', E)$-converges to 0. If $(h_\alpha)_\alpha$ is a net in H that $\sigma(E'', E')$-converges to h, then each $u_\beta(h)$ is the limit of the net $\left(u_\beta(h_\alpha)\right)_\alpha$ and

$$d_n(h, E) = u(h) = \lim_\beta \lim_\alpha u_\beta(h_\alpha), \quad 0 = \lim_\alpha u(h_\alpha) = \lim_\alpha \lim_\beta u_\beta(h_\alpha).$$

By Lemma 2.1 of [8] there exist sequences $(h_m)_m$ in H and $\left(u_p\right)_p$ in U_n^0 such that

$$d_n(h, E) = \lim_p \lim_m u_p(h_m), \quad 0 = \lim_m \lim_p u_p(h_m).$$

By $d_n(h, E) = \lim_p \lim_m u_p(h_m) - \lim_m \lim_p u_p(h_m)$ we have $k_n(H) \leq \gamma_n(H)$. □

Lemma 5.2 [3] *Let $(h_\alpha)_\alpha$ be a net in a bounded subset H of a Fréchet space E. Let h be a $\sigma(E'', E')$-cluster point of $(h_\alpha)_\alpha$. If $(v_\beta)_\beta$ is a net in U_n^0 such that the involved limits* $\lim_\beta \lim_\alpha v_\beta(h_\alpha)$ *and* $\lim_\alpha \lim_\beta v_\beta(h_\alpha)$ *exist, then*

$$\left|\lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\alpha \lim_\beta v_\beta(h_\alpha)\right| \leq 2d_n(h, E).$$

Hence $\gamma_n(H) \leq 2ck_n(H)$ for each $n \in \mathbb{N}$.

Proof If $(u_\beta)_\beta$ is a net in U_n^0 that $\sigma(E', E)$-converges to 0 and the involved limits in $\lim_\beta \lim_\alpha u_\beta(h_\alpha)$ exist, then

$$\left|\lim_\beta \lim_\alpha u_\beta(h_\alpha)\right| \leq d_n(h, E). \tag{5.1}$$

Indeed, for each $\varepsilon > 0$ let $h_\varepsilon \in E$ be such that

$$d_n(h, h_\varepsilon) < d_n(h, E) + \varepsilon.$$

By the hypothesis $\lim_\alpha u_\beta(h_\alpha) = u_\beta(h)$ and $\lim_\beta u_\beta(h_\varepsilon) = 0$. Then

$$\left|\lim_\beta \lim_\alpha u_\beta(h_\alpha)\right| = \left|\lim_\beta u_\beta(h)\right| = \left|\lim_\beta u_\beta(h - h_\varepsilon)\right| \leq d_n(h, h_\varepsilon) < d_n(h, E) + \varepsilon.$$

The inequality $\left|\lim_\beta \lim_\alpha u_\beta(h_\alpha)\right| < d_n(h, E) + \varepsilon$ is true for each positive number ε, so we have

$$\left|\lim_\beta \lim_\alpha u_\beta(h_\alpha)\right| \leq d_n(h, E).$$

To prove the main inequality pick v a $\sigma(E', E)$-cluster point of $(v_\beta)_\beta$. By hypothesis, $\lim_\alpha \lim_\beta v_\beta(h_\alpha)$ exists and $v(h_\alpha)$ is a cluster point of $(v_\beta(h_\alpha))_\beta$ so $\lim_\beta v_\beta(h_\alpha) = v(h_\alpha)$. Then $\lim_\alpha v(h_\alpha)$ exists and $v(h)$ is a cluster point of $(v(h_\alpha))_\alpha$ so $\lim_\alpha v(h_\alpha) = v(h)$. Therefore $u_\beta := 2^{-1}(v_\beta - v)$ is a net in U_n^0 such that 0 is a $\sigma(E', E)$-cluster point. Choosing a subnet we can suppose that $(u_\beta)_\beta$ $\sigma(E', E)$ converges to 0. The involved limits in $\lim_\beta \lim_\alpha u_\beta(h_\alpha)$ exist, because by hypothesis the limits in $\lim_\beta \lim_\alpha v_\beta(h_\alpha)$ exist and $\lim_\beta \lim_\alpha v(h_\alpha) = \lim_\beta v(h) = v(h)$. Then

$$\left|\lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\alpha \lim_\beta v_\beta(h_\alpha)\right| = \left|\lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\alpha v(h_\alpha)\right|$$
$$= \left|\lim_\beta \lim_\alpha v_\beta(h_\alpha) - \lim_\beta \lim_\alpha v(h_\alpha)\right| = 2\left|\lim_\beta \lim_\alpha 2^{-1}\left(v_\beta - v\right)(h_\alpha)\right| \leq 2d_n(h, E),$$

where the last inequality follows from (5.1). Hence $\gamma_n(H) \leq 2ck_n(H)$ for each $n \in \mathbb{N}$. □

Next result generalizes Theorem 5.4.

Theorem 5.6 [3] *$ck_n(H) \leq k_n(H) \leq \gamma_n(H) \leq 2ck_n(H)$ for a bounded subset H of a Fréchet space E and each $n \in \mathbb{N}$. Then $ck(H) = 0$ iff $k(H) = 0$.*

Proof The second and third inequalities follow from previous lemmas. The first inequality is obvious. □

Corollary 5.3 [3] *If H is a bounded subset of a Fréchet space E, the following conditions are equivalent:*

1. *$ck(H) = 0$.*
2. *$k(H) = 0$.*
3. *H is weakly relatively countably compact.*
4. *H is weakly relatively compact.*

Krein's theorem also holds for Fréchet spaces but since the metric is not a norm it seems that the inequality obtained in Theorem 5.3 should not hold here. So the natural problem here is to find a continuous function $F : [0, 1] \to [0, 1]$ such that $F(0) = 0$ and $k(coH) \leq F(k(H))$. The following theorem is a positive answer to this theorem.

Theorem 5.7 [4] *For a bounded set H in a Fréchet space E the following inequality holds*

$$k(coH) < (2^{n+1} - 2)k(H) + \frac{1}{2^n}$$

for all $n \in \mathbb{N}$. Consequently

$$k(coH) \leq \sqrt{k(H)}(3 - 2\sqrt{k(H)}).$$

Proof [Sketch of the proof] First we have to prove that $\gamma_n(H) = \gamma_n(coH)$. For this we can use the same ideas that were used in the proof of [9, Theorem 13] or the ideas of the proof of [7, Theorem 3.3]. Using the definition of d and that the function $f(x) = x/(1+x)$ is strictly increasing we get that

$$k(H') \leq \frac{2^n - 1}{2^n} \frac{k_n(H')}{1 + k_n(H')} + \frac{1}{2^n},$$

for all $H' \subset E$. In particular

$$k(coH) \leq \frac{2^n - 1}{2^n} \frac{k_n(coH)}{1 + k_n(coH)} + \frac{1}{2^n}.$$

By Theorem 5.6 we get that

$$k_n(coH) \leq \gamma_n(coH) = \gamma_n(H) \leq 2k_n(coH),$$

and then since $f(x) = x/(1+x)$ is strictly increasing

$$k(coH) \leq \frac{2^n - 1}{2^n} \frac{k_n(coH)}{1 + k_n(coH)} + \frac{1}{2^n} \leq \frac{2k_n(H)}{1 + 2k_n(H)} + \frac{1}{2^n}.$$

To finish the proof we have to prove that

$$\frac{1}{2^n} \frac{2k_n(c)}{1 + 2k_n(H)} < 2k(H).$$

For this we use again the definition of d and that $f(x)$ is strictly increasing. □

5.4 Distances to Spaces of Continuous Functions

In this section we collect several quantitative versions of classical theorems about compactness in spaces of continuous functions. First we provide a formula to measure distances to spaces of continuous functions from a normal space to $\mathbb{R}$ via oscillations, Theorem 5.9. This result appears in [6, Proposition 1.18] with proof in paracompact spaces but the authors also say that this result holds in normal spaces. We need the following theorem.

Theorem 5.8 ([13], Theorem 12.16) *Let X be a normal space and let $f_1 \leq f_2$ be to real functions on X such that f_1 is upper semicontinuous and f_2 is lower semicontinuous. Then, there exists a continuous function $f \in C(X)$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in X$.*

Definition 5.4 Let X be a topological. The oscillation of a bounded function $f \in \mathbb{R}^X$ at the point $x \in X$ is defined by

$$osc(f, x) = \inf_U \sup_{y,z \in U} |f(x) - f(y)|,$$

where the infimum is taken over the neighborhoods U of x in X.

Theorem 5.9 [6, Proposition 1.18] *Let X be a normal space. If $f \in \mathbb{R}^X$, then*

$$d(f, C(X)) = \frac{1}{2} osc(f)$$

where $osc(f) = \sup_{x \in X} osc(f, x)$ and

$$osc(f, x) := \inf_U \{ \sup_{y,z \in U} |f(y) - f(z)| : U \subset X \text{ open}, x \in U\}.$$

Proof Let us prove that $d(f, C(X)) \geq \frac{1}{2} osc(f)$. If $d(f, C(X))$ is infinite, clearly the inequality holds. Suppose that $d = d(f, C(X))$ is finite. Fix $\varepsilon > 0$ and $x \in X$. There exist $g \in C(X)$ such that $d(f, g) \leq d + \varepsilon/3$. Since g is continuous there is an open neighborhood U of x such that $diam(g(U)) < \varepsilon/3$. Then, if $y, z \in U$,

$$d(f(y), f(z)) \leq d(f(y), g(y)) + d(g(y), g(z)) + d(g(z), f(z)) < 2d + \varepsilon$$

so $osc(f, x) < 2d + \varepsilon$. Since we can do it for all $\varepsilon > 0$ we get that $osc(f, x) \leq 2d$.

Let us prove that $d(f, C(X)) \leq \frac{1}{2} osc(f)$. If $osc(f) = +\infty$ clearly the inequality holds so suppose that $\delta = \frac{1}{2} osc(f)$ is finite. For $x \in X$ denote by $\mathscr{U}_x$ the set of neighborhoods of x. Put

$$\mathscr{V}_x = \{U \in \mathscr{U}_x : diam(f(U)) < osc(f) + 1\}.$$

Clearly $\mathscr{V}_x$ is a basis of the neighborhoods of x and for each $U \in \mathscr{V}_x$, $f|_U$ is upper and lower bounded. Now

$$\begin{aligned} 2\delta \geq osc(f, x) &= \inf_{U \in \mathscr{U}_x} diam(f(U)) = \inf_{U \in \mathscr{V}_x} diam(f(U)) \\ &= \inf_{U \in \mathscr{V}_x} \sup_{y,z \in U} (f(y) - f(z)) \\ &\geq \inf_{U,V \in \mathscr{V}_x} \sup_{y \in U, z \in V} (f(y) - f(z)) \\ &= \inf_{U \in \mathscr{V}_x} \sup_{y \in U} f(y) - \sup_{U \in \mathscr{V}_x} \inf_{z \in U} f(z). \end{aligned}$$

So if we define

$$f_1(x) = \inf_{U \in \mathscr{V}_x} \sup_{z \in U} f(z) - \delta$$

$$f_2(x) = \sup_{U \in \mathscr{V}} \inf_{z \in U} f(z) + \delta$$

then $f_1 \le f_2$. It is easy to check that f_1 is upper semi-continuous and f_2 is lower semi-continuous. By Theorem 5.8, there is a continuous function $h \in C(X)$ such that

$$f_1(x) \le h(x) \le f_2(x)$$

for $x \in X$. Clearly

$$f_2(x) - \delta \le f(x) \le f_1(x) + \delta$$

so

$$h(x) - \delta \le f_2(x) - \delta \le f(x) \le f_1(x) + \delta \le h(x) + \delta$$

so $d(f, g) \le \delta$ and this proves the inequality. □

Previous result characterizes the normality of X:

Corollary 5.4 *Let X be a topological space. The following statements are equivalent:*

1. *X is normal.*
2. *For each $f \in \mathbb{R}^X$ there is $g \in C(X)$ such that $d(f, g) = \frac{1}{2} osc(f)$.*
3. *$d(f, C(X)) = \frac{1}{2} osc(f)$ for each function $f \in \mathbb{R}^X$.*

Proof By the previous result (and proof) we only have to prove that the third condition implies the first one. For this, if A and B are disjoint closed subsets of X, define $f = \chi_B - \chi_A$. Then $osc(f) \le 1$ so by hypothesis, $d(f, C(X)) = \frac{1}{2} osc(f) \le \frac{1}{2}$. Then we can pick a continuous function $g \in C(X)$ such that $d(f, g) < 1$. If we define $U = g^{-1}(-\infty, 0)$ and $V = g^{-1}(0, +\infty)$, then U and V are disjoint open subsets of X, $A \subset U$ and $B \subset V$ so X is normal. □

Definition 5.5 Let K be a compact topological space and let H be a uniformly bounded subset of $C(K)$. We define

$$ck(H) := \sup_{\varphi \in H^{\mathbb{N}}} d(clust_{\mathbb{R}^K}(\varphi), C(K))$$

and

$$\gamma_K(H) := \sup\{|\lim_n \lim_m f_m(x_n) - \lim_m \lim_n f_m(x_n)| : (f_m) \subset H, (x_n) \subset K\},$$

assuming the involved limits exist.

By definition $\inf \emptyset = +\infty$. Observe that if H is a τ_p-relatively countably compact subset of $C(K)$ then $ck(H) = 0$. Using γ_K, the usual notion of interchange of limits [12] can be defined as follows: H interchanges limits with K if, and only if $\gamma_K(H) = 0$.

Grothendieck proved in [12] that H is relatively compact in $(C(K), \tau_p)$ if, and only if, H interchanges limits with K. The following result establishes the relations between $\gamma_K(H)$, $ck(H)$ and $d(\overline{H}^{\mathbb{R}^K}, C(K))$.

Theorem 5.10 [1, 7] *Let K be a compact space and $H \subset C(K)$ a uniformly bounded set, then*

$$ck(H) \leq \hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)) \leq \gamma_K(H) \leq 2ck(H).$$

The following theorem is a quantitative version of the Krein-Šmulyan theorem and combined with Theorem 5.5 implies Theorem 5.3.

Theorem 5.11 [7] *Let K be a compact topological space and let H be a uniformly bounded subset of $\mathbb{R}^K$. Then*

$$\gamma_K(H) = \gamma_K(co(H))$$

and as a consequence for $H \subset C(K)$ we obtain that

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \leq 2\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K))$$

and if $H \subset \mathbb{R}^K$ is uniformly bounded then

$$\hat{d}(\overline{co(H)}^{\mathbb{R}^K}, C(K)) \leq 5\hat{d}(\overline{H}^{\mathbb{R}^K}, C(K)).$$

Now we are going to prove Lemma 5.4. This lemma is a very useful tool to study quantitative versions of angelicity in spaces of continuous functions and was proved in [1] in a more general case, when Z is a separable metric space. But since this result is a quantitative version of Theorem 5.2 we prefer to include here the compact metric space version and its proof.

Lemma 5.3 *Suppose that (Z, d) is a compact metric space and let X be a set. Given functions $f_1, \ldots, f_n \in Z^X$, $D \subset X$ and $\varepsilon > 0$ there is a finite subset $L \subset D$ such that for every $x \in D$*

$$\min_{y \in L} \max_{1 \leq k \leq n} d(f_k(y), f_k(x)) < \varepsilon.$$

Proof The metric

$$d_\infty\big((t_k), (s_k)\big) := \sup_{1 \leq k \leq n} d(t_k, s_k),$$

$(t_k), (s_k) \in Z^n$, defines the product topology of the space Z^n. Define

$$H = \{(f_1(x), f_2(x), \ldots, f_n(x)) : x \in D\}.$$

Now we only have to apply the compactness of (Z^n, d). □

Lemma 5.4 [1] *Let* (Z, d) *be a compact metric space,* X *a set and* H *a subset of the space* (Z^X, τ_p) *and* $\varepsilon \geq 0$. *We assume that:*

(i) There is $\Sigma \subset \mathbb{N}^{\mathbb{N}}$ *and a family* $\{A_\alpha : \alpha \in \Sigma\}$ *of non-void subsets of the set* X *such that* $X = \bigcup\{A_\alpha : \alpha \in \Sigma\}$.

(ii) For every $\alpha = (a_1, a_2, \ldots) \in \Sigma$ *the set* H ε*-interchanges limits in* Z *with every sequence* $(x_n)_n$ *in* X *that is eventually in each set*

$$C_{\alpha|m} = \bigcup\{A_\beta : \beta \in \Sigma, b = (b_m) \text{ and } b_j = a_j \text{ for } j = 1, 2, \ldots, m\},$$

for $m \in \mathbb{N}$.

Then for any $f \in \overline{H}^{Z^X}$ *there exists a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* H *such that*

$$\sup_{x\in X} d(g(x), f(x)) \leq \varepsilon$$

for any cluster point g *of* $(f_n)_{n\in\mathbb{N}}$ *in* Z^X.

Proof Define $f_0 := f$. Since $\mathrm{F}(\Sigma)$ is countable and infinite there is a bijection $\varphi : \mathbb{N} \to \mathrm{F}(\Sigma)$. We define $D_n := C_{\varphi(n)}$ for each $n \in \mathbb{N}$. We claim that there are a sequence of functions $f_0, f_1, \ldots, f_n, \ldots$ and a sequence of finite sets $L_1, L_2, \ldots, L_n, \ldots$ with the properties:

- for each $n \in \mathbb{N}$ and every $x \in D_n$ we have

$$\min_{y\in L_n} \max_{0\leq k<n} d(f_k(y), f_k(x)) < \frac{1}{n}; \tag{5.2}$$

- for each $n \in \mathbb{N}$ the function f_n belongs to H and

$$d(f_n(y), f_0(y)) < \frac{1}{n} \text{ for every } y \in \bigcup\{L_m : 1 \leq m \leq n\}. \tag{5.3}$$

We prove the existence of the above sequences of functions and sets by recurrence.
FIRST STEP. Applying Lemma 5.3 to $D := D_1$ and f_0 we obtain a finite subset L_1 of D_1 such that

$$\min_{y\in L_1} d(f_0(y), f_0(x)) < 1 \text{ for every } x \in D_1.$$

Since $f \in \overline{H}^{Z^X}$, there is $f_1 \in H$ such that

$$\max_{y\in L_1} d(f_1(y), f_0(y)) < 1.$$

INDUCTION STEP. Assuming we have produced $f_1, f_2, \ldots, f_n$ and $L_1, L_2, \ldots, L_n$ satisfying (5.2) and (5.3) we use Lemma 5.3 for $D := D_{n+1}$ and $f_0, f_1, \ldots, f_n$ to obtain $L_{n+1} \subset D_{n+1}$ satisfying

$$\inf_{y\in L_{n+1}} \max_{0\le k<n+1} d(f_k(y), f_k(x)) < \frac{1}{n+1} \text{ for every } x \in D_{n+1}.$$

Once again, since $f \in \overline{H}^{Z^X}$ we can take a function $f_{n+1} \in H$ satisfying

$$d(f_{n+1}(y), f_0(y)) < \frac{1}{n+1} \text{for every } y \in \bigcup\{L_m : 1 \le m \le n+1\}.$$

The constructed sequences $f_0, f_1, \ldots, f_n, \ldots$ and $L_1, L_2, \ldots, L_n, \ldots$ satisfy (5.2) and (5.3).

We shall prove now that $(f_n)_{n\in\mathbb{N}}$ has the property required in the thesis in the lemma: fix a cluster point g of $(f_n)_n$ in Z^X and fix a point $x \in X$ and let us prove that $d(g(x), f(x)) \le \varepsilon$. We note first that inequality (5.3) implies that

$$\lim_n f_n(y) = f(y) \text{ for every } y \in L = \bigcup_{n\in\mathbb{N}} L_n. \tag{5.4}$$

Now, we pick $\alpha = (a_1, a_2, \ldots) \in \Sigma$ such that $x \in A_\alpha$ and define

$$P := \varphi^{-1}(\{\alpha|n : n \in \mathbb{N}\}) \subset \mathbb{N}.$$

P is an infinite subset because φ is a bijection. Since the point $x \in \bigcap_{p\in P} D_p$, (5.2) for each $p \in P$ we can pick $y_p \in L_p$ with the property

$$d(f_k(y_p), f_k(x)) < \frac{1}{p} \text{ for } 0 \le k < p. \tag{5.5}$$

Being P infinite we can and do fix $p_1 < p_2 < \cdots < p_j < \cdots \nearrow +\infty$ a strictly increasing sequence in P. We claim that the sequence $(y_{p_j})_j$ is eventually in $C_{\alpha|n}$ for every $n \in \mathbb{N}$. Indeed, for a given $n \in \mathbb{N}$ take $p_{j(n)}$ an element of the sequence $(p_j)_j$, with $p_{j(n)} > \varphi^{-1}(\alpha|i)$, $i = 1, 2, \ldots, n$. Therefore, if $j > j(n)$ then $p_j \neq \varphi^{-1}(\alpha|i)$ for $i = 1, 2, \ldots, n$ and consequently $\varphi(p_j) = \alpha|n(p_j)$ for some $n(p_j) > n$. The latter implies

$$y_{p_j} \in D_{p_j} = C_{\alpha|n(p_j)} \subset C_{\alpha|n}, \text{ for } j > j(n),$$

proving that $(y_{p_j})_j$ is eventually in each $C_{\alpha|n}$.

Observe also that (5.5) implies that

$$\lim_j f_k(y_{p_j}) = f_k(x) \text{ for } k = 0, 1, 2, \ldots. \tag{5.6}$$

Since $g(x)$ is a cluster point of $(f_n(x))_n$ in the metric space (Z, d) we can choose a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that $\lim_k f_{n_k}(x) = g(x)$. With all the above we have

$$\lim_k \lim_j f_{n_k}(y_{p_j}) \overset{(5.6)}{=} \lim_k f_{n_k}(x) = g(x),$$

$$\lim_j \lim_k f_{n_k}(y_{p_j}) \overset{(5.4)}{=} \lim_j f(y_{p_j}) \overset{(5.6)}{=} f(x).$$

Being the sequence $(y_{p_j})_j$ eventually in every $C_{\alpha|n}$ the assumption (ii) in the lemma ensures us that H ε-interchanges limits with $(y_{p_j})_j$, consequently

$$d(g(x), f(x)) = d(\lim_k \lim_j f_{n_k}(y_{p_j}), \lim_j \lim_k f_{n_k}(y_{p_j})) \leq \varepsilon,$$

and the proof is over. □

Theorem 5.12 [3] Let X be a web-compact space with a representation $D = \bigcup\{A_\alpha : \alpha \in \Sigma\}$ with $X = \overline{D}$. Let (Z, d) be a compact metric space and $H \subset Z^X$. Then for each $f \in \overline{H}$ (the closure in Z^X) there exists a sequence $(f_n)_n$ in H such that

$$\sup_{x \in D} d(f(x), g(x)) \leq 2ck(H) + 2\hat{d}(H, C(X, Z)) \leq 4ck(H)$$

for any cluster point g of $(f_n)_n$ in Z^X.

Proof (Sketch of the proof) Let $\varepsilon := ck(H) + \hat{d}(H, C(X, Z))$ and let $\tilde{H} = \{f|_D : f \in H\}$. Then we have to prove that condition (ii) in Lemma 5.4 holds for D and $\tilde{H}$.

Now let $f \in \overline{H}$ (the closure in Z^X). Since $f|_D \in \overline{\tilde{H}}$ (the closure in Z^D), by Lemma 5.4 there exists a sequence $(g_n)_n$ in $\tilde{H}$ such that $\sup_{x \in D} d(f(x), h(x)) \leq 2\varepsilon$ for each cluster point h of $(g_n)_n$ in Z^X. For each g_n there exists $f_n \in H$ such that $f_n|_D = g_n$. If g is a cluster point of $(f_n)_n$ then $g|_D$ is cluster point of $(g_n)_n$ so $\sup_{x \in D} d(f(x), g(x)) \leq 2\varepsilon$. This yields the first inequality. The other inequality is trivial. □

Corollary 5.5 [1, 3] Let X be a strongly web-compact space, (Z, d) a compact metric space, and let $H \subset Z^X$. Then

$$ck(H) \leq \hat{d}(\overline{H}, C(X, Z)) \leq 3ck(H) + 2\hat{d}(H, C(X, Z))) \leq 5ck(H).$$

Proof We only have to prove the middle inequality: We may assume that $ck(H) < \infty$. Fix $t > 0$ with $ck(H) < t$ and $f \in \overline{H}$. Fix

$$\varepsilon := ck(H) + \hat{d}(H, C(X, Z)).$$

By Theorem 5.12 there exists a sequence $(f_n)_n$ in H such that $\sup_{x \in X} d(f(x), g(x)) \leq 2\varepsilon$ for any cluster point g of $(f_n)_n$ in Z^X. Since $ck(H) < t$, for this sequence $(f_n)_n$ there exists a cluster point g of $(f_n)_n$ such that $d(g, C(X, Z)) < t$. This yields the inequality. □

Proposition 5.2 [3] *Let X be a web-compact space with a representation $D = \bigcup\{A_\alpha : \alpha \in \Sigma\}$ with $X = \overline{D}$. Let (Z, d) be a compact metric space, let $H \subset Z^X$ and let $f \in \overline{H}$ (the closure in Z^X). Then for each $\delta > 0$ and $x \in X$ there exists $U \subset X$ a neighborhood of x such that*

$$d(f(x), f(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z)) + \delta$$

for every $d \in U \cap D$.

Proof (Sketch of the proof) Fik $x \in X$ and define $H' = \{j \in H : d(j(x), f(x)) < 4^{-1}\delta\}$. Since H' is the intersection of H and an open neigbourhood of f in Z^X, then $f \in \overline{H'}$. Now we obtain the sequence given by Theorem 5.12 and pick a cluster point g of this sequence such that such that $d(g, C(X, Z)) < ck(H) + 4^{-1}\delta$. Then we can pick $h \in C(X, Z)$ such that

$$d(g(z), h(z)) < ck(H) + \delta/4$$

for all $z \in X$. Using the continuity of h we can find $U \subset X$ a neighborhood of x such that $d(h(x), h(z)) < \delta/4$. We can check that

$$d(f(x), f(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z)) + \delta$$

for every $d \in U \cap D$. □

Theorem 5.13 [3] *Let X be a web-compact space, (Z, d) be a compact metric space and $H \subset Z^X$. Then for each $f \in \overline{H}$ (the closure in Z^X) there exists a sequence $(f_n)_n$ in H such that*

$$\sup_{x \in X} d(f(x), g(x)) \leq 10ck(H) + 6\hat{d}(H, C(X, Z)) \leq 16ck(H)$$

for any cluster point g of $(f_n)_n$ in Z^X.

Proof Let (f_n) be the sequence obtained from Theorem 5.12 and fix $x \in X$, g a cluster point of $(f_n)_n$ and $\delta > 0$. Since $f, g \in \overline{H}$ by Proposition 5.2, there exist $U, V \subset X$ neighborhoods of x such that

$$d(f(x), f(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z))) + \delta$$

for every $d \in U \cap D$ and

$$d(g(x), g(d)) < 4ck(H) + 2\hat{d}(H, C(X, Z))) + \delta$$

for every $d \in V \cap D$. Pick $d \in D \cap U \cap V$, then

$$d(f(x), g(x)) \leq d(f(x), f(d)) + d(f(d), g(d)) + d(g(d), g(x))$$
$$< 10ck(H) + 6\hat{d}(H, C(X, Z)) + 2\delta. \qquad \square$$

The following corollary follows from Theorem 5.13 like Corollary 5.5 from Theorem 5.12.

Corollary 5.6 [3] *Let X be a web-compact space, (Z, d) a separable metric space and let $H \subset Z^X$ be a τ_p-relatively compact set. Then*

$$ck(H) \leq \hat{d}(\overline{H}, C(X, Z)) \leq 11ck(H) + 6\hat{d}(H, C(X, Z))) \leq 17ck(H).$$

Corollary 5.7 *Let X be a web-compact space, (Z, d) a separable metric space and let $H \subset C(X, Z)$ be a τ_p-relatively compact set in X^Z. The following conditions are equivalent:*

1. *$ck(H) = 0$.*
2. *H is a relatively countably compact subset of $C(X, Z)$.*
3. *H is a relatively compact subset of $C(X, Z)$.*

References

1. Angosto, C., Cascales, B.: The quantitative difference between countable compactness and compactness. J. Math. Anal. Appl. **343**(1), 479–491 (2008)
2. Angosto, C., Cascales, B.: Measures of weak noncompactness in Banach spaces. Topology Appl. **156**, 1412–1421 (2009)
3. Angosto, C., Kąkol, J., López-Pellicer, M.: A quantitative approach to weak compactness in Fréchet spaces and spaces $C(X)$. J. Math. Anal. Appl. **403**(1), 13–22 (2013)
4. Angosto, C., Kąkol, J., Kubzdela, A., López-Pellicer, M.: A quantitative version of Krein's theorems for Fréchet spaces. Arch. Math. (Basel) **101**(1), 65–77 (2013)
5. Astala, K., Tylly, H.: Seminorms related to weak compactness to Tauberian operators. Math. Proc. Cambridge Philos. Soc. **107**, 367–375 (1990)
6. Benyamini, Y., Lindenstrauss, J.: Geometric nonlinear functional analysis. Vol. 1. American Mathematical Society Colloquium Publications, 48, American Mathematical Society, Providence, RI Springer-Verlag, New York (2000)
7. Cascales, B., Marciszewski, W., Raja, M.: Distance to spaces of continuous functions. Topology Appl. **153**, 2303–2319 (2006)
8. Eberlein, F.W.: Weak compactness in Banach spaces. Proc. Nat. Acad. Sci. U.S.A. **33**, 51–53 (1947)
9. Fabian, M., Hájek, P., Montesinos, V., Zizler, V.: A quantitative version of Krein's theorem. Rev. Mat. Iberoam. **21**, 237–248 (2005)
10. Floret, K.: Weakly compact sets. Lecture Notes in Mathematics, vol. 801, Springer, Berlin (1980)
11. Granero, A.S.: An extension of the Krein-Šmulian theorem. Rev. Mat. Iberoam. **22**, 93–110 (2006)
12. Grothendieck, A.: Critères de compacité dans les espaces fonctionnels généraux. Amer. J. Math. **74**, 168–186 (1952)
13. Jameson, G.J.O.: Topology and normed spaces. Chapman and Hall, London (1974)
14. Kelley, J.L., Namioka, I.: Linear topological spaces. Graduate Texts in Mathematics vol. 36, Springer-Verlag, New York (1976)

15. Orihuela, J.: Pointwise compactness in spaces of continuous functions. J. London Math. Soc. **36**, 143–152 (1987)
16. Pol, R.: On a question of H. H. Corson and some related problems. Fund. Math. **109**(2), 143–154 (1980)
17. Pryce, J.D.: A device of R. J. Whitley's applied to pointwise compactness in spaces of continuous functions. Proc. London Math. Soc. **23**(3), 532–546 (1971)
18. Šmulian, V.: Über lineare topologische Räume. Rec. Math. (Mat. Sbornik) N. S. 7(49), 425–448 (1940)