

Nikodym boundedness property in set-algebras

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March 2-4, 2016

XII Encuentro Red Análisis Funcional y Aplicaciones

Project Prometeo I/2013/058



UNIVERSITAT
POLITÈCNICA
DE VALÈNCIA



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Outline

1 Introduction

Notations

Let Ω be a set and \mathcal{B} a subset of a set-algebra \mathcal{A} of subsets of Ω .

$L(\mathcal{B})$ is linear hull of $\{e_C : C \in \mathcal{B}\}$ with the supremum norm $\|\cdot\|$
 $ba(\mathcal{A})$ (the Banach space of bounded variation finitely additive measures on \mathcal{A} with the variation norm ($|\cdot| := |\cdot|(\Omega)$)) is isometric to $L(\mathcal{A})'$ with the dual norm. Whence $L(\mathcal{A})'$ is identified with $ba(\mathcal{A})$.

The \mathcal{A} -supremum norm in $ba(\mathcal{A})$, i.e.,

$\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$, $\mu \in ba(\mathcal{A})$, is equivalent to the variation norm.

Nikodym property $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition (Schachermayer-Valdivia)

\mathcal{B} has *Nikodym property*, property N in brief, if each \mathcal{B} -pointwise bounded subset M of $ba(\mathcal{A})$ is bounded in $ba(\mathcal{A})$, i.e.,

$$\sup_{\mu \in M} |\mu(A)| < \infty, \forall A \in \mathcal{B} \implies \sup_{A \in \mathcal{A}, \mu \in M} |\mu(A)| < \infty$$

Then $L(\mathcal{B})$ is dense subspace of $L(\mathcal{A})$ ($\{e_C : C \in \mathcal{B}\}^\circ$ is bounded $\implies \{e_C : C \in \mathcal{B}\}^{\circ\circ}$ neighborhood of 0).

Definition (Valdivia)

\mathcal{B} has *strong Nikodym property* if for each increasing covering $\cup_m \mathcal{B}_m$ of \mathcal{B} there exists \mathcal{B}_n which has property N .

Web Nikodym property of $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition

Increasing web in a set A is a family

$\mathcal{W} := \{A_{m_1 m_2 \dots m_p} : (m_1, m_2, \dots, m_p) \in \cup_{s \in \mathbb{N}} \mathbb{N}^s\}$ of subsets of A such that

- 1 $A = \cup_{m_1} A_{m_1}$
- 2 $A_{m_1 m_2 \dots m_p} = \cup_{m_{p+1}} A_{m_1, m_2, \dots, m_p m_{p+1}}, \forall p, m_i \in \mathbb{N}, 1 \leq i \leq p + 1.$

Each sequence $(A_{m_1 m_2 \dots m_p})_p$ is call a *strand* in \mathcal{W} .

Definition (Kakol-LP)

\mathcal{B} has *web Nikodym property*, property wN in brief, if *each increasing web* $\{\mathcal{B}_t : t \in \cup_s \mathbb{N}^s\}$ in \mathcal{B} has a *strand* composed of sets which have property N

Properties N , sN and wN in σ -algebras

Theorem

For a σ -algebra \mathcal{S} of subsets of a set Ω it was shown:

- 1 \mathcal{S} has property N (Nikodym-Dieudonné-Grothendieck).
- 2 \mathcal{S} has property sN (Valdivia).
- 3 \mathcal{S} has property wN (Kakol-LP).

Properties N , sN and wN in algebras

Example

The algebra of finite and co-finite subsets of \mathbb{N} fails to have property N .

Example

The algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has property N (Schachermayer)

Example

The algebra $\mathcal{J}(K)$ of Jordan measurable subsets of $K := \prod_{1 \leq i \leq k} [a_i, b_i]$ has property sN (Valdivia 2013)

The aim of this talk is to prove that $\mathcal{J}(K)$ has property wN .

Outline

2 Deep unboundedness and property N

More notations

Let $t = (t_1, t_2, \dots, t_p) \in T \subset \mathbb{N}^p$ and

$u = (u_1, u_2, \dots, u_q) \in U \subset \mathbb{N}^q$.

p is the *length* of t . If $t = (t_1)$ then $t = t_1$.

$t(i) := (t_1, t_2, \dots, t_i)$ is the *section of length i* of t , $1 \leq i \leq p$.

$t(i) := \emptyset$ if $i > p$.

$t \times u := (t_1, t_2, \dots, t_p, t_{p+1}, t_{p+2}, \dots, t_{p+q})$, with $t_{p+j} := u_j$, for $1 \leq j \leq q$.

Definition

$(t^n = (t_1^n, t_2^n, \dots, t_n^n, \dots) \in T)_n$ is an *infinite chain* if

$t^{n+1}(n) = t^n(n) \neq \emptyset, \forall n \in \mathbb{N}$.

Last notations

$t \times s$ ($t \times s \in U$) is an extension of t (of t in U).

U is increasing at $t = (t_1, t_2, \dots, t_p) \in \cup_s \mathbb{N}^s$ if there exists

$$\{t^1 = (t_1^1, t_2^1, \dots), t^i = t(i-1) \times (t_i^i, t_{i+1}^i, \dots), 1 < i \leq p\} \subset U$$

such that $t_i < t_i^i$, for each $1 \leq i \leq p$.

Definition

U is increasing (increasing respect to a subset V of $\cup_s \mathbb{N}^s$) if U is increasing at each $t \in U$ (at each $t \in V$).

U is increasing if $|U(1)| = \infty$ and

$|\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty, \forall t = (t_1, t_2, \dots, t_p) \in U$ and $1 \leq i < p$.

NV-tree

Definition

An *NV-tree* T is an increasing subset of $\bigcup_{s \in \mathbb{N}} \mathbb{N}^s$ without infinite chains such that each $t = (t_1, t_2, \dots, t_p) \in T$ verifies that the length of each extension of $t(p-1)$ in T is p and $\{t(i) : 1 \leq i \leq p\} \cap T = \{t\}$.

Example

The infinite subsets of \mathbb{N} are *NV-trees*, named trivial *NV-trees*.

Example

\mathbb{N}^i , $i \in \mathbb{N} \setminus \{1\}$, and $\bigcup \{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are non trivial *NV-trees*.

Example

The product of a finite family of *NV-trees* is an *NV-tree*

Elementary properties of a NV -tree T

Proposition

Each increasing subset S of T is an NV -tree.

Whence if $S_n \subset T$ and S_{n+1} is increasing respect to S_n then $\bigcup_n S_n$ is an NV -tree.

Proposition

If no NV -tree is contained in U ($\subset T$) then $T \setminus U$ contains an NV -tree.

Elementary properties of a NV -tree T

Proposition

$$\{B_u : u \in \cup_s \mathbb{N}^s\} \uparrow \text{web in } B \implies B = \cup \{B_t : t \in T\}.$$

Proof.

This equality follows from the following trivial facts:

- 1 $B = \cup_{u \in T} B_{u(1)} \uparrow, B_{u(i)} = \cup_{u(i) \times n \in T(i+1)} B_{u(i) \times n}$
- 2 $b \in B \implies \exists t \in T : b \in B_t$, (T does not contain infinite chains).



Deep B -unbounded sets

Definition (Deep B -unbounded set)

Let $B \in \mathcal{A}$. $M \subset ba(\mathcal{A})$ is deep B -unbounded if for each finite subset Q of $\{e_A : A \in \mathcal{A}\}$

$$\sup\{|\mu(C)| : \mu \in M \cap Q^\circ, C \in \mathcal{A}, C \subset B\} = \infty.$$

Proposition

Let $M(\subset ba(\mathcal{A}))$ be deep B -unbd and $\{B_i \in \mathcal{A} : 1 \leq i \leq q\}$ a partition of B . There exists j , $1 \leq j \leq q$, such that M is deep B_j -unbounded.

Proof.

If $\exists Q^i \subset \{e_A : A \in \mathcal{A}\}$, finite, $\sup_{\mu \in M \cap (Q^i)^\circ} |\mu|(C_i) < H_i$,
 $i \in \{1, 2, \dots, q\}$, then $\sup_{\mu \in M \cap Q^\circ} |\mu|(B) < \sum_{1 \leq i \leq q} H_i$, with
 $Q = \cup_{1 \leq i \leq q} Q^i$. □

Webs without N -strands

Proposition

Let $\mathcal{B} := \{\mathcal{B}_{m_1 m_2 \dots m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ be an increasing web in \mathcal{A} without N -strands.

Then there exists an NV -tree T such that for each $t = (t_1, t_2, \dots, t_q) \in T$ the set \mathcal{B}_t does not have property N and if $p > 1$ then $\mathcal{B}_{t(i)}$ has property N if $i < p$.

Webs without N -strands

Proof.

Either \mathcal{B}_{m_1} , $m_1 \in \mathbb{N}$ not prop N (then $T := \mathbb{N}$)
or $\mathcal{B}_{m'_1}$ prop N . $Q_1 := \emptyset$ and $Q'_1 := \{t_1 \in \mathbb{N} : t_1 \geq m'_1\}$.

Suppose $j \leq i$, $t \in Q_j \cup Q'_j \implies t(j-1) \in Q'_{j-1}$,

- $t \in Q_j \implies \mathcal{B}_t$ not prop N ; $t(j-1) \times \mathbb{N} \subset Q_j$; $S_{t(j-1)} := \mathbb{N}$
and $S'_{t(j-1)} := \emptyset$,
- $t \in Q'_j \implies \mathcal{B}_t$ prop N ; $\exists \left| S'_{t(j-1)} \right| = \infty$ with
 $t(j-1) \times S'_{t(j-1)} \subset Q'_j$ and $(t(j-1) \times \mathbb{N}) \cap Q_j = \emptyset$. Then
 $S_{t(j-1)} := \emptyset$.



Webs without N -strands

Proof.

If $t \in Q'_i$ then \mathcal{B}_t prop N and $\mathcal{B}_t = \cup_n \mathcal{B}_{t \times n} \uparrow \implies$

- Either each $\mathcal{B}_{t \times n}$ not prop N . Then $S_{t_1 t_2 \dots t_i} := \mathbb{N}$ and $S'_{t_1 t_2 \dots t_i} := \emptyset$,
- or $\exists m'_{i+1} \in \mathbb{N} : \mathcal{B}_{t \times n}$ prop N if $n \geq m'_{i+1}$. Then $S_{t_1 t_2 \dots t_i} := \emptyset$ and $S'_{t_1 t_2 \dots t_i} := \{n \in \mathbb{N} : m'_{i+1} \leq n\}$.

$Q_{i+1} := \cup \{t \times S_t : t \in Q'_i\}$ and $Q'_{i+1} := \cup \{t \times S'_t : t \in Q'_i\}$.

\mathcal{B} without N -strands $\implies T := \cup \{Q_i : i \in \mathbb{N}\}$ does not contain infinite chains

$$(t \in Q_p \implies t(p-1) \in Q'_{t(p-1)} \implies \mathcal{B}_{t(p-1)} \text{ prop } N)$$



Webs without N -strands

Proof.

Each $t \in Q'_k$ has an extension $\tilde{t} \in Q_{k+q}$, hence
 $T(k) = Q_k \cup Q'_k, \forall k \in \mathbb{N}$. Hence T increasing prop, because

- $|T(1)| = |Q'_1| = \infty$
- $t = (t_1, t_2, \dots, t_p) \in T$ then $|S'_{t(i-1)}| = \infty, 1 < i < p$, and
 $|S_{t(p-1)}| = \infty$,

Further, $t(i) \in Q'_i \implies B_{t(i)}$ prop N and B_t not prop N ,
 $\{t(i) : 1 \leq i \leq p\} \cap T = \{t\}$ and

$$\{\text{ext } t(p-1) \text{ in } T\} = t(p-1) \times \mathbb{N}.$$



Increasing sequences without property N

Proposition

Let \mathcal{A} be an algebra of subsets of Ω and let $(\mathcal{B}_m)_m$ be an increasing sequence of subsets of \mathcal{A} such that each \mathcal{B}_m does not have property N and $\overline{\text{span}\{e_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$.

There exists $n_0 \in \mathbb{N}$ such that for each $m \geq n_0$ there exists a deep Ω -unbounded $\tau_s(\mathcal{A})$ -closed absolutely convex subset M_m of $\text{ba}(\mathcal{A})$ which is pointwise bounded in \mathcal{B}_m , i.e.,

$$\sup\{|\mu(C)| : \mu \in M_m\} < \infty \text{ for each } C \in \mathcal{B}_m.$$

In particular, this proposition holds if $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has property N .

Deep Ω -unbounded sets indexed by NV -trees

From the two preceding Propositions follows the next proposition.

Proposition

Let $\mathcal{B} := \{\mathcal{B}_{m_1 m_2 \dots m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ be an increasing web in a set-algebra \mathcal{A} .

If \mathcal{B} does not contain strands consisting of sets with property N then there exists an NV -tree T such that for each $t \in T$ there exists a deep Ω -unbounded $\tau_S(\mathcal{A})$ -closed absolutely convex subset M_t of $ba(S)$ which is \mathcal{B}_t -pointwise bounded.

Outline

3 Web Nikodym property in $\mathcal{J}(K)$

$$\mathcal{J}(K), K := \prod_{1 \leq i \leq k} [a_i, b_i] \subset \mathbb{R}^k$$

Definition

A bounded $B \subset \mathbb{R}^k$ is Jordan measurable whenever the Lebesgue measure of $\overline{B} \setminus \overset{\circ}{B}$ is zero.

$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K)$, $M \subset ba(\mathcal{J}(K))$, deep B -unbounded absolutely convex.

Proposition

If $\alpha > 0$ and $\epsilon > 0 \exists C_1, C'_1 \in \mathcal{J}(K)$, disjoint contained in B and $\mu \in M$ such that

- 1 $|\mu(C_1)| > \alpha$, $|\mu(C'_1)| > \alpha$, $\sum_{1 \leq j \leq r} \mu(Q_j) \leq 1$,
- 2 $C_1 \cup C'_1 \subset \prod_{1 \leq i \leq k} [c_i, d_i] \subset K$, $\prod_{1 \leq i \leq k} (d_i - c_i) < \epsilon$
- 3 M is deep C'_1 -unbounded.

$$B \subset K := \prod_{1 \leq i \leq k} [a_i, b_i] \subset \mathbb{R}^k$$

Proof.

Let $(c_i^j)_{1 \leq j \leq s(i)}$ strictly increasing, $a_i = c_i^1$, $b_i = c_i^{s(i)}$ and

$$\prod_{1 \leq i \leq k} (c_i^{j_i+1} - c_i^{j_i}) < \epsilon$$

Let $I_i^j := [c_i^{j-1}, c_i^j[$, with $1 < j < s(i)$, and $I_i^{s(i)} := [c_i^{s(i)-1}, c_i^{s(i)}]$,
and

let $\{B_1, B_2, \dots, B_m\}$ the non-void intersections of B with
 $I_1^{j_1} \times \dots \times I_k^{j_k}$, $1 \leq j_i \leq s(i)$, $1 \leq i \leq k$.

There exists B_n , $1 \leq n \leq m$, with M deep B_n -unbounded. \square

$$\{B_n \subset B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K)$$

$M \subset ba(\mathcal{J}(K))$, deep B_n -unbd absco, $\mathcal{Q} := \{Q_1, \dots, Q_r\}$

Proof.

$\sup\{|\mu(C)| : \mu \in rM \cap \mathcal{Q}^\circ, C \subset B_n, C \in \mathcal{A}\} = \infty,$
 $\exists D_1 \subset B_n, D_1 \in \mathcal{J}(K), \exists \lambda \in rM \cap \mathcal{Q}^\circ : |\lambda(D_1)| > r(1 + \alpha).$
 $\mu = r^{-1}\lambda \in M, |\mu(B_n)| \leq r^{-1} \leq 1$ and
 $\sum_{1 \leq j \leq r} |\mu(Q_j)| \leq r^{-1}r = 1.$
 $|\mu(B_n \setminus D_1)| \geq |\mu(D_1)| - |\mu(B)| > 1 + \alpha - 1 = \alpha.$
 M is deep D_1 -unbded or $B_n \setminus D_1$ -unbded. □

$$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K), M \subset ba(A)$$

Corollary

If M deep B -unbd absco, $q \in \mathbb{N}$, $\alpha > 0$, $\epsilon > 0$, \exists in B pairwise disjoint Jordan measurable subsets C_1, C_2, \dots, C_q and $\mu_1, \mu_2, \dots, \mu_q \in M$ such that:

- 1 $|\mu_i(C_i)| > \alpha$, $\sum_{1 \leq j \leq r} \mu_i(Q_j) \leq 1$, $i = 1, 2, \dots, q$,
- 2 M is deep C_q -unbounded
- 3 $\cup_{1 \leq l \leq q} C_l \subset \prod_{1 \leq i \leq k} [c_i, d_i] \subset K$, $\prod_{1 \leq i \leq k} (d_i - c_i) < \epsilon$.

Proof: Exist disjoint subsets $C_1, C'_1 \in \mathcal{J}(K)$, in B , and $\mu_1 \in M$:

- 1 $|\mu_1(C_1)| > \alpha$, $|\mu_1(C'_1)| > \alpha$, $\sum_{1 \leq j \leq r} \mu_1(Q_j) \leq 1$,
- 2 M is deep C'_1 -unbd
- 3 $C_1 \cup C'_1 \subset \prod_{1 \leq i \leq k} [c_i, d_i] \subset K$, $\prod_{1 \leq i \leq k} (d_i - c_i) < \epsilon \dots$

$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K), M_t \subset ba(\mathcal{A}), t \in T$

Proposition

If T is an NV-tree, each M_t is deep B -unbd absco, $\alpha > 0, \epsilon > 0$ and $\{t^j : 1 \leq j \leq k\} \subset T$, there exists $\{B_1, B'_1\} \subset \mathcal{J}(K)$, $B_1 \cap B'_1 = \emptyset$ and $B_1 \cup B'_1 \subset B$, $\mu_1 \in M_{t^1}$, an NV-tree T_1 and $\{\prod_{1 \leq i \leq k} [c_i^s, d_i^s], 1 \leq s \leq q\} \subset K$:

- 1 $\{t^j : 1 \leq j \leq k\} \subset T_1 \subset T$ and M_t is deep B'_1 -unbd, $\forall t \in T_1$.
- 2 $|\mu_1(B_1)| > \alpha$ and $\sum\{|\mu_1(Q_i)| : 1 \leq i \leq r\} \leq 1$.
- 3 $B_1 \cup B'_1 \subset \cup_{1 \leq s \leq q} \prod_{1 \leq i \leq k} [c_i^s, d_i^s] \subset K$,
 $\sum_{1 \leq s \leq q} (\prod_{1 \leq i \leq k} (d_i^s - c_i^s)) < \epsilon$.

$$t^j := (t_1^j, t_2^j, \dots, t_{p_j}^j), \quad 1 \leq j \leq k$$

Proof.

Let $q := 2 + \sum_{1 \leq j \leq k} p_j$, $(c_i^j)_{1 \leq j \leq s(i)} \uparrow$, $a_i = c_i^1$, $b_i = c_i^{s(i)}$,
 $1 \leq i \leq k$, $\prod_{1 \leq i \leq k} (c_i^{j+1} - c_i^j) < \epsilon/q$ for each $1 \leq j \leq s(i)$ and
 $1 \leq i \leq k$, $I_i^j := [c_i^{j-1}, c_i^j]$, with $1 < j < s(i)$, and
 $I_i^{s(i)} := [c_i^{s(i)-1}, c_i^{s(i)}]$.

Let $\{B_1, B_2, \dots, B_m\}$ be the non-void intersections of B with the
 products $I_1^{j_1} \times \dots \times I_k^{j_k}$.

$\exists B_n$, $1 \leq n \leq m$, such that M_{t^1} is deep B_n -unbounded.

Corollary applied to $\{B_n, Q_1, \dots, Q_r\}$, M_{t^1} , α and ϵ/q provides a
 Jordan measurable partition $\{C_1, C_2, \dots, C_q\}$ of B_n and $\{\lambda_1,$
 $\lambda_2, \dots, \lambda_q\} \subset M_{t^1}$ such that:

$$|\lambda_k(C_k)| > \alpha \quad \text{and} \quad \sum_{1 \leq i \leq r} |\lambda_k(Q_i)| \leq 1, \quad \text{for } k = 1, 2, \dots, q,$$

$$\{D_i : 1 \leq i \leq q'\} := \{B_i : i \neq n\} \cup \{C_i : 1 \leq i \leq q\}$$

Proof.

If $M \in ba(\mathcal{A})$ is deep B -unbd $\exists i_M \in \{1, 2, \dots, q'\} : M$ is deep D_{i_M} -unbd.

Whence if U is an NV -tree, M_u is deep B -unbd $\forall u \in U$ and

$$V_i := \{u \in U : M_u \text{ is deep } D_i\text{-unbd}\}$$

then $U = \bigcup_{1 \leq i \leq q'} V_i$. This implies that $\exists i_0$, with $1 \leq i_0 \leq q'$, such that V_{i_0} contains an NV -tree U_{i_0} . □

Deep D_j -unboundedness

Proof.

- $j \in \{1, 2, \dots, k\} \implies \exists i^j \in \{1, 2, \dots, q'\} : M_{t^j}$ deep D_{i^j} -unbd.
- $\exists i_0 \in \{1, 2, \dots, q'\}$ and an NV -tree $T_{i_0} \subset T : t \in T_{i_0}, M_t \implies D_{i_0}$ -unbounded.
- $t^j = (t_1^j, t_2^j, \dots, t_{p_j}^j) \notin T_{i_0}, 1 \leq j \leq k, \text{ and } 2 \leq m \leq p_j \implies$

$$W_m^j := \{v \in \cup_s \mathbb{N}^s : t^j(m-1) \times v \in T\}$$

is an NV -tree. $M_{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times w}$ is deep B -unbd, $\forall w \in W_m^j$,
whence $\exists i_m^j \in \{1, 2, \dots, q'\}$ and an NV -tree $V_m^j \subset W_m^j$
such that

$$M_{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times w} \text{ is deep } D_{i_m^j}\text{-unbd } \forall w \in V_m^j.$$

B_1 and μ_1

Proof.

Let D be the union of D_{i_0} and all D_{ij} and $D_{i_m^j}$ obtained. The number of sets defining D is less or equal than $q - 1$, hence $\exists C_h : D \subset B \setminus C_h$.

Let T_1 be the union of T_{i_0} and all $\{t^j\}$ and $\{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times V_m^j : 2 \leq m \leq p_j\}$ obtained. By construction T_1 has the increasing property ($\implies NV$ -set) and if $t \in T_1$ the set M_t is deep D -unbounded.

We get the proof with $B_1 := C_h$, $B'_1 := D$, $\mu_1 := \lambda_h$, because by construction

- 1 $|\mu_1(B_1)| > \alpha$, $\sum_{1 \leq j \leq r} |\mu_1(Q_j)| \leq 1$, and
- 2 $B_1 \cup B'_1 \subset \cup_{s \in J} \{\prod_{1 \leq i \leq k} [c_i^s, d_i^s]\}$, $|J| \leq q$, whence $\sum_{s \in J} (\prod_{1 \leq i \leq k} (d_i^s - c_i^s)) < (\epsilon/q)q = \epsilon$.

$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K), M_t \subset \text{ba}(\mathcal{A}), t \in T$

Corollary

If T is an NV-tree, each M_t is deep B -unbd absco, $\alpha > 0, \epsilon > 0$ and $\{t^j : 1 \leq j \leq k\} \subset T$, there exists in B a family of pairwise disjoint subsets $\{B_1, B_2, \dots, B_k, B'\} \subset \mathcal{J}(K)$, k measures $\mu_j \in M_{t^j}, 1 \leq j \leq k$, and an NV-tree T^* such that:

- 1 $\{t^j : 1 \leq j \leq k\} \subset T^* \subset T$ and M_t is deep B' -unbounded for each $t \in T^*$.
- 2 $|\mu_j(B_j)| > \alpha$ and $\Sigma\{|\mu_j(Q_i)| : 1 \leq i \leq r\} \leq 1$, for $j = 1, 2, \dots, k$.
- 3 $B' \subset \cup_{1 \leq s \leq q} \{\prod_{1 \leq i \leq k} [c_i^s, d_i^s]\}$ of compact intervals contained in K , such that $\Sigma_{1 \leq s \leq q} (d_1^s - c_1^s) \times \dots \times (d_k^s - c_k^s) < \epsilon$.

Proof.

Firstly apply Proposition.

Then apply again Proposition with $\{t^2, t^3, \dots, t^k, t^1\}$ and $B := B'_1$. Finish with k repetitions. □

The diagonal order in \mathbb{N}^2

The elements of \mathbb{N}^2 ordered with the diagonal order give the sequence

$$((1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), \dots)$$

We will need the sequence

$$(i_n)_n := (1, 1, 2, 1, 2, 3, \dots)$$

of its first components.

$\mathcal{J}(K)$ has property wN

Theorem

The algebra $\mathcal{J}(K)$ has property wN .

Proof.

Let us suppose that $\mathcal{J}(K)$ does not have property wN .

Then \exists exists in $\mathcal{J}(K)$ an increasing web

$\{\mathcal{B}_{m_1 m_2 \dots m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ without strands consisting of sets with Property N .

Whence \exists an NV -tree T such that for each $t \in T$ there exists a deep K -unbd $\tau_s(\mathcal{J}(K))$ -closed absco subset M_t of $ba(\mathcal{J}(K))$ which is \mathcal{B}_t -pointwise bounded. □

First induction

Proof.

By induction we determine an NV -tree $\{t^i : i \in \mathbb{N}\} \subset T$ and $(k_j \in \mathbb{N})_j \uparrow$ such that for each $(i, j) \in \mathbb{N}^2$ with $i \leq k_j$ there exists a set $B_{ij} \in \mathcal{J}(K)$ and $\mu_{ij} \in M_{t^i}$ that verify

- 1 $\sum_{s,v} \{|\mu_{ij}(B_{sv})| : s \leq k_v, 1 \leq v < j\} < 1,$
- 2 $|\mu_{ij}(B_{ij})| > j,$
- 3 $B_{ij} \cap B_{i'j'} = \emptyset$ if $(i, j) \neq (i', j')$ and
- 4 $\cup\{B_{sv} : s \leq k_v, j < v\} \subset \cup_{1 \leq s \leq q_j} (\prod_{1 \leq i \leq k} [c_{ij}^s, d_{ij}^s]) \subset K$
- 5 $\sum_{1 \leq s \leq q_j} (\prod_{1 \leq i \leq k} (d_{ij}^s - c_{ij}^s)) < 2^{-j},$ for each $j \in \mathbb{N}.$



First induction. Step I I

Proof.

Select $t^1 \in T$. Corollary with $B := \Omega$, $\alpha = 1$ and $\epsilon = 2^{-1}$ provides

- $B_{11}, B'_1 \in \mathcal{J}(K)$, $\mu_{11} \in M_{t^1}$ and an NV -tree T_1 such that
- $|\mu_{11}(B_{11})| > 1$, $t^1 \in T_1 \subset T$,
- M_t is deep B'_1 -unbd for each $t \in T_1$ and
- $B'_1 \subset \cup_{1 \leq s \leq q_1} \{\prod_{1 \leq i \leq k} [c_{i1}^s, d_{i1}^s]\} \subset K$, such that
- $\sum_{1 \leq s \leq q_1} (\prod_{1 \leq i \leq k} (d_{i1}^s - c_{i1}^s)) < 2^{-1}$.

Define $k_1 := 1$ and $S^1 := \{t^1\}$. □

First induction. Induction hypothesis

Proof: Let us suppose that we have obtained:

- The natural numbers $k_1 < k_2 < k_3 < \dots < k_n$,
- the NV-trees $T_1 \supset T_2 \supset T_3 \supset \dots \supset T_n$,
- $S^j := \{t^i : i \leq k_j\} \subset T_j$, $1 \leq j \leq n$, S^j incr respect S^{j-1} , $1 < j \leq n$,
- $\{\mu_{iv} \in M_{t^i} : i \leq k_v, 1 \leq v \leq j\}$
- $\{B'_j, B_{iv} : i \leq k_v, 1 \leq v \leq j\} \subset \mathcal{J}(K)$, pairwise disjoint, $\forall j \leq n$,
- $|\mu_{ij}(B_{ij})| > j$ and $\sum_{s \leq k_v, 1 \leq v < j} |\mu_{ij}(B_{sv})| < 1$, $i \leq k_j$ and $j \leq n$,
- $j \leq n$ and $t \in T_j \implies M_t$ is deep B'_j -unbd,
- $\cup_{s \leq k_v, j < v \leq n} B_{sv} \subset B'_j \subset \cup_{1 \leq s \leq q_j} \prod_{1 \leq i \leq k} [c_{ij}^s, d_{ij}^s] \subset K$,
- $\sum_{1 \leq s \leq q_j} (\prod_{1 \leq i \leq k} (d_{ij}^s - c_{ij}^s)) < 2^{-j}$.

First induction. End

Proof.

Select $S_{n+1} := \{t^{k_{n+1}}, \dots, t^{k_{n+1}}\} \subset T_n \setminus \{t^i : i \leq k_n\}$ incr respect S^n . Apply Corollary to $\{B'_n, B_{sv} : s \leq k_v, 1 \leq v \leq n\}$, T_n , the finite subset $S^{n+1} := \{t^i : i \leq k_{n+1}\}$ of T_n , $\alpha = n + 1$ and $\epsilon = 2^{-n-1}$ to obtain:

- $\{B'_{n+1}, B_{in+1} : 1 \leq i \leq k_{n+1}\} \subset \mathcal{J}(K)$, pairwise disjoint $\subset B'_n$,
- $\{\mu_{in+1} \in M_{t^i}, 1 \leq i \leq k_{n+1}\}$, $|\mu_{in+1}(B_{in+1})| > n + 1$,
 $\sum_{s \leq k_v, 1 \leq v \leq n} |\mu_{in+1}(B_{sv})| < 1$,
- $S^{n+1} \subset T_{n+1}$ (incr tree) $\subset T_n$, M_t deep B'_{n+1} -unbd $\forall t \in T_{n+1}$,
- $B'_{n+1} \subset \cup_{1 \leq s \leq q_{n+1}} \{\prod_{1 \leq i \leq k} [c_{i,n+1}^s, d_{i,n+1}^s]\} \subset K$
- $\sum_{1 \leq s \leq q_{n+1}} (\prod_{1 \leq i \leq k} (d_{i,n+1}^s - c_{i,n+1}^s)) < 2^{-n-1}$.

A Jordan measurable set

Proof.

If $H_j \subset \{1, 2, \dots, k_j\}$, $j \in \mathbb{N}$, then $B := \cup\{B_{ij} : i \in H_j, j \in \mathbb{N}\}$ is Jordan measurable.

In fact: For each $j_0 \in \mathbb{N}$ we have

$$\overline{B} \setminus \overset{\circ}{B} \subset \{\cup_{j \leq j_0} (\overline{B}_j \setminus \overset{\circ}{B}_j)\} \cup \overline{B}'_{j_0},$$

hence $\overline{B} \setminus \overset{\circ}{B}$ is a subset of

$\{\cup_{j \leq j_0} (\overline{B}_j \setminus \overset{\circ}{B}_j)\} \cup \{\cup_{1 \leq s \leq q_{j_0}} \prod_{1 \leq i \leq k} [c_{ij_0}^s, d_{ij_0}^s]\}$, which Lebesgue measure is $\leq 0 + 2^{-j_0}$. Therefore $\overline{B} \setminus \overset{\circ}{B}$ is a Lebesgue measurable set with Lebesgue measure zero. □

Second induction and basic idea

Proof.

With a new easy induction we obtain a strictly increasing subset $J := \{j_1, j_2, \dots, j_n, \dots\}$ of \mathbb{N} such that for each $n \in \mathbb{N}$ we have that

$$|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})| < 1.$$

Note that $|\mu_{i_n j_n}|(\Omega) < s_n \in \mathbb{N}$, $\{N_u, 1 \leq u \leq s_n\}$ partition of $J_n \subset \mathbb{N} \setminus \{1, 2, \dots, j_n\}$ in s_n infinite subsets and $B_u := \cup_{s \leq k_v, v \in N_u} B_{sv}$, then

$$\sum \{|\mu_{i_n j_n}|(B_u) : 1 \leq u \leq s_n\} < s_n$$

implies $\exists u'$, with $1 \leq u' \leq s_n$, such that

$$|\mu_{i_n j_n}|(B_{u'}) < 1.$$

Second induction. End

Proof.

Let $j_{n+1} \in N_{U'}$ and the induction is finish.

Then $\cup_{m>n} B_{i_m j_m} \subset B_{U'}$ implies that

$$|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})| < 1,$$

whence the Jordan measurable set $\cup_{m>n} B_{i_m j_m}$ verifies that

$$|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})| < 1.$$

Recall that $(B_{i_n j_n}, \mu_{i_n j_n})_n$ verifies for that:

$$\sum \{ |\mu_{i_n j_n}(B_{i_m j_m})| : m < n \} < 1,$$

$$|\mu_{i_n j_n}(B_{i_n j_n})| > j_n.$$

The contradiction

Proof.

S^{n+1} increasing respect $S^n \implies \{t^i : i \in \mathbb{N}\} (\subset T)$ is an NV -tree $\implies \cup\{\mathcal{B}_{t^i} : i \in \mathbb{N}\} = \mathcal{J}(K) \ni H := \cup_{s \in \mathbb{N}} \mathcal{B}_{i_s j_s} \implies \exists r \in \mathbb{N} : H \in \mathcal{B}_{t^r}$.





$(n_p)_p \uparrow, i_{n_p} = r \implies \{\mu_{i_{n_p} j_{n_p}} : p \in \mathbb{N}\} \subset M_{t^r}$ (\mathcal{B}_{t^r} -pointwise bounded) $\implies \sup\{|\mu_{i_{n_p} j_{n_p}}(H)| : p \in \mathbb{N}\} < \infty$
 in contradiction with:

$$\left| \mu_{i_{n_p} j_{n_p}} \left(\cup_{m < n_p} \mathcal{B}_{i_m j_m} \right) \right| < 1$$




$$\mu_{i_{n_p} j_{n_p}}(\mathcal{B}_{i_{n_p} j_{n_p}}) > j_{n_p} > n_p \quad \text{and}$$

$$\left| \mu_{i_{n_p} j_{n_p}} \left(\cup_{n_p < m} \mathcal{B}_{i_m j_m} \right) \right| < 1$$



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