20

Chapter 6
Two Classes of Metrizable Spaces ℓ_c-Invariant

Manuel López-Pellicer

Abstract Many properties of a Tychonoff space X have been characterized by properties of $C_p(X)$ or $C_c(X)$, the spaces of continuous real-valued functions on X provided with the topology of pointwise convergence or with the compact-open topology, respectively. The question of Arhangel'skii about preservation of metrizability by ℓ_p-equivalence in the class of first countable spaces has been partially answered by Valov in the class of those first countable spaces that are Čech complete. The preservation of complete metrizability by ℓ_p-equivalence in the class of metrizable spaces has been obtained by Baars, de Groot and Pelant. The ℓ_c-invariance of separable metrizability and separable complete metrizability (i.e., Polish spaces) in the class of spaces of pointwise countable type has been considered very recently by Kąkol, López-Pellicer and Okunev. These two ℓ_c-invariant properties were the aim of the talk given by the author in the First Meeting in Topology and Functional Analysis, dedicated to Professor Jerzy Kąkol on the occasion of his sixty birthday, September 27–28, in Elche (Spain). Here it is this talk with the proofs of properties needed to obtain these two ℓ_c-invariant properties. As additional motivation for non specialist readers three classical $C(X)$ theorems related with ℓ-equivalence's questions are also included.

Keywords Čech-complete space · Compact resolution · $C_p(X)$ and $L_p(X)$-spaces · ℓ_p-, ℓ_c- and t-equivalence · Metrizability · Polish space · Submetrizability · Spaces of pointwise countable type · μ-space · $\aleph_0$-space

2000 MSC Classification: 46E10, 54C35, 46A20, 46A50

Referencia al Prometeo página 115, ver dorso

M. López-Pellicer (✉)
Department of Applied Mathematics and IUMPA, Universitat Politècnica de València, 46022 València, Spain
e-mail: mlopezpe@mat.upv.es

J. C. Ferrando and M. López-Pellicer (eds.), *Descriptive Topology and Functional Analysis*, Springer Proceedings in Mathematics & Statistics 80, DOI: 10.1007/978-3-319-05224-3_6,

6.1 Introduction

All topological spaces (spaces, in brief) are assumed to be completely regular and Hausdorff. We use terminology and notation as in [9, 12, 13]. $C(X)$ is the set of all continuous real-valued functions on X, $C_p(X)$ is $C(X)$ endowed with the topology of pointwise convergence (also named simple convergence) and $C_c(X)$ is $C(X)$ with the compact open topology, which is the topology of uniform convergence on the compact subsets of X.

The symbol ω denotes the smallest infinite ordinal with the discrete topology and with the usual order, so ω is both in topology and order isomorphic to the set $\mathbb{N}^+$ of all non-negative integers.

Nagata's theorem [17] (see also [5, Theorem 0.6.1] or Sect. 6.2.2) states that two spaces X and Y are homeomorphic if and only if the topological rings $C_p(X)$ and $C_p(Y)$ are isomorphic. The spaces X and Y are said to be *t-equivalent* if the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic. A property $\mathscr{P}$ is preserved by t-equivalence if whenever X and Y are t-equivalent and X has the property $\mathscr{P}$ then Y has also the property $\mathscr{P}$. From Velichko's theorem (see [5, Theorem I.2.1] or Sect. 6.2.3) it follows that *finite cardinality* is preserved by t-equivalence.

Two spaces X and Y are said to be *ℓ_p-equivalent* if the spaces $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. A topological property $\mathscr{P}$ is invariant by ℓ_p-equivalence if whenever X has the property $\mathscr{P}$ and X is ℓ_p-equivalent to Y then Y has the property $\mathscr{P}$ too. Clearly, a property $\mathscr{P}$ is preserved by ℓ_P-equivalence if and only if *$\mathscr{P}$ admits a description in terms of the linear topological structure of $C_p(X)$*. There are many known results about preservation and non-preservation of various topological properties by ℓ_p-equivalence; see, e.g., [4–6, 18, 25, 26]. For example, the properties of being *hemicompact*, $\aleph_0$, *Lindelöf*, *Lindelöf-Σ*, *K-analytic* and *analytic* are preserved by ℓ_p-equivalence. On the other hand, *metrizability*, *local compactness*, *countable weight*, *normality* and *paracompactness* are not ℓ_p-invariant.

The spaces X and Y are *ℓ_c-equivalent* if the spaces $C_c(X)$ and $C_c(Y)$ are linearly homeomorphic. A topological property $\mathscr{S}$ is ℓ_c-invariant if for each space Y ℓ_c-equivalent to a space X with the property $\mathscr{S}$ then Y has also the property $\mathscr{S}$. This is the case if and only if *$\mathscr{S}$ is characterized by a property of $C_c(X)$*. For instance, Nachbin -Shirota theorem ([16, 21], see also Theorem 6.1) states that *μ-spaces* are preserved by ℓ_c-equivalence, because X is μ-space if and only if $C_c(X)$ is barrelled. Let us notice that in the class of μ-spaces the ℓ_p-equivalence implies ℓ_c-equivalence (see Sect. 6.2.6).

The space X *ℓ_c-covers* a space Y if there is a continuous open linear map from $C_c(X)$ onto $C_c(Y)$. If X and Y are ℓ_c-equivalent, then each of the two covers the other.

Arhangel'skii asked in [3, Problem 20] *if metrizability is preserved by ℓ_p-equivalence in the class of first countable spaces.* Baars, de Groot and Pelant proved that *complete metrizability is preserved by ℓ_p-equivalence in the class of metrizable spaces* [6, Theorem 3.3], given an alternative proof in the class of separable metrizable spaces by using Christensen's Theorem 6.4 [6, Theorems 5.1 and 5.3]. Later on, based on results in [24], Valov proved that a Čech-complete and first countable space Y is metrizable when it is ℓ_p-equivalent to a metrizable space X [25, Corollary 4.6].

In the general class of spaces of pointwise countable type the preservation by ℓ_c-equivalence of separable metrizability as well as the ℓ_c-invariance of Polish spaces have been considered very recently by Kąkol, López-Pellicer and Okunev in [11]. These two ℓ_c-invariant properties were the aim of the talk given by the author in the First Meeting in Topology and Functional Analysis, dedicated to Professor Jerzy Kąkol on the occasion of his sixty birthday, September 27–28, held in Elche (Spain). Here we complete the talk by collecting together all results needed to obtain, in Sect. 6.6.3, the two aforementioned ℓ_c-invariant properties. For the sake of completeness all proofs are developed. We have added three classical $C(X)$ theorems related to ℓ-equivalence as a motivation for readers non specialists in ℓ-equivalence. By these reasons, Sect. 6.2 contains the classical Nagata, Velichko and Nachbin-Shirota $C(X)$ theorems, Sect. 6.3 give some results of the Michael paper [15] and Sect. 6.5 collect some results on compact resolutions, including Christensen's theorem in Sect. 6.5.2.

The class of spaces of pointwise countable type contains the first countable spaces and the locally compact spaces among others. The ℓ_c-invariance of separable metrizability and of separable topologically complete metrizability for the spaces of pointwise countable type is consequence of its ℓ_c-invariance in the subclass of first countable spaces. Therefore, as in [11], we present first the ℓ_c-invariance of these two properties in the class of first countable spaces in Sects. 6.4 and 6.5.

6.2 Nagata, Velichko and Nachbin-Shirota Theorems in $C(X)$ Spaces

If D is a subset of a topological space X and S is a subset of $\mathbb{R}$ then $W(D, S) := \{f \in C(X) : f(D) \subset S\}$.

6.2.1 The Spaces $C_p(X)$ and $L_p(X)$

Recall that $C_p(X)$ is the space of continuous real-valued functions on X endowed with the topology of pointwise convergence. If $\mathscr{F}$ is the family of finite subsets of X then the family $\{W(F, (-\varepsilon, \varepsilon)) : F \in \mathscr{F}, \varepsilon > 0\}$ is a base of neighborhoods of 0 in $C_p(X)$.

$L_p(X)$ is the topological dual $C_p(X)'$ of $C_p(X)$ with the topology induced by $C_p(C_p(X))$. This topology, denoted by $\sigma\left(C_p(X)', C_p(X)\right)$, is named the weak*-topology on $C_p(X)'$. A net $(\varphi_i : i \in I)$ converges to φ in $L_p(X)$ if and only if $(\varphi_i(f) : i \in I)$ converges to $\varphi(f)$ for each $f \in C_p(X)$. The well known fact that a net $(x_i : i \in I)$ in X converges to x if and only if the limit of $(f(x_i) : i \in I)$ is $f(x)$, for each $f \in C(X)$, provides the standard embedding $\hat{\Delta}_X : X \to C_p(C_p(X))$ defined by

$\hat{\Delta}_X(x) = \hat{x}$, being $\hat{x}(g) = g(x)$ for each $g \in C(X)$. Clearly $\hat{\Delta}_X(X) \subset L_p(X)$ and if $\hat{X}$ is the set $\hat{\Delta}_X(X)$ with the topology induced by $C_p(C_p(X))$ then $\hat{\Delta}_X : X \to \hat{X}$ is a homeomorphism.

The space $L_p(X)$ is the linear hull of $\hat{X}$, (see [5, Proposition 0.5.7]), because if $\phi \in L_p(X)$ there exists a finite subset $F := \{x_1, x_2, \ldots, x_n\}$ in X, with $x_i \neq x_j$ if $i \neq j$, and a real number $\varepsilon > 0$ such that

$$\phi\left(W(F, (-\varepsilon, \varepsilon)\right) \subset [-1, 1].$$

Since for each $i \in \{1, 2, \ldots, n\}$ there exists $g_i \in C(X)$ with $g_i(x_i) = 1$ and $g_i(x_j) = 0$ if $j \neq i$, then for each $\lambda > 0$ and $f \in C(X)$ the function $g := f - \sum_{1 \le i \le n} f(x_i) g_i$ verifies that $\lambda g \in W(F, (-\varepsilon, \varepsilon))$, and therefore $\phi(g) = 0$. From

$$\begin{aligned} 0 = \phi(g) = \phi(f) - \sum_{1 \le i \le n} f(x_i)\phi(g_i) &= \\ = \phi(f) - \sum_{1 \le i \le n} \phi(g_i)\hat{x}_i(f) &= \\ = \left(\phi - \sum_{1 \le i \le n} \phi(g_i)\hat{x}_i\right)(f) & \end{aligned}$$

it follows that $\phi = \sum_{1 \le i \le n} \phi(g_i)\hat{x}_i$.

In [5, Proposition 0.5.9] it is also stated that $\hat{X}$ is a closed subset of $L_p(X)$. This property follows from the observation that if $\psi \in \overline{\hat{X}}^{L_p(X)} \backslash \hat{X}$ then $\psi = \Sigma_{1 \le i \le n} \lambda_i \hat{x}_i$, with $x_i \in X$, $\lambda_i \in \mathbb{R}$, $1 \leqslant i \leqslant n$, and there exists $f \in C_p(L_p(X))$ such that $f(\psi) = 1$ and $f(\hat{x}_i) = 0$, for each $1 \leqslant i \leqslant n$. The subset $\hat{A} := \left\{\hat{x} \in \hat{X} : f(\hat{x}) \geqslant 1/2\right\}$ verifies that $\psi \in \overline{\hat{A}}^{L_p(X)}$ and the function $g \in C_p(X)$ such that $g(x) := f(\hat{x})$, for each $x \in X$, verifies that $\{\hat{x}(g) : \hat{x} \in \hat{A}\} = f(\hat{A}) \subset [1/2, \infty]$. Therefore $\psi(g) \geqslant 1/2$. But this inequality contradicts that

$$\psi(g) = \left(\Sigma_{1 \le i \le n} \lambda_i \hat{x}_i\right)(g) = \Sigma_{1 \le i \le n} \lambda_i g(x_i) = \Sigma_{1 \le i \le n} \lambda_i f(\hat{x}_i) = 0.$$

6.2.2 Nagata Theorem [17]

Let $\check{X}$ be the topological subspace of $L_p(X)$ whose points are the nonzero continuous multiplicative linear forms on $C_p(X)$. Clearly $\hat{X} \subset \check{X} \subset L_p(X)\backslash\{0\}$. Nagata proved that $\hat{X} = \check{X}$ by the following two claims [17]:

If $\varphi = \sum\left\{\lambda_i \hat{x}_i : 1 \le i \le n\right\}$, with $x_j \neq x_k$ and $\lambda_j \lambda_k \neq 0$, then $\varphi \notin \check{X}$, because for the functions f_j and f_k in $C(X)$ such that $f_j(x_j) = \lambda_j^{-1}$, $f_k(x_k) = \lambda_k^{-1}$ and $f_i(x_m) = 0$

for the remaining values of i in $\{j, k\}$ and m in $\{1, 2, \ldots, n\}$, we have that $\varphi(f_j) = \varphi(f_k) = 1$ and $\varphi(f_j \times f_k) = 0$.

If $\varphi = \lambda\hat{x}$, with $0 \neq \lambda \neq 1$ then $\varphi \notin \check{X}$, since the function f identically equal to 1 verifies that $\varphi(f^2) = \varphi(f) = \lambda \neq \lambda^2 = \left[\varphi(f)\right]^2$.

Then the equality $\hat{X} = \check{X}$, the homeomorphism $\hat{\Delta}_X : X \to \hat{X}$ and the trivial fact that if the topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic then $\check{X}$ and $\check{Y}$ are homeomorphic imply Nagata's theorem, which states that if the topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic then X and Y are homeomorphic ([17] see also [5, Theorem 0.6.1]).

6.2.3 Velichko Theorem [22]

Recall that a topological space X is a P-space if the union of an increasing sequence of closed sets is closed. If X is a P-space and $\{f_i : i \in \mathbb{N}\}$ is a sequence of real continuous functions with pairwise disjoint supports then $\Sigma\{f_i : i \in \mathbb{N}\}$ is a continuous function. Velichko theorem states that X is finite if and only if $C_p(X)$ is σ-compact [22].

Suppose that $C_p(X) = \cup\{K_i : i \in \mathbb{N}\}$, each K_i being a countably compact subset of $C_p(X)$. Velichko's theorem follows from the next two claims stating firstly that X is a P-space and then that X must be finite.

In fact, if $(F_n)_n$ were an increasing sequence of closed subsets of X and there existed a point $x \in \overline{\cup_{n\in\omega} F_n} \setminus \cup_{n\in\omega} F_n$, then

$$\left\{f \in K_i : f(x) = 0,\ f\left(\cup_n F_n\right) \subset \mathbb{R}\setminus]-2^{-i}, 2^{-i}[\right\} = \emptyset$$

for each $i \in \mathbb{N}$. By the countably compactness of K_i, there would exist n_i such that

$$\left\{f \in K_i : f(x) = 0,\ f\left(F_{n_i}\right) \subset \mathbb{R}\setminus]-2^{-i}, 2^{-i}[\right\} = \emptyset.$$

Then if $\varphi_i \in C_p(X)$, $\varphi_i(X) \subset [0, 2^{-i}]$, $\varphi_i(x) = 0$ and $\varphi_i(F_{n_i}) = \{2^{-i}\}$ we would have that $\varphi_i \notin K_i$ and hence $\Sigma_n\varphi_n \notin \cup\{K_i : i \in \mathbb{N}\} = C_p(X)$. As this relation contradicts the continuity of $\Sigma_n\varphi_n$ we have that X is a P-space.

If $\{x_n : n \in \omega\}$ were an infinite subset of the P-space X then, by induction, each x_n would have a neighborhood V_n such that $\overline{V_n} \cap \overline{V_m} = \emptyset$ when $n \neq m$. For each $n \in \omega$ let $\alpha_n = 1 + \sup\{|f(x_n)| : f \in K_n\}$ and let $g_n \in C_p(X)$ be such that $g_n(X) \subset [0, \alpha_n]$, $g_n(x_n) = \alpha_n$ and $g_n(X\setminus V_n) = \{0\}$. The contradiction that the continuous function $g = \Sigma_n g_n \notin \cup\{K_i : i \in \mathbb{N}\} = C_p(X)$ proves that X is finite.

Conversely, if the cardinal of X is n then $C_p(X)$ is the σ-compact space $\mathbb{R}^n$. Therefore the following conditions are equivalent:

1. $C_p(X)$ is σ-compact.
2. $C_p(X)$ is σ-countably compact.
3. X is finite.

The equivalence between 1 and 3 was obtained by Velichko [5, Theorem I.2.1] and the equivalence between 2 and 3 was due to Tkachuk and Shakhmatov in [22].

6.2.4 The Support of a Linear Continuous Functional on $C_c(X)$

Let $\mathscr{K}$ be the family of compact subsets of X. In the space $C_c(X)$ of continuous real-valued functions on X endowed with the compact open topology, the family $\{W(K, (-\varepsilon, \varepsilon)) : K \in \mathscr{K}, \varepsilon > 0\}$ is a base of neighborhoods of 0 in $C_c(X)$. Let us observe the next Claim, as easy as helpful:

Claim If the sets $K \in \mathscr{K}$ and $D \subset X$ verify inclusion

$$W(K, (-\varepsilon, \varepsilon)) \subset W(D, [-m, m]),$$

with $m > 0$, then $D \subset K$.

If $x \in D \backslash K$ then there exists $f \in C(X)$ such that $f(K) = \{0\}$ and $f(x) = 2m$. Hence $f \in W(K, (-\varepsilon, \varepsilon)) \backslash W(D, [-m, m])$. This contradiction implies the Claim.

The name *compact open topology* comes from the fact that if $\mathscr{B}$ is a base of the topology of $\mathbb{R}$ then the family of sets $\{W(K, V) : K \in \mathscr{K}, V \in \mathscr{B}\}$ is a subbase of $C_c(X)$.

Recall that a subset in a locally convex space E is a *barrel* if it is absorbing, closed and absolutely convex. If V is a subset of a locally convex space E its polar is the set $V^{\circ} := \{f \in E' : f(V) \subset [-1, 1]\}$ and if V is a barrel then $V = V^{\circ\circ}$ [13, 20.8 (5)]. E is *barrelled* if every barrel is a neighborhood of 0, or, equivalently, if each bounded subset of $(E', \sigma(E', E))$ is equicontinuous, where the weak* topology $\sigma(E', E)$ is the topology induced on the topological dual E' of E by $C_p(E)$ [13, 21.2 and 27]. Hence a locally convex space E is barrelled if and only if its topology coincides with the strong topology $\beta(E, E')$, which is the topology of the uniform convergence on the bounded subsets of $(E', \sigma(E', E))$ [13, 21.2 (2)].

By Baire's category theorem it follows that Banach and Fréchet spaces are barrelled spaces. Then, since the set of the real continuous bounded functions in X with the supremum norm is a Banach space, if V is a barrel in $C_c(X)$ there exists a $\delta_V > 0$ such that $W(X, [-\delta_V, \delta_V]\} \subset V$. If additionally there exists in X a subset K such that $W(K, \{0\}\} \subset V$ then

$$W(K, [-\delta_V/2, \delta_V/2]) \subset V \tag{6.1}$$

because from $f(K) \subset [-\delta_V/2, \delta_V/2]$ it follows that the functions $g := \sup(f, \delta_V/2) + \inf(f, -\delta_V/2)$ and $f - g$ verify that $2g \in W(K, \{0\}\}(\subset V)$ and $2(f-g) \in W(X, [-\delta_V, \delta_V]\}(\subset V)$. Therefore $f = g + (f-g) \in 2^{-1}V + 2^{-1}V = V$. From (6.1) follows the next sufficient condition for a barrel to be a neighborhood of zero.

Remark 6.1 If a barrel V in $C_c(X)$ contains a set $W(K, \{0\})$, with K compact, then V is a neighborhood of zero in $C_c(X)$.

For each $\varphi \in C_c(X)'$ the barrel $V := \{f \in C_c(X) : |\varphi(f)| \leqslant 1\}$ is a neighborhood of 0 in $C_c(X)$ and then there exists a compact subset K in X and a positive number ε such that $W(K, (-\varepsilon, \varepsilon)) \subset V$ and, in particular, $W(K, \{0\}) \subset \varphi^{-1}\{0\}$. Therefore the family $\mathscr{K}_\varphi := \{K_i : i \in I\}$ of compact subsets of X such that $W(K_i, \{0\}) \subset \varphi^{-1}\{0\}$ for each $i \in I$ is nonvoid and it verifies the following two claims:

Claim $K := \cap\{K_i : i \in I\} \in \mathscr{K}_\varphi$.

If this Claim were not true then there would exist $f \in C_c(X)$ with $f(K) = \{0\}$ and $\varphi(f) = 2$. Let δ_V be the number considered in (6.1) for the barrel $V := \{f \in C_c(X) : |\varphi(f)| \leqslant 1\}$. Then $\{x \in X : |f(x)| < \delta_V/2\}$ is a neighborhood of K and, by compactness, there exists $K_i \subset \{x \in X : |f(x)| < \delta_V/2\}$. From $f \in W(K_i, [-\delta_V/2, \delta_V/2]\}$ we deduce by (6.1) that $f \in V$ obtaining the contradiction $|\varphi(f)| \leqslant 1$. Therefore $K \in \mathscr{K}_\varphi$ and K is the least compact subset of X such that $W(K, \{0\}) \subset \varphi^{-1}\{0\}$. It is said that K is the support $s(\varphi)$ of φ.

Claim If $K \in \mathscr{K}_\varphi$ and F is a closed subset of X with the property that $W(F, \{0\}) \subset \varphi^{-1}(0)$ then $K \cap F \in \mathscr{K}_\varphi$.

If this were not true then there would exist $f \in W(K \cap F, \{0\})$ such that $\varphi(f) = 2$. Again let δ_V be the number considered in (6.1) for the barrel $V := \{f \in C_c(X) : |\varphi(f)| \leqslant 1\}$. For the set $K_{\delta_V/2} := \{x \in K : |f(x)| \geqslant \delta_V/2\}$ there exists $n \in C_c(X)$ such that $n(X) \subset [0, 1]$, $n(K_{\delta_V/2}) = 0$ and $n(F) = \{1\}$. Then $f - f \times n \in W(F, \{0\})$ implies that $\varphi(f - f \times n) = 0$ and then $\varphi(f \times n) = \varphi(f) = 2$. From $(f \times n) \in W(K, [-\delta_V/2, \delta_V/2]\}$ and (6.1) follows that $f \times n \subset V$ and we get the contradiction $|\varphi(f \times n)| \leqslant 1$.

The relation $K \cap F \in \mathscr{K}_\varphi$ and the minimality of the $s(\varphi)$ implies that $s(\varphi) \subset K \cap F$.

6.2.5 Nachbin-Shirota Barrelledness Theorem [16, 21]

Examples of properties of X that may be characterized by properties of $C_c(X)$ may be seen in [12, Theorems 2.10, 2.11, 2.13 and 2.14 and Propositions 2.15 and 2.16]. For instance the space X is realcompact if and only if $C_c(X)$ is an inductive limit of normed (or Banach) spaces [12, Proposition 2.16]. In the next theorem, obtained independently by Nachbin and Shirota in [16] and [21], it was proved that $C_c(X)$ is barrelled if and only X is a μ-space (see also [12, Proposition 2.15] and [20, Theorem 10.1.20]).

Recall that a set A in a space X is *functionally bounded* if every continuous real-valued function on X is bounded on A. A space X is a μ-space if every closed functionally bounded subspace of X is compact. As a realcompact space X embeds into a closed subset of $\mathbb{R}^{C(X)}$ by the map defined by $\varphi(x) := \{f(x) : f \in C(X)\}$ then each functionally bounded subset of a realcompact space X is relatively compact and therefore X is a μ-space [20, Observation 10.1.19]. In particular, Lindelöf spaces are μ-spaces [12, Proposition 3.12]. Examples of μ-spaces which are not realcompact can be found in [16] and [21].

Theorem 6.1 *The topological space X is a μ-space if and only if $C_c(X)$ is barrelled.*

Proof Let us suppose that X is a μ-space. If V is a barrel in $C_c(X)$ and

$$K := \overline{\cup\{s(\varphi) : \varphi \in V^{\mathrm{o}}\}}$$

then from $\{f \in C(X) : f(K) = \{0\}\} \subset V^{\mathrm{oo}} = V$ and Remark 6.1 it follows that in order to prove the barrelledness of $C_c(X)$ it suffices to show that K is compact.

If K were a non compact subset of the μ-space X then there would exist $f \in C_c(X)$ unbounded on K. By an easy induction, we may obtain an increasing sequence $(n_i)_i$ of natural numbers, a sequence $(\varphi_i)_i$ in V^{o} and a sequence (g_i) in $C_c(X)$, with $\varphi_i(g_i) = 1$, such that for each $i \in \mathbb{N}$ the sets $A_i = \{x \in K : |f(x)| > n_i\}$ verify that $s(\varphi_i) \cap A_i \neq \emptyset$, $s(\varphi_j) \cap A_i = \emptyset$, when $j < i$, and $g_i(X \backslash A_i) = \{0\}$. In fact, for $n_1 = 1$ there exists $\varphi_1 \in V^{\mathrm{o}}$ such that $s(\varphi_1) \cap A_1 \neq \emptyset$. Recall that if $g \in C_c(X)$ and $g(X \backslash A_1) = \{0\}$ would imply $\varphi_1(g) = 0$ then, by Claim 6.2.4, $s(\varphi_1) \subset s(\varphi_1) \cap (X \backslash A_1)$, in contradiction with $s(\varphi_1) \cap A_1 \neq \emptyset$. Therefore there exists $g_1 \in C_c(X)$ such that $g_1(X \backslash A_1) = \{0\}$ and $\varphi_1(g_1) = 1$. Now fix n_2 $(> n_1)$ such that $s(\varphi_1) \cap A_2 = \emptyset$ and then choose $\varphi_2 \in V^{\mathrm{o}}$ with $s(\varphi_2) \cap A_2 \neq \emptyset$; as in preceding case there exists $g_2 \in C_c(X)$ such that $g_2(X \backslash A_2) = \{0\}$ and $\varphi_2(g_2) = 1$. The induction follows in an obvious way.

The function $g := \sum_{i=1}^{\infty} c_i g_i \in C_c(X)$ for any choice of coefficients c_i, $i \in \mathbb{N}$, because $\left(\sum_{i=j+1}^{\infty} c_i g_i\right)(X \backslash A_{j+1}) = \{0\}$. Then the relation $\varphi_j(g) = \sum_{i=1}^{i=j} c_i \varphi_j(g_i) = \sum_{i=1}^{i=j-1} c_i \varphi_j(g_i) + c_j$ enables to select $(c_i)_i$ such that $\varphi_j(g) = j+1$. As each $\varphi_j \in V^o$ we get that $g \notin iV$, for each $i \in \mathbb{N}$, contradicting that V is absorbing. Therefore K is compact and $C_c(X)$ is barrelled by Remark 6.1.

To prove the converse let D be a closed noncompact subset in X. If the barrel $W(D, [-1, +1])$ were a neighborhood of zero, then there would exist a compact subset K and a positive number ε such that $W(K, (-\varepsilon, \varepsilon)) \subset W(D, [-1, +1])$. Then Claim 6.2.4 implies that $D \subset K$, which contradicts the noncompactness of D. Therefore if $C_c(X)$ is barrelled the set $W(D, [-1, +1])$ fails to be absorbent and hence there exists $f \in C_c(X)$ unbounded in D. This proves that X is a μ-space. □

6.2.6 μ-Spaces and ℓ-Equivalence

In the class of μ-spaces we have that if $h : C_p(X) \to C_p(Y)$ is a linear homeomorphism then $h : C_c(X) \to C_c(Y)$ is also a linear homeomorphism, hence ℓ_p-equivalence implies ℓ_c-equivalence. This result follows from the following facts:

If $h : C_p(X) \to C_p(Y)$ is a linear homeomorphism then its dual map $h' : L_p(Y) \to L_p(X)$ is also a linear homeomorphism by [13, 20.4.6]. This implies that a subset

B of $L_p(Y)$ is bounded if and only if $h'(B)$ is bounded and then by [13, 21.2] we have that $h : (C(X), \beta(C_p(X), L_p(X))) \to (C(Y), \beta(C_p(Y), L_p(Y)))$ is also a linear homeomorphism.

If Z is a μ-space and $\tau_c(Z)$ is the topology of $C_c(Z)$ then $\tau_c(Z) = \beta(C_p(Z), L_p(Z))$ because:

1. The homeomorphism from Z onto the subspace $\hat{Z}$ of $L_p(Z)$ imply that $\tau_c(Z) \leqslant \beta(C_p(Z), L_p(Z))$.
2. The identity map is a continuous immersion from $C_c(Z)$ onto $C_p(Z)$ and then $\beta(C_p(Z), L_p(Z)) \leqslant \beta(C_c(Z), C_c(Z)')$.
3. Finally, Theorem 6.1 implies that $\beta(C_c(Z), C_c(Z)') = \tau_c(Z)$.

In [4, Theorems 1 and 3], by a combination of Milyutin's and Pestov's results, it is proved that [0, 1] and [0, 1] $\times$ [0, 1] are ℓ_c-equivalent but not ℓ_p-equivalent.

6.3 Michael's Results on $\aleph_0$-Spaces

All the propositions of this section were obtained by E. Michael in [15].

6.3.1 Elementary Properties. Heritability by k-Equivalence

A family $\mathscr{N}$ of subsets of a space X is called a k-network in X if, whenever $K \subset U$, with K compact and U open in X, then $K \subset \cup\{F : F \in \mathscr{F}\} \subset U$ for some finite family $\mathscr{F} \subset \mathscr{N}$. A topological space X is called $\aleph_0$-space if it is regular and it has a countable k-network. First-countable or locally compact $\aleph_0$-spaces are separable and metrizable. Any subspace of an $\aleph_0$-space is an $\aleph_0$-space and the countable product of a family of $\aleph_0$-spaces is an $\aleph_0$-space. Every $\aleph_0$-space X is Lindelöf (hence normal [12, Lemma 6.1]), hereditary separable and every open subset is an F_σ. These direct properties are proved in [15].

Two Hausdorff topological spaces X and Y are k-equivalent if there exists a bijection $f : X \to Y$ such that f and f^{-1} preserve the compact subsets. More precisely, it is said that X and Y are k-equivalent by the bijection f. Two Hausdorff topologies τ_1 and τ_2 on a set X are *k-equivalent* if they yield the same compact subsets. Let (X, τ) be a topological space; from Tychonoff's product theorem it follows that the topology τ is k-equivalent to the supremum τ_k of the family of all topologies k-equivalent to τ. The topology τ_k is the finest topology k-equivalent to τ and it is said that τ_k is the k-topology associated to τ and that $X_k := (X, \tau_k)$ is the k-space associated to (X, τ).

A topological space (X, τ) is a k-space if $\tau = \tau_k$ and (X, τ) is a k-space if and only if a subset A is open when for each compact subset C the intersection $C \cap A$ is an open subset in C with the induced topology.

A function $f : (X, \tau) \to Y$ is k-continuous if its restriction to every compact subset of X is continuous. This happens if and only if $f : (X, \tau_k) \to Y$ is continuous.

It is obvious that if the Hausdorff topological spaces (X, τ) and Y are k-equivalent by the bijection f, (X, τ) is a k-space and f^{-1} is continuous then f is a homeomophism, because $f : (X, \tau_k) \to Y$ is continuous and $\tau = \tau_k$.

Proposition 6.1 ([15, Proposition 8.2]) *If $\mathscr{J}_1$ and $\mathscr{J}_2$ are two k-equivalent topologies on X then $(X, \mathscr{J}_2)$ has a countable k-network if and only if $(X, \mathscr{J}_1)$ does.*

Proof Let $\mathscr{P}$ be a countable k-network for $\mathscr{J}_1$ closed under finite intersections and suppose that $\mathscr{P}$ is not a countable k-network for $\mathscr{J}_2$. Then there exists a compact subset C contained in a $\mathscr{J}_2$-open subset U_2 such that, if $(P_n)_n$ is an enumeration of the elements of $\mathscr{P}$ containing C, then for each n there exists $x_n \in P'_n \backslash U_2$, with $P'_n = \cap \{P_m : 1 \leqslant m \leqslant n\}$. If $C \subset U$ and U is a $\mathscr{J}_1$ open there exists $P'_{n(U)}$ such that $C \subset P'_{n(U)} \subset U$. Therefore $\{x_m : n(U) \leqslant m\} \subset U$ and from this inclusion it follows that $C \cup \{x_n : n \in \omega\}$ is a compact set. Let U_1 be the $\mathscr{J}_1$ open set such that

$$U_1 \cap [C \cup \{x_n : n \in \omega\}] = U_2 \cap [C \cup \{x_n : n \in \omega\}] .$$

From this equality it follows a contradiction because $\{x_m : n(U_1) \leqslant m\} \subset U_1$ and $U_2 \cap [C \cup \{x_n : n \in \omega\}] = C$. □

6.3.2 $C_c(X)$ $\aleph_0$-Spaces

In [15, Theorem 9.3] Michael proves that if X is a Hausdorff space with countable k-network (named pseudobase in [15]) and if Y is and $\aleph_0$-space then $C_c(X, Y)$ is and $\aleph_0$-space. The below Theorem 6.2 is the particular case of this result needed to proof Theorem 6.3.

Lemma 6.1 ([15, Lemma 9.1]) *Let M be a compact subset of X. Then the mapping $\psi : M \times C_c(X) \to C_c(X)$ such that $\psi(x,f) = f(x)$ is continuous.*

Proof Fix $(x,f) \in M \times C_c(X)$. Let V be a neighborhood of $f(x)$ and D be a closed neighborhood of x in M such that $f(D) \subset V$. Then D and $W(D, V)$ verify that $\psi((D \times W(D, V))) \subset V$. Therefore ψ is continuous in (x,f). □

Lemma 6.2 ([15, Lemma 9.2]) *Let K be a compact subset of $C_c(X)$ and let U be an open subset of $\mathbb{R}$. If X is a k-space then $V := \{x \in X : K(x) \subset U\}$ is open.*

Proof It is enough to prove that for each compact subset M of X the set

$$V \cap M = \{x \in M : K(x) \subset U\}$$

is open in M. Let x be a point in $V \cap M$. Then, by Lemma 6.1, for each $f \in K$ there exists a M-neighborhood D_f of x and a neighborhood W_f of f such that $W_f(D_f) \subset U$. By compactness there exists a finite set $\{f_1, f_2, \ldots, f_n\} \subset K$ such that $K \subset \cup \left\{W_{f_i} : 1 \leqslant i \leqslant n\right\}$. This implies that $K \left(\cap D_{f_i}\right) \subset U$ and then $\cap D_{f_i} \subset V \cap M$. This proves that $V \cap M$ is open in M. □

To proof the next Lemma 6.3 we need to observe that if a normal space X is the union of a finite family $\{A_i : 1 \leqslant i \leqslant n\}$ of open subsets, then X is covered by a family $\{B_i : 1 \leqslant i \leqslant n\}$ of open subsets such that $\overline{B}_i \subset A_i$, for each $1 \leqslant i \leqslant n$. In fact, as A_1 is an open neighborhood of the closed set $X \setminus \cup\{A_i : 2 \leqslant i \leqslant n\}$ then, by normality, there exists an open set B_1 such that $X \setminus \cup\{A_i : 2 \leqslant i \leqslant n\} \subset B_1 \subset \overline{B}_1 \subset A_1$. Hence $X = B_1 \cup \{A_i : 2 \leqslant i \leqslant n\}$. After n-steps of an obvious induction process we get the open covering $X = \cup\{B_i : 1 \leqslant i \leqslant n\}$, with $\overline{B}_i \subset A_i$, for $1 \leqslant i \leqslant n$.

Lemma 6.3 ([15, Lemma 5.1]) *Let $\mathscr{S}$ be a subbase of the Hausdorff topological space X and let $\mathscr{P}$ be a collection of subsets of X such that for each compact set C and each open set $U \in \mathscr{S}$ there exists $P \in \mathscr{P}$ such that $C \subset P \subset U$. Then X has a countable k-network.*

Proof Let $\mathscr{R}$ be the family of finite unions of finite intersections of elements of $\mathscr{P}$. We will prove that the countable family $\mathscr{R}$ is a k-network for X. Let C be a compact subset of X and let U be an open neighborhood of C. The proof is trivial if U is a finite intersection $\cap\{S_i : 1 \leqslant i \leqslant n\}$ of elements of $\mathscr{S}$, because then for each i there exist P_i such that $C \subset P_i \subset S_i$. Hence $C \subset \cap\{P_i : 1 \leqslant i \leqslant n\} \subset \cap\{S_i : 1 \leqslant i \leqslant n\} = U$ and $R := \cap\{P_i : 1 \leqslant i \leqslant n\} \in \mathscr{R}$. To finish the proof recall that the base $\mathscr{B}$ generated by $\mathscr{S}$ is the family formed by the finite intersections of elements of $\mathscr{S}$. By compactness we may suppose that $C \subset \cup\left\{A_j : 1 \leqslant j \leqslant p\right\}$, with $A_j \in \mathscr{B}$ and $A_j \subset U$ for each $1 \leqslant j \leqslant p$. By the observation preceding this lemma we deduce that $C = \cup\left\{\overline{B}_j : 1 \leqslant j \leqslant p\right\}$, where each $\overline{B}_j$ is a compact set contained in A_j. We have proved that there exists $R_j \in \mathscr{R}$ such that $\overline{B}_j \subset R_j \subset A_j$ for each $1 \leqslant j \leqslant p$. From these relations it follows that the set $R = \cup\left\{R_j : 1 \leqslant j \leqslant p\right\} \in \mathscr{R}$ and verifies the inclusions $C \subset R \subset U$. □

Theorem 6.2 ([15, Theorem 9.3]) *If X is an $\aleph_0$-space then $C_c(X)$ is an $\aleph_0$-space.*

Proof By the Proposition 6.1 we know that kX is an $\aleph_0$-space. As $C_c(X)$ is a subspace of $C_c(kX)$ it is enough to prove the theorem in the case that X is a k-space. Let $\{Q_m : m \in \omega\}$ be a countable k-network in X and let $\mathscr{B} = \{B_n : n \in \omega\}$ be a base of the topology of $\mathbb{R}$. By Lemma 6.3 it is enough to prove that there exists a countable family $\mathscr{P}$ of subsets of $C_c(X)$ such that for each compact subset K of $C_c(X)$, each compact subset M of X and each $B_n \in \mathscr{B}$ such that $K \subset W(M, B_n)$ there exists $P \in \mathscr{P}$ that verifies $K \subset P \subset W(M, B_n)$. Clearly $M \subset \{x \in X : K(x) \subset B_n\}$ and then, by Lemma 6.2 there exists Q_m such that $M \subset Q_m \subset \{x \in X : K(x) \subset B_n\}$. Hence $K \subset W(Q_m, B_n) \subset W(M, B_n)$ and then $\mathscr{P} := \{W(Q_m, B_n) : (m, n) \in \omega^2\}$ is the countable family we are looking for. □

6.4 Preservation of $\aleph_0$-Spaces by ℓ_c-Equivalence

The main result in this section is that $\aleph_0$-spaces are preserved by ℓ_c-equivalence. It was obtained in [12, Theorem 21].

As $C_p(C_p(X))$ is a topological subspace of $C_p(C_c(X))$ we have that the map $\hat{\Delta}_X : X \to C_p(C_c(X))$, defined by $\hat{\Delta}_X(x) := \hat{x}$ with $\hat{x}(f) = f(x)$ for each $f \in C(X)$, is an embedding. Then X is homeomorphic to the space $\hat{X}$, which is the set $\hat{\Delta}_X(X)$ endowed with the topology induced by $C_p(C_c(X))$.

In general, the map $\tilde{\Delta}_X : X \to C_c(C_c(X))$ defined by $\tilde{\Delta}_X(x) = \tilde{x}$, where also $\tilde{x}(f) = f(x)$ for each $f \in C(X)$, is not continuous because the topology induced in $\tilde{\Delta}_X(X)$ by $C_c(C_c(X))$ is finer that the topology induced by $C_p(C_c(X))$. Therefore the map $\tilde{\Delta}_X$ is continuous if and only if it is an embedding. The space $\tilde{X}$ is the set $\tilde{\Delta}_X(X)$ equipped with the topology induced by $C_c(C_c(X))$.

Lemma 6.4 ([12, Lemma 20]) *The map $\tilde{\Delta}_X$ is k-continuous. Therefore the topological spaces $\hat{X}$ and $\tilde{X}$ are k-equivalents by the map i defined by $i(\hat{x}) = \tilde{x}$ for each $x \in X$ and if X is a k-space then the map $\tilde{\Delta}_X$ is an embedding.*

Proof Let M be a compact subset of X and $F = C_c(C_c(X)) \setminus W(K, V)$, where K is a compact subset of $C_c(X)$ and V is an open subset of $\mathbb{R}$. Let x be the limit of a convergent net $(x_l : l \in L)$, with each $x_l \in M \cap \tilde{\Delta}_X^{-1}(F)$. For each $l \in L$ fix $f_l \in K$ with $f_l(x_l) \notin V$. By compactness we may suppose, taking a subnet if necessary, that the net $(f_l : l \in L)$ converges uniformly on M to some $f \in K$. Then $(f_l(x_l) : l \in L)$ converges to $f(x)$ and the relation $f_l(x_l) \notin V$, for each $l \in l$, implies $f(x) \notin V$. Therefore $x \in M \cap \Delta_X^{-1}(F)$. This proves that the map $\tilde{\Delta}_X$ is k-continuous, and then the remaining assertions are obvious. □

The first part of the next theorem extends Theorem 21 in [11]. It is well known that the $\aleph_0$-space property in not preserved by open maps.

Theorem 6.3 *If X is an $\aleph_0$-space and there is a continuous linear mapping h from $C_c(X)$ onto $C_c(Y)$, then Y is an $\aleph_0$-space. In particular the property of being $\aleph_0$-space is preserved by ℓ_c-equivalence.*

Proof The injective dual mapping $h^* : C(C_c(Y)) \to C(C_c(X))$ verifies that $h^* : C_c(C_c(Y)) \to C_c(C_c(X))$ is continuous and that $h^* : C_p(C_c(Y)) \to C_p(C_c(X))$ is an embedding. We endow $h^*(\tilde{Y})$ and $h^*(\hat{Y})$ with the topologies induced by $C_c(C_c(X))$ and $C_p(C_c(X))$, respectively. Let $j : h^*(\hat{Y}) \to h^*(\tilde{Y})$ be the map defined by $j(h^*(\hat{y})) = h^*(\tilde{y})$, for each $y \in Y$. Since j is the restriction to $h^*(\hat{Y})$ of the natural embedding of $C_p(C_c(X))$ onto $C_c(C_c(X))$, then the inverse map j^{-1} is continuous. By Lemma 6.4, $\hat{Y}$ and $\tilde{Y}$ are k-equivalent by the map $i : \hat{Y} \to \tilde{Y}$ such that $i(\hat{y}) = \tilde{y}$ for each $y \in Y$. By definition $h^*i = jh^*|_{\hat{Y}}$.

If M is a compact subset of $\hat{Y}$ then, by k-equivalence and continuity, we have that $h^*i(M)$ is a compact subset of $h^*(\tilde{Y})$. If K is a compact subset of $h^*(\tilde{Y})$ then by continuity $\left(jh^*|_{\hat{Y}}\right)^{-1}(K)$ is a compact subset of $\hat{Y}$. Therefore $h^*(\tilde{Y})$ is k-equivalent to $\hat{Y}$ and, by homeomorphism is also k-equivalent to Y.

From Michael's Theorem 6.2 it follows that $C_c(C_c(X))$ is an $\aleph_0$-space. So its subspace $h^*(\tilde{Y})$ is an $\aleph_0$-space. Proposition 6.1 implies that Y is an $\aleph_0$-space. □

Let $\mathfrak{D}$ be the class of first countable or locally compact spaces. From this theorem and Michael results asserting that $\aleph_0$-spaces in $\mathfrak{D}$ are metrizable [15], it follows the next corollary (the first countable case is in [11, Carollary 22]).

Corollary 6.1 *If X is an $\aleph_0$-space, $Y \in \mathfrak{D}$ and there exists a continuous linear map h from $C_c(X)$ onto $C_c(Y)$, then Y is metrizable and separable. In particular, the property of being metrizable and separable is preserved by ℓ_c-equivalence in the class $\mathfrak{D}$.*

As a regular space is second countable if and only if it is metrizable and separable we get that this corollary applies when X is second countable, providing and extension of Arhangel'skii's [3, Theorem 16].

In the next section we are going to prove that the property of being a Polish space is also invariant by ℓ_c-equivalence in this class $\mathfrak{D}$, extending [11, Corollary 23].

6.5 Preservation of Polish Spaces by ℓ_c-Equivalence

Recall that a topological space X is a *Polish space* if it is separable and completely metrizable.

6.5.1 Compact Resolutions and $\mathfrak{G}$-Bases

A family $C = \left\{C_{n_1 \cdots n_{p-1} n_p} : n_i \in \omega, 1 \leqslant i \leqslant p\right\}$ of subsets of X is a *web* if $X = \cup_{n_1} C_{n_1}$ and $C_{n_1 \cdots n_{p-1}} = \cup_{n_p} C_{n_1 \cdots n_{p-1} n_p}$, for each $p \geqslant 2$.

A *resolution* X is a family $\mathscr{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ of subsets of X such that $X = \cup_\alpha K_\alpha$ and $K_\alpha \subset K_\beta$ if $\alpha \leqslant \beta$. For $\eta = (n_i : i \in \omega) \in \omega^\omega$ and $p \in \omega$ we write that $\eta|p := (n_i : 1 \leqslant i \leqslant p)$ and then the sets $C_{n_1 \cdots n_{p-1} n_p} = \cup \{A_\alpha : \alpha \in \omega^\omega, \alpha|p = \eta|p\}$ determine the web $\mathscr{C}$ associate to the resolution $\mathscr{K}$. A resolution $\{K_\alpha : \alpha \in \omega^\omega\}$ in a topological space X is compact if each K_α is compact. If, additionally, for each compact subset K of X there exists $K_\alpha \in \mathscr{K}$ such that $K \subset K_\alpha$ then it is said that $\mathscr{K}$ is a *compact resolution swallowing compact sets*.

The name resolution appears in [8]. In [23] a space endowed with a compact resolution is called a *space dominated by irrationals*.

A base $\{U_\alpha : \alpha \in \omega^\omega\}$ of neighborhoods of zero in the topological vector space E is a $\mathfrak{G}$-*base* if $U_\beta \subset U_\alpha$ whenever $\alpha \leq \beta$ [12, Lemma 15.2 (iii)]. If f is a linear continuous open map from the topological vector space E onto the topological vector space F and $\{U_\alpha : \alpha \in \omega^\omega\}$ is a $\mathfrak{G}$- base in E then $\{f(U_\alpha) : \alpha \in \omega^\omega\}$ is a $\mathfrak{G}$-base in F.

The following proposition given in [10, Theorem 2] furnishes a relation between resolutions in X and $\mathfrak{G}$-bases in $C_c(X)$.

Proposition 6.2 ([10, Theorem 2]) *A completely regular topological space X has a compact resolution $\mathscr{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ swallowing compact sets if and only if $C_c(X)$ has a $\mathfrak{G}$-base.*

Proof If $\mathscr{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ is a compact resolution on X swallowing compact sets then for each $K \in \mathscr{K}$ and $\varepsilon > 0$ there exists K_α, with $\alpha = (a_n)_n \in \omega^\omega$, such that $K \subset K_\alpha$ and $a_1^{-1} < \varepsilon$. Let $U_\alpha := W(K_\alpha, (-a_1^{-1}, a_1^{-1}))$. As $U_\alpha \subset W(K, \varepsilon)$ and $U_\beta \subset U_\alpha$ when $\alpha \leq \beta$ we have that $\mathscr{U} = \{U_\alpha : \alpha \in \omega^\omega\}$ is a $\mathfrak{G}$-base in $C_c(X)$.

Conversely, let $\mathscr{U} = \{U_\alpha : \alpha \in \omega^\omega\}$ be a $\mathfrak{G}$-base of $C_c(X)$. For each $\alpha = (a_n)_n \in \omega^\omega$ there exists $K \in \mathscr{K}$ and $\varepsilon > 0$ such that $W(K, (-\varepsilon, \varepsilon)) \subset U_\alpha$. As the set $K_\alpha := \{x \in X : |f(x)| \leq a_1 \text{ for all } f \in U_\alpha\}$ verifies that $U_\alpha \subset W(K_\alpha, [-a_1, a_1])$ then $W(K, (-\varepsilon, \varepsilon)) \subset W(K_\alpha, [-a_1, a_1])$ and Claim in 6.2.4 imply that $K_\alpha \subset K$. Therefore the closed set K_α is compact. If D is a compact subset of X then there exists $\alpha = (a_n)_n \in \omega^\omega$ such that $U_\alpha \subset W(D, [-1, 1]\}$. Therefore for each $x \in D$ and each $f \in U_\alpha$ we have that $|f(x)| \leq 1$ ($\leq a_1$), hence, by definition, $x \in K_\alpha$. The obtained inclusion $D \subset K_\alpha$ and and the fact $K_\alpha \subset K_\beta$ when $\alpha \leqslant \beta$ in ω^ω imply that $\mathscr{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ is a compact resolution swallowing compact subsets of X. □

6.5.2 Christensen Theorem

Let A be an uncountable subset of X endowed with a resolution $\mathscr{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ and let $\mathscr{C}$ be the web associate to $\mathscr{K}$. By induction we get a sequence $\eta \in \omega^\omega$ such that $A \cap C_{\eta|p}$ is uncountable for each $p \in \omega$. Therefore there exists an infinite subset $\{x_p : p \in \omega\}$ in X and a sequence $\{\beta(p) : p \in \omega\}$ in ω^ω such that $x_p \in K_{\beta(p)}$ and $\beta(p)|p = \eta|p$ for each $p \in \omega$. If $\alpha := \sup\{\beta(p) : p \in \omega\}$ we have that $\{x_p : p \in \omega\} \subset K_\alpha$. This implies that if (X, d) is a metric space with a compact resolution $\mathscr{K}$ and A_n is a maximal subset of X such that $d(x, y) \geqslant n^{-1}$, for each $(x, y) \in A_n^2$ with $x \neq y$, then A_n is countable, because the intersection of A_n with every compact subset must be finite. The maximality of each A_n implies that $\cup\{A_n : n \in \omega\}$ is a dense subset of the space (X, d). Therefore a metric space with a compact resolution is separable.

A separable and complete metric space (X, d) has a compact resolution $\mathscr{K}$ swallowing compact sets, because if $\{x_n : n \in \omega\}$ is a dense subset of (X, d), $B(x, \varepsilon)$ is the closed ball of center x and radius ε and for $\alpha = (a_n)_n \in \omega^\omega$, $K_{a_n} := \cup\left\{B(x_m; n^{-1}) : m \leqslant a_n\right\}$ and $K_\alpha := \cap\left\{K_{a_n} : n \in \omega\right\}$ we get that K_α is compact, because is closed and precompact, and if K is a compact subset of X then, by compact definition, for each n there exists $b_n \in \omega$ such that $K \subset K_{b_n}$. From this inclusion it follows that $K \subset K_\beta$, with $\beta := (b_n : n \in \omega)$. Then $\{K_\alpha : \alpha \in \omega^\omega\}$ is a compact resolution in X swallowing compact subsets.

This proves that each Polish space X has a compact resolution swallowing compact sets. To proof Theorem 6.5 we need the converse of this result, which is a deep theorem due to Christensen [7]. A detailed proof of Christensen's theorem may be found in [12, Theorem 6.1] and the essential parts of this proof are below in Theorem 6.4.

If F is a subset of a metric space (X, d) then the equality $d(x, F) = \inf_{y \in F} d(x, y)$ defines a real continuous function on X, because from $d(x, F) \leqslant d(x, y) \leqslant d(x, z)$

$+\, d(z, y)$ for each $y \in F$ it follows that $d(x, F) \leqslant d(x, z) + d(z, F)$, and then, by symmetry, $|d(z, F) - d(x, F)| \leqslant d(z, x)$. The continuity of $d(x, F)$ is the key in the proof of the next lemma.

Lemma 6.5 ([9, 4.3.22 Lemma]) *Every G_δ-set A in a metric space (X, d) is topologically homeomorphic to a closed subspace of the cartesian product $X \times \mathbb{R}^\omega$. Therefore every G_δ-set A in a complete metrizable space (X, d) is topologically complete.*

Proof $X \backslash A$ is the union of a sequence $(F_n)_{n\in\omega}$ of closed subsets of X. The continuity of each $d(x, F_n)$ implies that the injective map f from A into $X \times \mathbb{R}^\omega$ defined by $f(x) = \big(x, \big([d(x, F_n)]^{-1}, n \in \omega\big)\big)$ is a homeomorphism from A onto $f(A)$. If there exists a sequence $(f(x_p))_{p\in\omega}$ in $f(A)$ that converges to $(x, (r_n, n \in \omega))$ then $x = \lim_p x_p$ and $\lim_p [d(x_p, F_n)]^{-1} = r_n$ for each $n \in \omega$. If $x \notin A$ then there exists $m \in \omega$ such that $x \in F_m$ and we obtain the contradiction $\lim_p \big[d(x_p, F_m)\big]^{-1} = \infty$. Therefore $f(A)$ is closed. If (X, d) is complete then A is homeomorphic to the closed subspace $f(A)$ of the complete metric space $(X \times \mathbb{R}^\omega, \rho)$, being ρ the usual metric associated to a countable product of metric spaces. Hence A is complete with the metric ρ' induced by f (i.e., $\rho'(x, y) := \rho(f(x), f(y))$ for each $x, y \in A$). The lemma follows from the fact that the topologies induced by d and ρ' on A are the same. □

Theorem 6.4 ([12, Theorem 6.1, Christensen Theorem]) *A metrizable topological space (X, d) has a compact resolution swallowing compact sets if and only if (X, d) is a Polish space.*

Proof Let's suppose that (X, d) has a compact resolution $\mathscr{K} = \{K_\alpha : \alpha \in \omega^\omega\}$ swallowing compact subsets and let $\{x_{n_1} : n_1 \in \omega\}$ be a dense subset in X.

For each x_{n_1} there exists an open ball $B_{n_1} = B(x_{n_1}, p_1(n_1)^{-1})$, with $p_1(n_1) \in \omega$, such that each compact subset K of B_{n_1} is contained in a K_α with $\alpha|1 = p_1(n_1)$. In fact, if this were not true then for each $m \in \omega$ there would exist a compact subset K_m in $B(x_{n_1}, m^{-1})$ not contained in every K_α with $\alpha|1 = m$. As there exists $\beta \in \omega^\omega$ such that the compact set $\{x_{n_1}\} \cup \{\cup K_m : m \in \omega\}$ is contained in K_β, we get contradiction for $m = \beta|1$. Let $p_1 : \omega \to \omega$ be the map whose values are $p_1(n_1)$, $n_1 \in \omega$.

Let $\{x_{n_1 n_2} : n_2 \in \omega\}$ be a dense subset in B_{n_1}. Again for each $x_{n_1 n_2}$ there exists an open ball $B_{n_1 n_2} = B(x_{n_1 n_2}, p_2(n_1, n_2)^{-1})$ contained in B_{n_1} with $p_2(n_1, n_2) \in \omega$, such that each compact subset K of $B_{n_1 n_2}$ is contained in a K_α with $\alpha|2 = (p_1(n_1), p_2(n_1, n_2))$. If this were not true then for each m such that $B(x_{n_1 n_2}, m^{-1}) \subset B_{n_1}$ there would exist a compact subset K_m in $B(x_{n_1 n_2}, m^{-1})$ not contained in each K_α with $\alpha|2 = (p_1(n_1), m)$. The set $H = \{x_{n_1 n_2}\} \cup \big\{\cup K_m : m \in \omega, B(x_{n_1 n_2}, m^{-1}) \subset B_{n_1}\big\}$ is a compact subset of B_{n_1} and we have proved that there exists $\beta \in \omega^\omega$ with $\beta|1 = p_1(n_1)$ such that $H \subset K_\beta$. We may suppose that $\beta|2 = (p_1(n_1), m)$ verifies that $B(x_{n_1 n_2}, m^{-1}) \subset B_{n_1}$ and then the inclusion $K_m \subset K_\beta$ is a contradiction. We denote by p_2 the map defined in ω^2 and generated by the natural numbers $p_2(n_1, n_2)$ obtained in this second step of this induction process.

By a clear induction process we obtain an open web

$$\mathscr{B} = \big\{B_{n_1 \cdots n_{q-1} n_q} : n_i \in \omega, 1 \leqslant i \leqslant q\big\}$$

and a sequence of mappings $(p_i)_i$ such that for each sequence $\eta = (n_i)_i \in \omega^\omega$ and each sequence $(x_q)_q$, with $x_q \in B_{n_1 \cdots n_{q-1} n_q}$ for each $q \in \omega$, there exists a sequence $(\alpha(q))_q$ in ω^ω such that $x_q \in K_{\alpha(q)}$ and

$$\alpha(q)|q = (p_1(n_1), p_2(n_1, n_2), \ldots, p_q(n_1 \cdots n_{q-1} n_q)).$$

From $\sup_q \alpha(q) = \beta \in \omega^\omega$ it follows that the sequence $(x_q)_q$ is contained in the compact set K_β and therefore it has an adherent point in K_β.

Let $\left(\tilde{X}, \tilde{d}\right)$ be the $\tilde{d}$-completion of (X, d). For each $B_{n_1 \cdots n_{q-1} n_q}$ let $A_{n_1 \cdots n_{q-1} n_q}$ be an open subset of $\widetilde{X}$ such that

$$A_{n_1 \cdots n_{q-1} n_q} \cap X = B_{n_1 \cdots n_{q-1} n_q}$$

and let

$$M := \cup \left\{ \cap \{A_{n_1 \cdots n_{q-1} n_q}, q \in \omega\}, \eta = (n_i)_i \in \omega^\omega \right\}.$$

By web's definition it follows that $X \subset M$. For each $y \in M$ we may choose $\eta = (n_i)_i \in \omega^\omega$ such that $y \in A_{n_1 \cdots n_{q-1} n_q}$ for each $q \in \mathbb{N}$. Then, for each $q \in \mathbb{N}$ there exists

$$x_q \in B_{n_1 \cdots n_{q-1} n_q}$$

such that

$$d(y, x_q) < q^{-1}.$$

We have proved that the sequence $(x_q)_g$ has an adherent point in X. Therefore $y = \lim_q x_q \in X$ and then $X = M$. Clearly $\widetilde{X} \backslash X = \widetilde{X} \backslash M$ is the union of the closed set

$$\widetilde{X} \backslash \left\{ \cup [A_{n_1} : n_1 \in \omega] \right\},$$

and the sets

$$A_{n_1 \cdots n_{q-1} n_q} \backslash \left\{ \cup [A_{n_1 \cdots n_{q-1} n_q n_{q+1}} : n_{q+1} \in \omega] \right\}$$

for each $q \in \omega$, $(n_1 \cdots n_{q-1} n_q) \in \omega^q$. By the separability of X we note that each set $A_{n_1 \cdots n_{q-1} n_q}$ is a countable union of closed sets. Hence $\widetilde{X} \backslash X$ is a F_σ- subset of $\widetilde{X}$. Therefore X is a G_δ subset of the complete space $\left(\tilde{X}, \tilde{d}\right)$ and, by Lemma 6.5, (X, d) is a Polish space. □

Let us notice that the web $\mathscr{B} = \left\{ B_{n_1 \cdots n_{q-1} n_q} : n_i \in \omega, 1 \leqslant i \leqslant q \right\}$ in Theorem 6.4 was constructed in such a way that for each $\eta = (n_i : i \in \omega) \in \omega^\omega$ and each $x_q \in B_{n_1 \cdots n_{q-1} n_q}$, with $q \in \omega$, we have that the sequence $(x_q)_q$ has a cluster point in

X. This condition reminds Orihuela's definition of *web-compact spaces* [19] that has many interesting applications (see also [12, Sects. 4.3 and 5.1]).

6.5.3 Preservation of Polish Spaces by ℓ_c-Equivalence in the Class $\mathfrak{D}$

The next theorem on the preservation of Polish spaces under ℓ_c-equivalence in the class $\mathfrak{D}$ of first-countable or locally compact spaces extends Pelant's result [4, Theorem 3.27]. It completes [11, Corollary 23] for locally compact spaces.

Theorem 6.5 *Let X be a Polish space and $Y \in \mathfrak{D}$. If X ℓ_c-cover Y then Y is a Polish space. Therefore the property of being Polish space is invariant by ℓ_c-equivalence in the class $\mathfrak{D}$.*

Proof The space Y is $\aleph_0$-space by Theorem 6.3 and then Y is metrizable by [15]. As X has a compact resolution swallowing compact sets then $C_c(X)$ has a $\mathfrak{G}$-base by Proposition 6.2. Also $C_c(Y)$ has a $\mathfrak{G}$-base, because there exits an open continuous linear map from $C_c(X)$ onto $C_c(Y)$. The Proposition 6.2 implies that Y has a compact resolution swallowing compact sets and then Christensen's Theorem 6.4 implies that Y is a Polish space. □

6.6 Extension to Spaces of Pointwise Countable Type

Corollary 6.1 and Theorem 6.5 stated that second countability and the property of being Polish space are invariants by ℓ_c-equivalence in the class $\mathfrak{D}$ of first countable or locally compact spaces. The ℓ_c-invariance of Polish spaces was extended in [11, Corollary 24] to the class of pointwise countable type spaces. This result (see Corollary 6.3) and an extension of preservation of second countability by ℓ_c-equivalence in the class of pointwise countable type spaces (see Corollary 6.2) are provided in this section.

6.6.1 Spaces of Pointwise Countable Type

A space Y is *of pointwise countable type* [5, Chap. 0, Sect. 2] if for every $y \in Y$ there exists a compact set K that contains y such that K has a countable base of neighborhoods in Y. From this definition it follows that the class of spaces of pointwise countable type contains the first countable and the locally compact spaces.

Let us notice that if Y is a Čech-complete space then Y is a space of pointwise countable type. In fact, as $Y = \cap O_n$, being each O_n an open subset of the Stone-Čech compactification βY of Y, then for each $y \in Y$ and each n there exists in βY a closed neighborhood A_n of y such that $A_n \subset O_n$. The set $K := \cap A_n$ is a compact subset of X. We may suppose that $A_{n+1} \subset A_n$ for each $n \in \omega$. Clearly $y \in K$ and by compactness if A is an open neighborhood of K in βY there exists $n \in \omega$ such that $A_n \subset A$. Then $\{A_n \cap X : n \in \omega\}$ is a countable base of neighborhoods of K and hence Y is a space of pointwise countable type.

The next Propositions 6.3 and 6.4 are the Exercise 3.3.I of [9] and the Theorem 5.6.2 of [14]. We recall that the proofs are given for the sake of completeness.

Proposition 6.3 *Every space X of pointwise countable type is a k-space.*

Proof Let us suppose that X is a topological space of pointwise countable type that it is not k-space. Then there exists a point $x \in \overline{A} \backslash A$, being A a subset such that the intersection of A with every compact subset of X is closed. By hypothesis the point x is contained in a compact subset K that has a decreasing base of neighborhoods $(V_n)_n$. Since the point x does not belong to the closed set $A \cap K$ then there exists a closed neighborhood V of x such that $V \cap A \cap K = \emptyset$.

For each n there exists $x_n \in V \cap V_n \cap A$. If U is a neighborhood of K there exists m such that $K \subset V_m \subset U$. Therefore from $K \cup \{x_n : n \geqslant m\} \subset U$ it follows that $K \cup \{x_n : n \in \omega\}$ is a compact subset of X. Hence the set $V \cap A \cap (K \cup \{x_n : n \in \omega\})$ is closed and equal to $\{x_n : n \in \omega\}$. This implies that the set $X \backslash \{x_n : n \in \omega\}$ is open and contains K. Therefore there exists $q \in \omega$ such that $K \subset V_q \subset X \backslash \{x_n : n \in \omega\}$. This inclusion yields the contradiction $x_q \in X \backslash \{x_n : n \in \omega\}$, so X is a k-space. □

6.6.2 Two Results on Submetrizability

A topological space X is submetrizable if there exists a metric space M and a continuous bijective map $\Phi : X \to M$. It is clear that if a topological space Y has a dense σ-compact subset A, i.e. $A = \cup\{K_n : n \in \omega\}$, $\overline{A} = Y$ and each K_n compact, then $C_c(Y)$ is submetrizable, because the topology in $C(Y)$ of the uniform convergence on each K_n, $n \in \omega$, is metrizable and weaker than the topology of $C_c(Y)$. It is not so obvious to proof the next proposition ([14, Theorem 5.6.2]).

Proposition 6.4 *If X is submetrizable then $C_c(X)$ has a dense σ-compact subset.*

Proof By hypothesis there exists a metric space M and a continuous bijective mapping $\Phi : X \to M$. Then the mapping $\Phi^* : C_c(M) \to C_c(X)$ defined by $\Phi^*(g) = g \cdot \Phi$ is continuous. From the fact that the restriction of Φ to a compact subset K of X is a homeomorphism it follows that $\Phi^*(C_c(M))$ is a dense subset of $C_c(X)$; in fact, if $f \in C_c(X)$ and K is a compact subset of X then the map h defined on $\Phi(K)$ by $h(\Phi(x)) = f(x)$ is continuous and hence it has a continuous extension $g \in C_c(M)$. For each $x \in K$ we have that $(\Phi^*(g))(x) = h(\Phi(x)) = f(x)$ and then from $\sup\{|f(x) - (\Phi^*(g))(x)| : x \in K\} = 0$ it follows that $\Phi^*(C_c(M))$ is a dense subset of $C_c(X)$. Therefore it is enough to prove this proposition when X is metrizable.

The paracompactness of the metric space X implies that for each $n \in \omega$ there exists a locally finite partition of the unit $\mathscr{F}_n$ subordinated to the open subsets of diameter less than $1/n$. This means that $f(X) \subset [0, 1]$, for each $f \in \mathscr{F}_n$, being $\{f^{-1}((0, 1]) : f \in \mathscr{F}_n\}$ a locally finite family of sets with diameter less than $1/n$ and such that $\sum\{f(x) : f \in \mathscr{F}_n\} = 1$ for each $x \in X$. The locally finite condition implies that for each point x there exists a neighborhood $V(x)$ and a finite subset $F_n(x)$ of $\mathscr{F}_n$ such that $f(V(x)) = \{0\}$ for each $f \in \mathscr{F}_n \setminus F_n(x)$. Therefore each net $\mathscr{N}$ in $\mathscr{F}_n$ has a subnet that converges uniformly on $V(x)$, hence if K is a compact subset of X the net $\mathscr{N}$ has also a subnet that converges uniformly in K. By Tychonov theorem each net $\mathscr{N}$ in $\mathscr{F}_n$ has a subnet that converges uniformly in each compact subset K of X to a real function g. Since X is a k-space we get that $g \in C(X)$. Hence $\mathscr{F}_n$ is a relatively compact subset of $C_c(X)$.

Let $e \in C_c(X)$ be such that $e(X) = \{1\}$. From the continuity of the sum, products (of two functions as well as the product of a function by scalar) and the compactness of finite subsets it follows that the sets

$$\mathscr{F}'_n := \{rf : -1 \leqslant r \leqslant 1,\ f \in \mathscr{F}_n \cup \{e\}\}$$

$$\mathscr{G}_n := \left\{f_1 f_2 \cdots f_k : 1 \leqslant k \leqslant n,\ \{f_i : 1 \leqslant i \leqslant k\} \subset \cup\{\mathscr{F}'_j : 1 \leqslant j \leqslant n\}\right\}$$

$$\mathscr{H}_n := \{f_1 + f_2 + \cdots + f_k : 1 \leqslant k \leqslant n,\ \{f_i : 1 \leqslant i \leqslant k\} \subset \mathscr{G}_n\}$$

are relatively compact subsets of $C_c(X)$. $\mathscr{H} = \cup\{\mathscr{H}_n : n \in \omega\}$ is a subalgebra of $C_c(X)$ since from $f \in \mathscr{H}_m$, $g \in \mathscr{H}_n$ and $t \in \mathbb{R}\setminus\{0\}$ with $|t| \leqslant p$ it follows that $f + g \in \mathscr{H}_{m+n}, fg \in \mathscr{H}_{mn}$ and $tf = (tp^{-1})pf \in \mathscr{H}_{mp}$.

If $x \neq y$ are two points of X then there exists n such that $d(x, y) > n^{-1}$ and since $\mathscr{F}_n$ is a partition of the unit there exists $f \in \mathscr{F}_n$ such that $f(x) \neq 0$. By construction, the diameter of $f^{-1}((0, 1])$ is less than $1/n$ and therefore $f(y) = 0$. Then $\mathscr{H}$ separates points of X and the Stone-Weierstrass theorem assures that the algebra $\mathscr{H}$ is dense in $C_c(X)$. Therefore $\cup\{\overline{\mathscr{H}_n}^{C_c(X)} : n \in \omega\}$ is a dense σ-compact subset of $C_c(X)$. □

The *transitivity of character for compact sets* implies that if F_1 and F_2 are compact subsets of a topological space Y such that $F_1 \subset F_2$, F_1 has countable character in F_2 and F_2 has countable character in Y then F_1 has countable character in Y, see [1, Proposition 3.3]. In particular, a submetrizable space Y of pointwise countable type is first countable. To proof this property apply the transitivity of character for a compact set $F_1 := \{y_0\}$ and a compact set F_2 of countable character in Y that contains y_0. Nevertheless an easy direct proof of this property is given in Proposition 6.5.

Proposition 6.5 *Let Y be a submetrizable space of pointwise countable type. Then Y is first countable.*

Proof Each $y \in Y$ is contained in a compact subset K which has a decreasing base $(V_n)_n$ of closed neighborhoods. As K is metrizable there exists in Y a decreasing sequence $(W_n)_n$ of closed neighborhoods of y such that $(W_n \cap K)_n$ is a base of neighborhoods of y in K. We may suppose that $W_n \subset V_n$ for each $n \in \omega$. Therefore if V is an open neighborhood of y there exist m such that $W_m \cap K \subset V$.

Let us suppose that for each $n \in \omega$ there exists $x_n \in W_n \backslash V$. Then from $x_p \in V_n$ for each $p \geqslant n$ it follows that $K \cup \{x_n : n \in \omega\}$ is compact. If x is an adherent point of $(x_n)_n$ we have that $x \in W_n$ for each $n \in \omega$, and then $x \in \cap\{V_n : n \in \omega\} = K$. Therefore $x \in W_m \cap K \subset V$. But the relations $x_n \in W_n \backslash V \subset E \backslash V$ for each $n \in \omega$ yield the contradiction $x \in E \backslash V$. Hence there exists $W_p \subset V$, and the sequence $(W_n)_n$ is a base of neighborhoods of y. □

6.6.3 Preservation of Second Countable and Polish Spaces by ℓ_c-Equivalence for Spaces of Pointwise Countable Type

Proposition 6.6 ([11, Proposition 18]) *Let X be a submetrizable space and let Y be a space of pointwise countable type. If there exists a continuous map from $C_c(X)$ onto $C_c(Y)$ then Y is submetrizable and first countable.*

Proof By Proposition 6.4 the space $C_c(X)$ has a dense σ-compact subset. Hence by continuity $C_c(Y)$ also has a dense σ-compact subset, which implies that $C_c(C_c(Y))$ is submetrizable. From Proposition 6.3 it follows that Y is a k-space and then, by Lemma 6.4, Y is embeds in $C_c(C_c(Y))$. Therefore Y is a submetrizable space. From Proposition 6.5 we get that Y is first countable. □

The next corollary extends [11, Corollary 22] for spaces of pointwise countable type.

Corollary 6.2 *If X is metrizable and separable, Y is a space of pointwise countable type and h is a continuous linear mapping from $C_c(X)$ onto $C_c(Y)$, then Y is metrizable and separable. Therefore second countability is preserved by ℓ_c-equivalence for spaces of pointwise countable type.*

Proof From Proposition 6.6 it follows that Y is first countable. Then from Corollary 6.1 it is deduced the statement of this corollary. □

The property that Polish spaces are preserved by ℓ_c-equivalence in the spaces of countable type may be obtained by a combination of results of Baars, de Groot, Pelant and Valov on preservation of complete metrizability by ℓ_p-equivalence in the class of metrizable spaces [6, 25]. The next corollary is the simple proof given in [11, Corollay 24].

Corollary 6.3 ([11, Corollay 24]) *Let X be a Polish space that ℓ_c-covers the space Y. If Y is of pointwise countable type then Y is a Polish space. Thus the property of being a Polish space is preserved by ℓ_c-equivalence for spaces of pointwise countable type.*

Proof From Proposition 6.6 it follows that Y is first countable. So Theorem 6.5 applies. □

Acknowledgments Supported by Generalitat Valenciana, Conselleria d'Educació Cultura i Esport, Spain, Grant PROMETEO/2013/058.

References

1. Arhangel'skii, A.V.: Bicompact sets and the topology of spaces. Trudy Moskov. Mat. Obšč. **13**, 3–55 (1965) (in Russian)
2. Arhangel'skii, A.V.: On linear homomorphisms of function spaces. Dokl. Akad. Nauk SSSR. **264**, 1289–1292 (1982)
3. Arhangel'skii, A.V.: A survey of C_p-theory. Questions Answers Gen. Topology. **5**, 1–109 (1987)
4. Arhangel'skii, A.V.: General Topology III. Encyclopaedia of Mathematical Sciences, 51. Springer, Berlin (1991)
5. Arhangel'skii, A.V.: Topological Function Spaces. Mathematics and its Applications 78. Kluwer Academic Publishers, Dordrecht (1992)
6. Baars, J., de Groot, J., Pelant, J.: Function spaces of completely metrizable spaces. Trans. Amer. Math. Soc. **340**, 871–883 (1993)
7. Christensen, J.P.R.: Topology and Borel Structure. North-Holland Mathematics Studies 10, North-Holland, Amsterdam (1974)
8. Drewnowski, L.: Resolutions of topological linear spaces and continuity of linear maps. J. Math. Anal. Appl. **335**, 1177–1195 (2007)
9. Engelking, R.: General Topology. Monografie Matematiczne 60. PWN, Warszawa (1975)
10. Ferrando, J.C., Kąkol, J.: On precompact sets in spaces $C_c(X)$. Georgian Math. J. **20**, 247–254 (2013)
11. Kąkol, J., López-Pellicer, M., Okunev, O.: Compact covers and functions spaces. J. Math. Anal. Appl. **411**, 372–380 (2014)
12. Kąkol, J., Kubiś, W., López-Pellicer, M.: Descriptive Topology in Selected Topics of Functional Analysis. Developments in Mathematics 24. Springer, New York (2011)
13. Köthe, G.: Topological Vector Spaces. Springer, Berlin (1969)
14. McCoy, R.A., Ntantu, I.: Topological Properties of Spaces of Continuous Functions. Lecture Notes in Mathematics 1315. Springer (1988)
15. Michael, E.: $\aleph_0$-spaces. J. Math. Mech. **15**, 983–1002 (1966)
16. Nachbin, L.: Topological vector spaces of continuous functions. Proc. Nat. Acad. Sci. **40**, 471–474 (1945)
17. Nagata, J.: On lattices of functions on topological spaces and of functions on uniform spaces. Osaka Math. J. **1**, 166–181 (1949)
18. Okunev, O.: A relation between spaces implied by their t-equivalence. Topology Appl. **158**, 2158–2164 (2011)
19. Orihuela, J.: Pointwise compactness in spaces of continuous functions. J. Lond. Math. Soc. **36**, 143–152 (1987)
20. Pérez Carreras, P.; Bonet, J.: Barrelled Locally Convex Spaces. North-Holland Mathematics Studies 131. North-Holland, Amsterdam (1987)
21. Shirota, T.: On locally convex vector spaces of continuous functions. Proc. Japan Acad. **30**, 294–298 (1954)
22. Tkachuk, V.V., Shakhmatov, D.B.: When is space $C_p(X)$ σ-countably compact? Moscow Univ. Math. Bull. **42**, 23–26 (1987)
23. Tkachuk, V.V.: A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic. Acta Math. Hungar. **107**, 253–265 (2005)

24. Uspenski, V.: On the topology of a free locally convex space. Soviet Math. Dokl. **27**, 781–785 (1983)
25. Valov, V.: Function spaces. Topology Appl. **81**, 1–22 (1997)
26. Velichko, N.V.: The Lindelöf property is *l*-invariant. Topology Appl. **89**, 277–283 (1998)