

Nikodym boundedness property in set-algebras

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- 2 Web Nikodym property in $\mathcal{J}(K)$
- 3 Open question and application

Outline

1 Introduction

Notations

Let \mathcal{B} a subset of a set-algebra \mathcal{A} of subsets of Ω .

$L(\mathcal{B}) := \text{span}\{e_C : C \in \mathcal{B}\}$ with the supremum norm $\|\cdot\|$.

Its dual with dual norm is (isometric) the Banach space $ba(\mathcal{A})$ of bounded variation finitely additive measures on \mathcal{A} with the variation norm ($\|\cdot\| := |\cdot|(\Omega)$)).

Variation norm is equivalent to the \mathcal{A} -supremum norm

$$\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$$

Nikodym property of $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition

\mathcal{B} has property N if each \mathcal{B} -pointwise bounded subset M of $ba(\mathcal{A})$ is bounded in $ba(\mathcal{A})$, i.e.,

$$\sup_{\mu \in M} |\mu(\mathcal{A})| < \infty, \forall \mathcal{A} \in \mathcal{B} \implies \sup_{\mathcal{A} \in \mathcal{A}, \mu \in M} |\mu(\mathcal{A})| < \infty$$

Then $\{e_C : C \in \mathcal{B}\}^{\circ\circ}$ neighborhood of 0 $\implies \overline{L(\mathcal{B})} = L(\mathcal{A})$.

Definition (Valdivia)

\mathcal{B} has property sN if $\mathcal{B} = \cup_m \mathcal{B}_m \uparrow \implies \exists \mathcal{B}_n$ which has property N .

Web Nikodym property of $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition

A family $\mathcal{W} := \{A_{m_1 m_2 \dots m_p} : (m_1, m_2, \dots, m_p) \in \cup_{s \in \mathbb{N}} \mathbb{N}^s\}$ of subsets of A is *increasing web* if

① $A = \cup_{m_1} A_{m_1}$

② $A_{m_1 m_2 \dots m_p} = \cup_{m_{p+1}} A_{m_1, m_2, \dots, m_p, m_{p+1}}, \forall p, m_1, \dots, m_{p+1} \in \mathbb{N}.$

Each sequence $(A_{m_1 m_2 \dots m_p})_p$ is called a *strand* in \mathcal{W} .

Definition (Kakol-LP)

\mathcal{B} has property *wN* if each increasing web $\{\mathcal{B}_t : t \in \cup_s \mathbb{N}^s\}$ in \mathcal{B} has a *N-strand* (=its sets have property *N*).

Properties N , sN and wN in σ -algebras

Theorem

For a σ -algebra S of subsets of a set Ω it was shown:

- 1 S has property N (Nikodym-Dieudonné-Grothendieck).
- 2 S has property sN (Valdivia).
- 3 S has property wN (Kakol-LP).

Properties N , sN and wN in algebras

Example

The algebra of finite and co-finite subsets of \mathbb{N} fails to have property N .

Example

The algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has property N (Schachermayer)

Example

The algebra $\mathcal{J}(K)$ of Jordan measurable subsets of $K := \prod_{1 \leq i \leq k} [a_i, b_i]$ has property sN (Valdivia 2013)

First we show that $\mathcal{J}(K)$ has property wN .

Outline

- 2 Web Nikodym property in $\mathcal{J}(K)$
 - Deep unboundedness and property N
 - Deep unboundedness in $ba(\mathcal{J}(K))$
 - Web Nikodym property in $\mathcal{J}(K)$

Notations in $\cup_s \mathbb{N}^s$

$\{t = (t_1, t_2, \dots, t_p), u = (u_1, u_2, \dots, u_q)\} \subset \cup_s \mathbb{N}^s$.

p is the *length* of t .

$t(i) := (t_1, t_2, \dots, t_i)$ is the *section of length i* of t , $1 \leq i \leq p$.

$t(i) := \emptyset$ if $i > p$. If $t = (t_1)$ then $t = t_1$.

$t \times u := (t_1, t_2, \dots, t_p, t_{p+1}, t_{p+2}, \dots, t_{p+q})$, with $t_{p+j} := u_j$.

Definition

$(t^n = (t_1^n, t_2^n, \dots, t_n^n, \dots)) \in \cup_s \mathbb{N}^s)_n$ is an *infinite chain* if for each $n \in \mathbb{N}$

$$t^{n+1}(n) = t^n(n) \neq \emptyset.$$

Notations in $\cup_s \mathbb{N}^s$

$t \times s$ ($t \times s \in U \subset \cup_s \mathbb{N}^s$) is an extension of t (of t in U).
 U is increasing at $t = (t_1, t_2, \dots, t_p) \in \cup_s \mathbb{N}^s$ if there exists

$$\{t^1 = (t_1^1, t_2^1, \dots), t^i = t(i-1) \times (t_i^i, t_{i+1}^i, \dots), 1 < i \leq p\} \subset U$$

such that $t_i < t_i^i$, for each $1 \leq i \leq p$.

Definition

U is increasing (increasing respect to $V \subset \cup_s \mathbb{N}^s$) if U is increasing at each $t \in U$ (at each $t \in V$).

U is increasing if $\forall t = (t_1, t_2, \dots, t_p) \in U$,

$$|U(1)| = |\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty, 1 \leq i < p$$

NV-tree

Definition

An *NV-tree* T is an increasing subset of $\cup_{s \in \mathbb{N}} \mathbb{N}^s$ without infinite chains such that each $t = (t_1, t_2, \dots, t_p) \in T$ verifies that

- the length of each extension of $t(p-1)$ in T is p
- and $\{t(i) : 1 \leq i \leq p\} \cap T = \{t\}$.

Example

The infinite subsets of \mathbb{N} are *NV-trees*, named trivial *NV-trees*.

Example

$\mathbb{N}^i, i \in \mathbb{N} \setminus \{1\}$, and $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are non trivial *NV-trees*.

Example

The product $T \times U$ of *NV-trees* is an *NV-tree*

Elementary properties of a NV-tree T

Proposition

Each increasing subset S of T is an NV-tree. Whence if $S_n \subset T$ and S_{n+1} is increasing respect to S_n then $\cup_n S_n$ is an NV-tree.

Proposition

If non NV-tree is contained in $U (\subset T)$ then $T \setminus U$ contains an NV-tree.

Proposition

$\{B_u : u \in \cup_s \mathbb{N}^s\} \uparrow \text{web in } B \implies B = \cup \{B_t : t \in T\}.$

Proof.

This equality follows from the following trivial facts:

- 1 $B = \cup_{u \in T} B_{u(1)} \uparrow, B_{u(i)} = \cup_{u(i) \times n \in T(i+1)} B_{u(i) \times n}$
- 2 $b \in B \implies \exists t \in T : b \in B_t, (T \text{ does not contain infinite chains}).$

Deep B -unbounded sets

Definition

Let $B \in \mathcal{A}$. $M \subset ba(\mathcal{A})$ is deep B -unbounded if for each finite subset Q of $\{e_A : A \in \mathcal{A}\}$

$$\sup\{|\mu(C)| : \mu \in M \cap Q^\circ, C \in \mathcal{A}, C \subset B\} = \infty.$$

Proposition

Let $M(\subset ba(\mathcal{A}))$ be deep B -unbounded and $\{B_i \in \mathcal{A} : 1 \leq i \leq q\}$ a partition of B . There exists j , $1 \leq j \leq q$, such that M is deep B_j -unbounded.

Proof.

If there exist finite $Q^i \subset \{e_A : A \in \mathcal{A}\}$ with $\sup\{|\mu|(C_i) : \mu \in M \cap (Q^i)^\circ\} < H_i$, $1 \leq i \leq q$, then for $Q = \bigcup_{1 \leq i \leq q} Q^i$ we get $\sup\{|\mu|(B) : \mu \in M \cap Q^\circ\} < \sum_{1 \leq i \leq q} H_i$. \square

Deep Ω -unbounded sets and sequences

Proposition

M is deep Ω -unbounded if and only if M° is not neighborhood of zero in its linear hull or if its $\text{span}\{M^\circ\}$ has infinite codimension in $L(\mathcal{A})$. In particular, if $\overline{\text{span}\{M^\circ\}} = L(\mathcal{A})$ then M is deep Ω -unbounded if and only if M is unbounded.

Proposition (Increasing sequences without N property)

Let \mathcal{A} be an algebra of subsets of Ω and let $(\mathcal{B}_m)_m$ be an increasing sequence of subsets of \mathcal{A} . If each \mathcal{B}_m does not have N -property and $\overline{\text{span}\{e_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$ then $\exists n_0 \in \mathbb{N}$ such that for each $m \geq n_0$ there exists a deep Ω -unbounded subset M_m of $ba(\mathcal{A})$ which is pointwise bounded in \mathcal{B}_m .

In particular, this proposition holds if $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has N -property.

Webs without N -strands and NV -trees

Proposition

Let $\mathcal{B} := \{\mathcal{B}_{m_1 m_2 \dots m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ be an increasing web in \mathcal{A} without N -strands.

- 1 Then there exists an NV -tree T' such that for each $t = (t_1, t_2, \dots, t_q) \in T'$ the set \mathcal{B}_t does not have property N and if $p > 1$ then $\mathcal{B}_{t(i)}$ has property N if $i < p$.
- 2 There exists an NV -tree $T \subset T'$ such that for each $t \in T$ there exists a deep Ω -unbounded subset M_t of $ba(S)$ which is \mathcal{B}_t -pointwise bounded.

$$\mathcal{J}(K), K := \prod_{1 \leq i \leq k} [a_i, b_i] \subset \mathbb{R}^k$$

Definition

A bounded $B \subset \mathbb{R}^k$ is Jordan measurable whenever the Lebesgue measure of $\overline{B} \setminus \overset{\circ}{B}$ is zero.

$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K)$, $M \subset ba(\mathcal{J}(K))$, deep B -unbounded.

Proposition

If $\alpha > 0$ and $\epsilon > 0 \exists C_1, C'_1 \in \mathcal{J}(K)$, disjoint contained in B and $\mu \in M$ such that

- 1 $|\mu(C_1)| > \alpha$, $|\mu(C'_1)| > \alpha$, $\sum_{1 \leq j \leq r} \mu(Q_j) \leq 1$,
- 2 $C_1 \cup C'_1 \subset \prod_{1 \leq i \leq k} [c_i, d_i] \subset K$, $\prod_{1 \leq i \leq k} (d_i - c_i) < \epsilon$
- 3 M is deep C'_1 -unbounded.

$$B \subset K := \prod_{1 \leq i \leq k} [a_i, b_i] \subset \mathbb{R}^k$$

Proof.

There exists $B_n \subset B$, M deep B_n -unbounded and $B_n \subset k$ -compact interval of measure $< \epsilon$. Let $\mathcal{Q} := \{B_n, Q_1, \dots, Q_r\}$. As

$$\sup\{|\mu(C)| : \mu \in rM \cap \mathcal{Q}^\circ, C \subset B_n, C \in \mathcal{A}\} = \infty,$$

$\exists D_1 \subset B_n, D_1 \in \mathcal{T}(K), \exists \lambda \in rM \cap \mathcal{Q}^\circ : |\lambda(D_1)| > r(1 + \alpha)$. Hence

$$\mu = r^{-1}\lambda \in M, |\mu(B_n)| \leq r^{-1} \leq 1 \text{ and } \sum_{1 \leq j \leq r} |\mu(Q_j)| \leq r^{-1}r = 1$$

we get $|\mu(B_n \setminus D_1)| \geq |\mu(D_1)| - |\mu(B_n)| > 1 + \alpha - 1 = \alpha$.

M is deep D_1 -unbounded or $B_n \setminus D_1$ -unbounded. □

$$\{B, Q_1, \dots, Q_r\} \subset \mathcal{T}(K), M \subset ba(\mathcal{T}(K))$$

Corollary

If M deep B -unbounded, $q \in \mathbb{N}$, $\alpha > 0$, $\epsilon > 0$, \exists in B pairwise disjoint Jordan measurable subsets C_1, C_2, \dots, C_q and $\mu_1, \mu_2, \dots, \mu_q \in M$ such that:

- 1 $|\mu_i(C_i)| > \alpha$, $\sum_{1 \leq j \leq r} \mu_i(Q_j) \leq 1$, $i = 1, 2, \dots, q$,
- 2 M is deep C_q -unbounded
- 3 $\cup_{1 \leq i \leq q} C_i \subset \prod_{1 \leq i \leq k} [c_i, d_i] \subset K$, $\prod_{1 \leq i \leq k} (d_i - c_i) < \epsilon$.

Proof.

There exists two disjoint subsets $C_1, C'_1 \in \mathcal{T}(K)$, in B , and $\mu_1 \in M$:

- 1 $|\mu_1(C_1)| > \alpha$, $|\mu_1(C'_1)| > \alpha$, $\sum_{1 \leq j \leq r} \mu_1(Q_j) \leq 1$,
- 2 M is deep C'_1 -unbounded
- 3 $C_1 \cup C'_1 \subset \prod_{1 \leq i \leq k} [c_i, d_i] \subset K$, $\prod_{1 \leq i \leq k} (d_i - c_i) < \epsilon \dots$



$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K), M_t \subset ba(\mathcal{J}(K)), t \in T$

Proposition

If T is an NV-tree, each M_t is deep B -unbounded, $\alpha > 0$, $\epsilon > 0$ and $\{t^j : 1 \leq j \leq k\} \subset T$, then

- there exists in B two disjoint subsets $B_1, B'_1 \in \mathcal{J}(K)$,
- and there exists $\mu_1 \in M_{t^1}$, an NV-tree T_1 and k intervals $\prod_{1 \leq i \leq k} [c_i^s, d_i^s] \subset K$, $1 \leq s \leq q$,

such that:

- 1 $\{t^j : 1 \leq j \leq k\} \subset T_1 \subset T$ and M_t is deep B'_1 -unbounded, $\forall t \in T_1$.
- 2 $|\mu_1(B_1)| > \alpha$ and $\sum\{|\mu_1(Q_i)| : 1 \leq i \leq r\} \leq 1$.
- 3 $B_1 \cup B'_1 \subset \bigcup_{1 \leq s \leq q} \prod_{1 \leq i \leq k} [c_i^s, d_i^s] \subset K$,
 $\sum_{1 \leq s \leq q} (\prod_{1 \leq i \leq k} (d_i^s - c_i^s)) < \epsilon$.

$$t^j := (t_1^j, t_2^j, \dots, t_{p_j}^j), \quad 1 \leq j \leq k$$

Proof.

Let $q := 2 + \sum_{1 \leq j \leq k} p_j$, and let $\{B_1, B_2, \dots, B_m\}$ be a $\mathcal{T}(K)$ partition of B with each $B_i \neq \emptyset$ and contained in a k -interval of measure $< \epsilon/q$.
Exists

$B_n, 1 \leq n \leq m$, such that M_{t^1} is deep B_n -unbounded.

By Corollary (with $\{B_n, Q_1, \dots, Q_r\}, M_{t^1}, \alpha$ and ϵ/q) exists a $\mathcal{T}(K)$ partition $\{C_1, C_2, \dots, C_q\}$ of B_n and $\{\lambda_1, \lambda_2, \dots, \lambda_q\} \subset M_{t^1}$ such that:

$$|\lambda_k(C_k)| > \alpha \quad \text{and} \quad \sum_{1 \leq i \leq r} |\lambda_k(Q_i)| \leq 1, \quad \text{for } k = 1, 2, \dots, q,$$



$$\{D_i : 1 \leq i \leq q'\} := \{B_i : i \neq n\} \cup \{C_i : 1 \leq i \leq q\}$$

Proof.

- $j \in \{1, 2, \dots, k\} \implies \exists i^j \in \{1, 2, \dots, q'\}$ such that

M_{t^j} is deep D_{i^j} -unbounded.

- $t^j = (t_1^j, t_2^j, \dots, t_{p_j}^j)$, $1 \leq j \leq k$, and $2 \leq m \leq p_j$ then $M_{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times w}$ is deep B -unbounded if $t^j(m-1) \times v \in T$.
 Whence $\exists i_m^j \in \{1, 2, \dots, q'\}$ and an NV -tree V_m^j ($t^j(m-1) \times V_m^j \subset T$) such that

$M_{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times w}$ is deep $D_{i_m^j}$ -unbounded $\forall w \in V_m^j$.

- $\exists i_0 \in \{1, 2, \dots, q'\}$ and an NV -tree $T_{i_0} \subset \{t \in T : t_1^j < t(1), 1 \leq j \leq k\}$ such that

M_t is deep D_{i_0} -unbounded $\forall t \in T_{i_0}$.

B_1 and μ_1

Proof.

Let D be the union of D_{i_0} and all D_{ij} and D_{ij}^m .

D is a union of less than q sets, whence there exists C_h such that

$$C_h : D \subset B \setminus C_h.$$

The union T_1 of T_{i_0} and all t^j and $(t_1^j, t_2^j, \dots, t_{m-1}^j) \times V_m^j$ has the increasing property.

As M_t is deep D -unbounded for each $t \in T_1$ we get the proof with

$$B_1 := C_h, \quad B'_1 := D, \quad \text{and} \quad \mu_1 := \lambda_h,$$

because by construction

- 1 $|\mu_1(B_1)| > \alpha$, $\sum_{1 \leq i \leq r} |\mu_1(Q_i)| \leq 1$, and
- 2 $B_1 \cup B'_1 \subset \bigcup_{s \in J} \{\prod_{1 \leq i \leq k} [c_i^s, d_i^s]\}$, $|J| \leq q$, whence $\sum_{s \in J} (\prod_{1 \leq i \leq k} (d_i^s - c_i^s)) < (\epsilon/q)q = \epsilon$.

$$\{B, Q_1, \dots, Q_r\} \subset \mathcal{J}(K), M_t \subset ba(\mathcal{J}(K)), t \in T$$

Corollary

If T is an NV-tree, each M_t is deep B -unbounded, $\alpha > 0$, $\epsilon > 0$ and $\{t^j : 1 \leq j \leq k\} \subset T$, then there exists

- a subset $\{B_1, B_2, \dots, B_k, B'\} \subset \mathcal{J}(K)$ of pairwise disjoint subsets of B ,
- k measures $\mu_j \in M_{t^j}$, $1 \leq j \leq k$, and an NV-tree T^*

such that:

- 1 $\{t^j : 1 \leq j \leq k\} \subset T^* \subset T$ and M_t is deep B' -unbounded $\forall t \in T^*$.
- 2 $|\mu_j(B_j)| > \alpha$, $\sum\{|\mu_j(Q_i)| : 1 \leq i \leq r\} \leq 1$, for $1 \leq j \leq k$.
- 3 B' contained in a finite union of k -intervals of measure $< \epsilon$.

Apply Proposition.

Apply Proposition with $\{t^2, t^3, \dots, t^k, t^1\}$ and $B := B'_1 \dots$

$\mathcal{J}(K)$ has property wN

Theorem

The algebra $\mathcal{J}(K)$ has property wN .

Proof.

If $\mathcal{J}(K)$ does not have property wN there exists in $\mathcal{J}(K)$ an increasing web

$$\{\mathcal{B}_t : t \in \cup_s \mathbb{N}^s\}$$

without N -strands.

Whence there exists an NV -tree T and a family

$$\{M_t : t \in T\}$$

of deep K -unbounded subsets of $ba(\mathcal{J}(K))$ which are \mathcal{B}_t -pointwise bounded. □

The first induction

Proof.

By induction we determine

- an NV -tree $\{t^i : i \in \mathbb{N}\} \subset T$ and $(k_j \in \mathbb{N})_j \uparrow$

such that for each $(i, j) \in \mathbb{N}^2$ with $i \leq k_j$

- there exists a set $B_{ij} \in \mathcal{T}(K)$ and $\mu_{ij} \in M_{t^i}$

that verify

- 1 $\sum_{s,v} \{|\mu_{ij}(B_{sv})| : s \leq k_v, 1 \leq v < j\} < 1,$
- 2 $|\mu_{ij}(B_{ij})| > j,$
- 3 $B_{ij} \cap B_{i'j'} = \emptyset$ if $(i, j) \neq (i', j')$ and
- 4 $\cup\{B_{sv} : s \leq k_v, j < v\} \subset \cup_{1 \leq s \leq q_j} (\prod_{1 \leq i \leq k} [c_{ij}^s, d_{ij}^s]) \subset K$
- 5 $\sum_{1 \leq s \leq q_j} (\prod_{1 \leq i \leq k} (d_{ij}^s - c_{ij}^s)) < 2^{-j},$ for each $j \in \mathbb{N}.$



First induction. Step I

Proof.

Select $t^1 \in T$.

By Corollary with

$$B := \Omega, \quad \alpha = 1 \quad \text{and} \quad \epsilon = 2^{-1}$$

there exists

$$B_{11}, B'_1 \in \mathcal{J}(K), \quad \mu_{11} \in M_{t^1} \quad \text{and an } NV\text{-tree } T_1$$

such that

- $|\mu_{11}(B_{11})| > 1, t^1 \in T_1 \subset T,$
- M_{t^1} is deep B'_1 -unbounded for each $t \in T_1$ and
- $B'_1 \subset \cup_{1 \leq s \leq q_1} \{\prod_{1 \leq i \leq k} [c_{i1}^s, d_{i1}^s]\} \subset K,$ such that
- $\sum_{1 \leq s \leq q_1} (\prod_{1 \leq i \leq k} (d_{i1}^s - c_{i1}^s)) < 2^{-1}.$

Define $k_1 := 1$ and $S^1 := \{t^1\}.$



First induction. Induction hypothesis

Proof.

Let us suppose that we have obtained for each $1 < j \leq n$:

- The natural numbers $k_1 < k_2 < k_3 < \dots < k_n$,
- the NV -trees $T_1 \supset T_2 \supset T_3 \supset \dots \supset T_n$,
- $S^j := \{t^i : i \leq k_j\} \subset T_j$, S^j increasing respect S^{j-1} ,
- $\{\mu_{iv} \in M_{t^i} : i \leq k_v, 1 \leq v \leq j\}$
- $\{B'_j, B_{iv} : i \leq k_v, 1 \leq v \leq j\} \subset \mathcal{T}(K)$, pairwise disjoint,
- $|\mu_{ij}(B_{ij})| > j$ and $\sum_{s \leq k_v, 1 \leq v < j} |\mu_{ij}(B_{sv})| < 1, i \leq k_j$,
- $t \in T_j \implies M_t$ is deep B'_j -unbounded,
- $\cup_{s \leq k_v, j < v \leq n} B_{sv} \subset B'_j \subset \cup_{1 \leq s \leq q_j} \prod_{1 \leq i \leq k} [c_{ij}^s, d_{ij}^s] \subset K$,
- $\sum_{1 \leq s \leq q_j} (\prod_{1 \leq i \leq k} (d_{ij}^s - c_{ij}^s)) < 2^{-j}$.



First induction. End

Proof.

Select $S_{n+1} := \{t^{k_{n+1}}, \dots, t^{k_{n+1}}\} \subset T_n \setminus \{t^i : i \leq k_n\}$ increasing respect to S^n and apply Corollary with

$$\{B'_n, B_{sv} : s \leq k_v, 1 \leq v \leq n\}, T_n, \alpha = n + 1 \quad \epsilon = 2^{-n-1}$$

and $S^{n+1} := \{t^i : i \leq k_{n+1}\} \subset T_n$ to get for each $1 \leq i \leq k_{n+1}$:

- The $\mathcal{J}(K)$ -measurable pairwise disjoint subsets B'_{n+1} , $B_{1,n+1}, \dots, B_{k_{n+1},n+1}$ of B'_n ,
 - $\mu_{in+1} \in M_{t^i}$ and the increasing tree T_{n+1} such that
- 1 $|\mu_{in+1}(B_{in+1})| > n + 1, \sum_{s \leq k_v, 1 \leq v \leq n} |\mu_{in+1}(B_{sv})| < 1,$
 - 2 $S^{n+1} \subset T_{n+1} \subset T_n, M_t$ deep B'_{n+1} -unbounded $\forall t \in T_{n+1},$
 - 3 $B'_{n+1} \subset \cup_{1 \leq s \leq q_{n+1}} \{\prod_{1 \leq h \leq k} [c_{h,n+1}^s, d_{h,n+1}^s]\} \subset K$ and
 - 4 $\sum_{1 \leq s \leq q_{n+1}} (\prod_{1 \leq i \leq k} (d_{i,n+1}^s - c_{i,n+1}^s)) < 2^{-n-1}.$

Induction result

$$\begin{array}{llll} B_{11, \mu_{11}} \in M_1 & B_{12, \mu_{12}} \in M_1 & B_{13, \mu_{13}} \in M_1 & \dots \\ & B_{22, \mu_{22}} \in M_2 & B_{23, \mu_{23}} \in M_2 & \dots \\ & \vdots & \vdots & \\ & B_{k_2 2, \mu_{k_2 2}} \in M_{k_2} & B_{k_2 3, \mu_{k_2 3}} \in M_{k_2} & \dots \\ & & \vdots & \dots \\ & B_{k_2 2, \mu_{k_2 2}} \in M_{k_2} & B_{k_2 3, \mu_{k_2 3}} \in M_2 & \dots \\ & & \vdots & \\ & & B_{k_3 3, \mu_{k_3 3}} \in M_{k_3} & \dots \end{array}$$

Some Jordan measurable sets

Proof.

For each $H_j \subset \{1, 2, \dots, k_j\}$, $j \in \mathbb{N}$, the set

$$B := \cup\{B_{ij} : i \in H_j, j \in \mathbb{N}\}$$

is Jordan measurable, because for each $j_0 \in \mathbb{N}$

$$\overline{B} \setminus \overset{\circ}{B} \subset \{\cup_{i \in H_j, j \leq j_0} (\overline{B}_{ij} \setminus \overset{\circ}{B}_{ij})\} \cup \overline{B}'_{j_0}$$

$$\overline{B}'_{j_0} \subset \cup_{1 \leq s \leq q_{j_0}} \{\prod_{1 \leq i \leq k} [c_{ij_0}^s, d_{ij_0}^s]\},$$

the Lebesgue measure of each $\overline{B}_{ij} \setminus \overset{\circ}{B}_{ij}$ is 0 and

\overline{B}'_{j_0} is contained in a Lebesgue measurable set of measure less or equal than 2^{-j_0} . □

Second induction: Basic idea

Proof.

For $(i_n)_n := (1, 1, 2, 1, 2, 3, \dots)$ there exists a $(j_n)_n \uparrow$ in \mathbb{N} such that

$$|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})| < 1, \text{ for each } n \in \mathbb{N}.$$

In fact, if $|\mu_{i_n j_n}|(\Omega) < s_n \in \mathbb{N}$, J_n is an infinite subset of $\mathbb{N} \setminus \{1, 2, \dots, j_n\}$, $\{L_u, 1 \leq u \leq s_n\}$ is a partition of J_n in infinite subsets and $B_u := \cup_{s \leq k_v, v \in L_u} B_{sv}$, then as

$$\sum \{|\mu_{i_n j_n}|(B_u) : 1 \leq u \leq s_n\} \leq \mu(\Omega) < s_n$$

there exists u' , with $1 \leq u' \leq s_n$, such that

$$|\mu_{i_n j_n}|(B_{u'}) < 1.$$

To finish the induction let j_{n+1} be the first element of $L_{u'}$ and $J_{n+1} := L_{u'} \setminus \{j_{n+1}\}$. From $\cup_{m>n} B_{i_m j_m} \subset B_{u'}$ it follows that $|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})| < 1$.



The contradiction

Proof.

S^{n+1} increasing respect $S^n \implies \{t^i : i \in \mathbb{N}\}$ is an NV -tree.

$$H := \bigcup_{s \in \mathbb{N}} B_{i_s j_s} \in \mathcal{T}(K) = \bigcup \{B_{t^i} : i \in \mathbb{N}\} \implies \exists r : H \in \mathcal{B}_{t^r}.$$

Let $(n_p)_p \uparrow$, $i_{n_p} = r$ then $\{\mu_{i_{n_p} j_{n_p}} : p \in \mathbb{N}\} \subset M_{t^r}$, which is \mathcal{B}_{t^r} -pointwise bounded. The inequality

$$\sup \{ |\mu_{i_{n_p} j_{n_p}}(H)| : p \in \mathbb{N} \} < \infty$$

contradicts:

$$\begin{aligned} & \left| \mu_{i_{n_p} j_{n_p}} \left(\bigcup_{m < n_p} B_{i_m j_m} \right) \right| < 1 \\ & \mu_{i_{n_p} j_{n_p}}(B_{i_{n_p} j_{n_p}}) > j_{n_p} > n_p \quad \text{and} \\ & \left| \mu_{i_{n_p} j_{n_p}} \left(\bigcup_{n_p < m} B_{i_m j_m} \right) \right| < 1 \end{aligned}$$

Set-algebras with wN -property

Proposition

Let $(\mathcal{A}_n)_n$ be a decreasing sequence of subsets of an algebra \mathcal{A} of subsets of Ω such that for each $n \in \mathbb{N}$,

- 1 Ω is covered by a finite subset of \mathcal{A}_n .
- 2 $\{A \cap B : A \in \mathcal{A}_n, B \in \mathcal{A}\} \subset \mathcal{A}_n$.
- 3 $\{C \cap D : C, D \in \mathcal{A}_{n+1}\} \subset \mathcal{A}_n$.

Then \mathcal{A} has wN -property when the following condition hold:

- If a pairwise disjoint sequence $(A_n)_n$, with each $A_n \in \mathcal{A}_n$ and such that for each n there exists $B_n \in \mathcal{A}_n$ that verify

$$\bigcup_{n < m} A_m \subset B_n$$

for each n , then $\bigcup_n A_n \in \mathcal{A}$.

Follow the scheme to prove wN -property in $J(K)$ to get the proof. Clearly $J(K)$ and each σ -algebra verify this proposition.

Outline

- 3 Open question and application
 - Uniform bounded deciding sets
 - On bounded vector measures in $\mathcal{J}(K)$

Uniform bounded deciding sets

The fact that a set algebra \mathcal{A} has N -property if each \mathcal{A} -pointwise bounded subset M of $ba(\mathcal{A})$ is uniformly bounded in the unit ball of $(L(\mathcal{A}), \|\bullet\|)$, motivates the *ubd* definition (Shapiro and others).

Definition

A subset A of a normed space $(E, \|\bullet\|)$ is uniform bounded deciding (*ubd*) if each \mathcal{A} -pointwise bounded subset M of E' is uniformly bounded in its unit ball.

- A is *ubd* $\iff \{\|x\|^{-1} x : x \in A \setminus \{0\}\}$ is *ubd*,
- E is *ubd* $\iff (E, \|\bullet\|)$ is barrelled and
- \mathcal{A} has N -property $\iff \{e(A) : A \in \mathcal{A}\}$ is *ubd* subset of $(L(\mathcal{A}), \|\bullet\|)$.

Characterization of *ubd* subsets of $(E, \|\bullet\|)$

- 1 If B be a non-*ubd* subset of $(E, \|\bullet\|)$,
- 2 there exist an unbounded B -pointwise bounded subset M in the dual Banach space.
- 3 Then $B = \cup_n B_n$, with $B_n := \{x \in B : |x^*(x)| \leq n, x^* \in M\}$ non-norming.
- 4 $\text{span } B = \text{span } \{\cup_n n^{-1} B_n\}$, and $A := \cup_n n^{-1} B_n$ non-norming ($M \subset A^\circ$).

For each $m \in \mathbb{N}$ there exists $x'_m \in (m(\text{absco } A))^\circ$ with $\|x'_m\| > m$, whence

5. $\text{span } B = \cup_m m(\text{absco } A)$ is a non-*ubd* (unbounded set $\{x'_m : m \in \mathbb{N}\}$ is pointwise bounded in $\text{span } B$).
6. B be a non-*ubd* subset of $(E, \|\bullet\|)$.

Valdivia sN open question

Open question (Valdivia 2013). *Let \mathcal{A} be a set-algebra that has property N . Has the set-algebra \mathcal{A} the property sN ?*

This question motivated the following analogous problem:

Open question. Assume that an *ubd* subset B of a normed space is written as a countable, increasing union $B := \cup_n B_n$. Does there exist an index m so that B_m is a *ubd* set?

A $\mathcal{J}(K)$ property

A *bounded finitely additive vector measure*, or simply a *bounded vector measure*, μ defined in an algebra \mathcal{A} of subsets of Ω with values in a topological vector space $E(\tau)$ is a map $\mu: \mathcal{A} \rightarrow E$ such that $\mu(\mathcal{A})$ is a bounded subset of E and $\mu(B \cup C) = \mu(B) + \mu(C)$, for each pairwise disjoint subsets $B, C \in \mathcal{A}$.

The set $\mu(\mathcal{A})$ is a bounded subset of $E(\tau)$ if and only if the E -valued linear map $\mu: L(\mathcal{A}) \rightarrow E$ defined by $\mu(e_B) := \mu(B)$, for each $B \in \mathcal{A}$, is continuous.

Proposition

Let $(\mathcal{B}_{m_1})_{m_1}$ be an increasing covering of the algebra $\mathcal{J}(K)$. There exists $n_1 \in \mathbb{N}$ such that \mathcal{B}_{n_1} has property wN .

Proof.

If $\{\mathcal{B}_t^{m_1} : t \in \cup_s \mathbb{N}^s\}$ is an increasing web in \mathcal{B}_{m_1} without N -strands, for each $m_1 \in \mathbb{N}$, then $\{\mathcal{B}_t^{m_1} : (m_1, t) \in \mathbb{N} \times \cup_s \mathbb{N}^s\}$ is an increasing web without N -strands, contradiction. □

A localization property

Proposition

Let μ be a bounded vector measure defined in $\mathcal{J}(K)$ with values in $E(\tau)$, inductive limit $(E_{m_1}(\tau_{m_1}))_{m_1}$ of (LF)-spaces. There exists E_{m_1} such that for each defining sequence $(E_{n_1 m_2}(\tau_{n_1 m_2}))_{m_2}$ there exists $E_{n_1 n_2}$ such that $\mu(\mathcal{S})$ is $\tau_{n_1 n_2}$ -bounded.






Proof.

$(\mathcal{B}_{m_1} := \mu^{-1}(E_{m_1}))_{m_1} \uparrow$ covers $\mathcal{J}(K)$. Exists \mathcal{B}_{n_1} with property wN. If $(E_{n_1 m_2}(\tau_{n_1 m_2}))_{m_2}$ defines $E_{n_1}(\tau_{n_1})$ then $(\mathcal{B}_{n_1 m_2} := \mu^{-1}(E_{n_1 m_2}))_{m_2} \uparrow$ covers \mathcal{B}_{n_1} . If $\mathcal{B}_{n_1 n_2}$ has property N, the closed map





$$\mu|_{L(\mathcal{B}_{n_1 n_2})} : (L(\mathcal{B}_{n_1 n_2}), \|\bullet\|) \rightarrow E_{n_1 n_2}(\tau_{n_1 n_2})$$

has a continuous extension ν to $(L(\mathcal{J}(K)), \|\bullet\|)$ in $E_{n_1 n_2}(\tau_{n_1 n_2})$.
 $\nu(A) = \mu(A)$, for each $A \in \mathcal{J}(K)$ (by continuity of $\mu : L(\mathcal{J}(K)) \rightarrow E(\tau) \implies \mu(\mathcal{J}(K))$ is a $\tau_{n_1 n_2}$ -bounded. □

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