Topological groups and spaces $C(X)$ with ordered bases

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Abstract

An index set $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ is boundedly complete if each bounded subset of Σ has an upper bound at Σ. If Σ is unbounded and directed (and if additionally Σ is boundedly complete) a base $\{U_\alpha : \alpha \in \Sigma\}$ of neighborhoods of the identity of a topological group G with $U_\beta \subseteq U_\alpha$, whenever $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$, is called in [7] a Σ-base (a long Σ-base). The case $\Sigma = \mathbb{N}^{\mathbb{N}}$ has been noticed for topological vector spaces under the name of $\mathfrak{G}$-base at [2]. If X is a separable, metrizable and not Polish space, the space $C_c(X)$ has a Σ-base but does not admit any $\mathfrak{G}$-base ([7]). Under an appropriate ZFC model the space $C_c(\omega_1)$ has a long Σ-base which is not a $\mathfrak{G}$-base ([7]).

In [7] we proved that (i) if G is a topological group with a long Σ-base then every compact subset of G is metrizable and (ii) that a Fréchet-Urysohn topological group is metrizable if and only if it has a long Σ-base. This result improve the recent result in [9] stating that a Fréchet-Urysohn topological group with $\mathfrak{G}$-base is metrizable.

By (i) if $C_c(X)$ has a long Σ-base then every compact subset of $C_c(X)$ is metrizable (i.e., $C_c(X)$ is strictly angelic). Then X is a C-Suslin space, and we get that $C_p(X)$ is angelic by Orihuela's theorem at [12], whence $C_c(X)$ is also angelic. Also we show in [7] that a $C_p(X)$ space has a long Σ-base if and only if X is countable.

Problem We do not know whether there exists a topological group with a long Σ-base that admits no $\mathfrak{G}$-base.

Problem Let X be a separable metric space admitting a compact ordered covering of X indexed by an unbounded and boundedly complete proper subset of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact subsets of X. Is then X a Polish space?

1 Σ-bases in topological groups

1.1 $\mathfrak{G}$-bases and quasi-$\mathfrak{G}$-bases

Definition A topological group G is said to have a $\mathfrak{G}$-base if there is a base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighborhoods of the identity e in G such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$.

- Metrizable topological group $\Longrightarrow$ $\mathfrak{G}$-base.
- Fréchet-Urysohn topological group with a $\mathfrak{G}$-base $\Longrightarrow$ metrizable (Grabriyelyan ..., Fundamenta Math. 2015).

Definition A compact resolution on a topological space X is a compact covering $\mathcal{K} = \left\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ of X such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$. If for each compact subset K of X there exists K_α such that $K \subset K_\alpha$, then $\mathcal{K}$ is a compact resolution swallowing compact subsets.

Theorem A space $C_c(X)$ has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of (absolutely convex) neighborhoods of the origin if and only if X has a compact resolution $\mathcal{K} = \left\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ swallowing compact subsets

Corollary If X is a Polish space the $C_c(X)$ has a $\mathfrak{G}$-base. Whence $C_c\left(\mathbb{R}^{\mathbb{N}}\right)$ is a non-metrizable locally convex space with a $\mathfrak{G}$-base.

Definition [Tsaban and Zdomskyy, 2009] A topological group G has the strong Pytkeev property if there exists a sequence $\mathcal{D}$ of subsets of G satisfying the property: for each neighborhood U of the unit e and each $A \subseteq G$ with $e \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite.

Proposition [Gabriyelyan, Kąkol and Leiderman, 2014] Any topological group G with the strong Pytkeev property admits a quasi-$\mathfrak{G}$-base $\{U_\alpha : \alpha \in \Sigma\}$ of the identity, i.e., an ordered base of neighborhoods $\{U_\alpha : \alpha \in \Sigma\}$ of e over some $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$.

Proposition [Banakh, 2015] For every separable metrizable space X the space $C_c(X)$ has the strong Pytkeev property; therefore such $C_c(X)$ admits a quasi-$\mathfrak{G}$-base.

Remark Let X be a separable metric space which is not a Polish space. Then $C_c(X)$ has a quasi-$\mathfrak{G}$-base but $C_c(X)$ does not admit a $\mathfrak{G}$-base.

1.2 Σ-bases and $C_c(X)$ with Σ-base

Definition If $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ is an unbounded (i.e., $\sup\{\alpha(k) : \alpha \in \Sigma\} = \infty$ for some $k \in \mathbb{N}$) and directed subset of $\mathbb{N}^{\mathbb{N}}$, a base $\{U_\alpha : \alpha \in \Sigma\}$ of neighborhoods of the neutral element of a topological group G is a Σ-base if $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$.

Theorem [For a completely regular space X are equivalent:]

1. The locally convex space $C_c(X)$ has a Σ-base of absolutely convex neighborhoods of the origin.
2. There is a compact covering $\{K_\alpha : \alpha \in \Sigma\}$ of X that swallows the compact sets of X, with Σ unbounded, directed and such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$ in Σ.

Theorem If (X, d) is a separable and not Polish, then $C_c(X)$ admits Σ-base and it does not admit any $\mathfrak{G}$-base.

2 Boundedly complete sets and long Σ-bases

2.1 Boundedly complete subsets of $\mathbb{N}^{\mathbb{N}}$

In this section we are going to consider a special class of Σ-bases, which we denominate long Σ-bases, and study some properties of them quite close to those of $\mathfrak{G}$-bases.

Definition A subset Σ of $\mathbb{N}^{\mathbb{N}}$ will be called boundedly complete if each bounded set Δ of Σ has a bound at Σ.

- Σ boundedly complete $\Longrightarrow$ Σ is directed.
- If $\{U_\alpha : \alpha \in \Sigma\}$ is an infinite base of neighborhoods of a (Hausdorff) locally convex space and Σ is a boundedly complete subset of $\mathbb{N}^{\mathbb{N}}$ then Σ must be unbounded. (Otherwise $\sup\{\alpha(k) : \alpha \in \Sigma\} < \infty$ for every $k \in \mathbb{N} \Longrightarrow$ there exists $\gamma \in \Sigma$ with $\alpha \leq \gamma$ for every $\alpha \in \Sigma$. Hence

$$U_\gamma \subseteq \cap_{\alpha \in \Sigma} U_\alpha, \text{ a contradiction)}$$

Example Every cofinal subset Σ of $\mathbb{N}^{\mathbb{N}}$ with respect to the partial order '$\leq$' is boundedly complete.

Proposition If X is a topological space with a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ that swallows the compact sets indexed by a boundedly complete subset Σ of $\mathbb{N}^{\mathbb{N}}$ and such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ in Σ, then X is strongly dominated by a second countable space.

2.2 Long Σ-bases

Definition A Σ-base of neighborhoods of the unit element of a topological group G indexed by a boundedly complete subspace Σ of $\mathbb{N}^{\mathbb{N}}$ will be referred to as a long Σ-base.

Of course, every $\mathfrak{G}$-base of neighborhoods of the origin of a locally convex space E is a long Σ-base, with $\Sigma = \mathbb{N}^{\mathbb{N}}$. The proof of the next theorem uses the following

Proposition [Cascales, Orihuela, Tkachuk, 2011] A compact topological space K is metrizable if and only if the space $(K \times K) \setminus \Delta$ is strongly dominated by a second countable space, where here $\Delta := \{(x, x) : x \in K\}$.

Theorem If a topological group G has a long Σ-base $\{U_\alpha : \alpha \in \Sigma\}$ then every compact subset K in G is metrizable. Consequently, G is strictly angelic.

Corollary If there exists a family $\{A_\alpha : \alpha \in \Sigma\}$ made up of compact sets, indexed by a boundedly complete set Σ such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and satisfying that $\cup\{A_\alpha : \alpha \in \Sigma\} = X$, then $C_c(X)$ is strictly angelic.

Theorem If $C_c(X)$ has a long Σ-base of neighborhoods of the origin, then X is a C–Suslin space. Consequently $C_c(X)$ is angelic.

A limit property in Fréchet-Urysohn topological groups

Let $\{U_\alpha : \alpha \in \Sigma\}$ be a long Σ-base in a topological group G. For every $\alpha = (a_i)_{i \in \mathbb{N}} \in \Sigma$ and each $k \in \mathbb{N}$, set

$$\alpha(k) := (a_1, a_2, \ldots, a_k)$$

$$D_k(\alpha) := \cap\{U_\beta : \beta \in \Sigma, \ \beta(k) = \alpha(k)\}.$$

Clearly, $\{D_k(\alpha)\}_{k \in \mathbb{N}}$ is an increasing and $e \in D_k(\alpha)$.

Proposition [Chasco, Martín-Peinador and Tarieladze, 2007] Let $\{x_{n,k} : (n, k) \in \mathbb{N} \times \mathbb{N}\}$ a subset of a Fréchet-Urysohn topological group G such that $\lim_n x_{n,k} = x \in G$, $k = 1, 2, \ldots$. There exists two increasing sequences of natural numbers $(n_i)_{i \in \mathbb{N}}$ and $(k_i)_{i \in \mathbb{N}}$, such that $\lim_i x_{n_i, k_i} = x$.

Metrizability in Fréchet-Urysohn topological groups

Theorem Each Fréchet-Urysohn topological group G with a long Σ-base $\{U_\alpha : \alpha \in \Sigma\}$ is metrizable.

Corollary Let $\{G_t\}_{t \in T}$ be a family of metrizable topological groups. Then the product $G := \prod_{t \in T} G_t$ has a long Σ-base if and only if T is countable, i.e., when G is metrizable.

Corollary The space $C_p(X)$ has a long Σ-base if and only if X is countable.

3 Existence of proper long Σ-bases on $C_c([0, \omega_1))$

The dominating cardinal

In $\left(\mathbb{N}^{\mathbb{N}}, \leq^*\right)$

- $\alpha \leq^* \beta$ stands for the *eventual dominance preorder* defined so that $\alpha(n) \leq \beta(n)$ for almost all $n \in \mathbb{N}$, i.e., for all but finitely many values of n.
- $\alpha <^* \beta$ means that there exists $m \in \mathbb{N}$ such that $\alpha(n) < \beta(n)$ for every $n \geq m$.

ω_1 is the first ordinal of uncountable cardinal, whose cardinality we denote by $\aleph_1$.

ZFC model means Zermelo–Fraenkel model + axiom of choice.

Definition The *dominating cardinal* $\mathfrak{d}$ is the least cardinality for cofinal subsets of the preordered space $\left(\mathbb{N}^{\mathbb{N}}, \leq^*\right)$.

One has $\aleph_1 \leq \mathfrak{d} \leq \mathfrak{c}$.

Lemma If $\aleph_1 = \mathfrak{d}$ there exists a cofinal ω_1-sequence $\Gamma := \{\beta_\kappa : \kappa < \omega_1\}$ in $\left(\mathbb{N}^{\mathbb{N}}, \leq^*\right)$ such that

1. $\kappa_1 < \kappa_2$ implies that $\beta_{\kappa_1} <^* \beta_{\kappa_2}$,
2. for each $\alpha \in \mathbb{N}^{\mathbb{N}}$ the subset
$$\Delta_\alpha := \{\kappa < \omega_1 : \beta_\kappa \leq^* \alpha\}$$
of $[0, \omega_1)$ is countable,
3. if $\alpha \leq^* \gamma$ then $\Delta_\alpha \subseteq \Delta_\gamma$, and
4. every countable subset of $[0, \omega_1)$ is contained in some Δ_γ; in particular, $\bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \Delta_\alpha = [0, \omega_1)$.

Example In any ZFC model for which $\aleph_1 = \mathfrak{d} < \mathfrak{c}$ there exists a completely regular space X and a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ of X, with $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and indexed by an unbounded, directed and boundedly complete proper subset Σ of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact sets of X.

Corollary In any ZFC model for which $\aleph_1 = \mathfrak{d} < \mathfrak{c}$ there exists a long Σ-base of absolutely convex neighborhoods of the origin of the space $C_c([0, \omega_1))$ which is not a $\mathfrak{G}$-base.

Open question Let X be a separable metric space admitting a compact ordered covering of X indexed by an unbounded and boundedly complete proper subset of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact sets of X. Is then X a Polish space?

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