Metrizability and angelicity in topological groups with bases

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Outline

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1. Σ-bases in topological groups
   - $\mathfrak{G}$-bases and quasi-$\mathfrak{G}$-bases
   - Σ-bases and $C_c(X)$ with Σ-base

$\mathfrak{G}$-bases

Definition

A topological group G is said to have a $\mathfrak{G}$-base if there is a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods of the neutral e in G such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$.

- Metrizable topological group $\Longrightarrow$ $\mathfrak{G}$-base.

Let us see that the converse is not true, even in $C_c(X)$

Definition

A compact resolution on a topological space X is a compact covering $\mathcal{K} = \left\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\right\}$ of X such that $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$. If for each compact subset K of X there exists K_α such that $K \subset K_\alpha$, then $\mathcal{K}$ is a compact resolution swallowing compact subsets.

$\mathfrak{G}$-bases in $C_c(X)$

Theorem

A space $C_c(X)$ has a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ if and only if X has a compact resolution $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ swallowing compact subsets

Proof.

Let $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base and K a compact subset of X.

$$\exists\, U_\alpha \subset W(K,[-1,1]) \Longrightarrow K \subset K_\alpha := \cap_{f\in U_\alpha} f^{-1}([-1,1])$$

Exists a compact K_{U_α} and $\varepsilon_\alpha > 0$ such that

$$W(K_{U_\alpha},(-\varepsilon,\varepsilon)) \subset U_\alpha \subset W(K_\alpha,[-1,1]) \subset W(K,[-1,1])$$

whence $K \subset K_\alpha \subset K_{U_\alpha} \Longrightarrow \{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a c r of X s c s.
The converse follows from

$$W(K_{\alpha=(a_1,..)},[-a_1^{-1},a_1^{-1}]) \subset W(K,[-\varepsilon,\varepsilon]) \quad \text{if} \quad K \subset K_\alpha \quad \text{and} \quad a_1^{-1} < \varepsilon.$$

$\mathfrak{G}$-bases in non-metrizable $C_c(X)$

Corollary

If (X,d) is a Polish space the $C_c(X)$ has a $\mathfrak{G}$-base.
The non-metrizable $C_c(\mathbb{R}^{\mathbb{N}})$ has $\mathfrak{G}$-base.

Proof.

If $\overline{\{x_n : n \in \mathbb{N}^{\mathbb{N}}\}} = X$ and $\alpha := (a_n)_n \in \mathbb{N}^{\mathbb{N}}$ then

$$K_\alpha := \cap_{n\in\mathbb{N}}[\cup_{1\leq m\leq a_n}\overline{B(x_m, n^{-1})}]$$

and $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a c r of X s c s.
Finally, $\mathbb{R}^{\mathbb{N}}$ is Polish but not hemicompact. □

Theorem (Grabriyelyan-Kąkol-Leiderman, Fundamenta Math. 2015)

Fréchet-Urysohn topological group with a $\mathfrak{G}$-base $\Longrightarrow$ metrizable.

Strong Pytkeev property and quasi-$\mathfrak{G}$-bases

Definition (Tsaban and Zdomskyy, 2009)

A t g G has strong Pytkeev property if $\exists\ (D_n \subset G)_n$ for each e-neighborhood U and each $\overline{A} \setminus A \ni e$, exists

$$D_n \subset U \quad \text{with} \quad |D_n \cap A| = \infty.$$

Proposition (Gabriyelyan, Kąkol and Leiderman, 2014)

G with the strong Pytkeev property admits a quasi-$\mathfrak{G}$-base $\{U_\alpha : \alpha \in \Sigma\}$ of e, i.e., an ordered base ($U_\beta \subseteq U_\alpha$ if $\alpha \leq \beta$ and $\alpha, \beta \in \Sigma$) of e-neighborhoods with $\Sigma \subseteq \mathbb{N}^\mathbb{N}$.

Proposition (Banakh, 2015)

(X, d) separable $\Longrightarrow C_c(X)$ s P p.

Remark

Let (X, d) separable not Polish space $\Longrightarrow C_c(X)$ has quasi-$\mathfrak{G}$-base

$C_c(X)$ with Σ-base

Definition

A quasi-$\mathfrak{G}$-base $\{U_\alpha : \alpha \in \Sigma\}$ of e-neigh of G is a Σ-base if Σ is directed and unbounded ($\exists k$ with $\sup_{\alpha\in\Sigma} \alpha(k) = \infty$).

Theorem

$C_c(X)$ has Σ-base $\Longleftrightarrow \exists$ compact X-covering $\{K_\alpha : \alpha \in \Sigma\}$ swallowing X-comp subsets, with Σ unbounded, directed and

$$K_\alpha \subseteq K_\beta \text{ whenever } \alpha \leq \beta \text{ in } \Sigma.$$

Proof ($\Longrightarrow$like when $\Sigma = \mathbb{N}^{\mathbb{N}}$).

The converse follows from given K and $\varepsilon > 0$ there exists $\alpha \in \Sigma$ such that $K \subset K_\alpha$ and $a_n^{-1} < \varepsilon$, whence

$$W(K_{\alpha=(a_1,..)}, [-a_1^{-1}, a_1^{-1}]) \subset W(K, [-\varepsilon, \varepsilon]) \quad \text{if} \quad K \subset K_\alpha \quad \text{and} \quad a_1^{-1} < \varepsilon.$$

Σ-base $\not\Longrightarrow$ $\mathfrak{G}$-base

Theorem

(X, d) sep not Polish $\Longrightarrow$ $C_c(X)$ Σ-base not admit $\mathfrak{G}$-base.

Proof.

Let K be τ_d-compact, $\overline{\{x_m : m \in \mathbb{N}\}} = X$ and $\overline{\{y_n : n \in \mathbb{N}\}} = K$.
If $\lim_p x_{n(p)} = y_n$ and $d(x_{n(p)}, y_n) < n^{-1}$, for each $p \in \mathbb{N}$,then

$$K \subset \overline{\{x_{n(p)} : (n, p) \in \mathbb{N}^2\}}, \quad \text{compact.}$$

If $\alpha \in \Sigma := \cup_{p \in \mathbb{N} \setminus \{1\}} \{1, p\}^{\mathbb{N}}$ and

$$K_\alpha := \overline{\{x_m : m \in \mathbb{N},\ a_m \neq 1\}}$$

then the compact covering $\{K_\alpha : \alpha \in \Sigma\}$ gives a Σ-base of $C_c(X)$.
If $C_c(X)$ has $\mathfrak{G}$-base $\Longrightarrow$ X compact resolution swallowing compact subsets $\Longrightarrow$ X Polish (contradiction). □

Outline

2. Long Σ-bases. Metrization of topological groups
   - Boundedly complete subsets of $\mathbb{N}^{\mathbb{N}}$ and long Σ-bases
   - Metrization of topological groups
   - Existence of proper long Σ-bases on $C_c([0, \omega_1))$

Boundedly complete subsets of $\mathbb{N}^{\mathbb{N}}$

Long Σ-bases enable to improve G-K-L group metrizability theorem.

Definition

Σ ($\subset \mathbb{N}^{\mathbb{N}}$) is boundedly complete if each bounded subset Δ of Σ has a bound at Σ.

$\sup\{\alpha(k) : \alpha \in \Delta\} < \infty$, for each $k \in \mathbb{N} \Longrightarrow \exists\gamma \in \Sigma$, $\alpha \leq \gamma$, for each $\alpha \in \Delta$.

- Σ boundedly complete ($\Longrightarrow \Sigma$ is directed) and $\{U_\alpha : \alpha \in \Sigma\}$ infinite base of neighborhoods of a topological group then Σ unbounded.

Proof.

$\sup\{\alpha(k) : \alpha \in \Sigma\} < \infty$, $\forall k \in \mathbb{N} \Longrightarrow \exists\gamma \in \Sigma$ with $\alpha \leq \gamma$, $\forall\alpha \in \Sigma$. Hence $U_\gamma \subseteq \cap_{\alpha\in\Sigma} U_\alpha$, contradiction. □

Example

Σ cofinal in $(\mathbb{N}^{\mathbb{N}}, \leq) \Longrightarrow \Sigma$ boundedly complete.

Long Σ-bases

Definition

A Σ-base of neighborhoods of the unit element of a topological group G indexed by a boundedly complete subspace Σ of $\mathbb{N}^{\mathbb{N}}$ will be referred to as a long Σ-base.

Of course, every $\mathfrak{G}$-base of neighborhoods of the origin of a locally convex space E is a long Σ-base, with $\Sigma = \mathbb{N}^{\mathbb{N}}$.

A limit property in Fréchet-Urysohn topological groups

Let $\{U_\alpha : \alpha \in \Sigma\}$ be a long Σ-base in a topological group G. For every $\alpha = (a_i)_{i\in\mathbb{N}} \in \Sigma$ and each $k \in \mathbb{N}$, set $\alpha(k) := (a_1, a_2, \ldots, a_k)$ and

$$D_k(\alpha) := \cap\{U_\beta : \beta \in \Sigma,\ \ \beta(k) = \alpha(k)\}.$$

Clearly, $\{D_k(\alpha)\}_{k\in\mathbb{N}}$ is increasing and $e \in D_k(\alpha)$.

Proposition (Chasco, Martín-Peinador and Tarieladze, 2007)

Let $\{x_{n,k} : (n,k) \in \mathbb{N}\times\mathbb{N}\}$ a subset of a Fréchet-Urysohn topological group G such that

$$\lim_n x_{n,k} = x \in G, \text{ for each } k = 1, 2, \ldots.$$

There exists two increasing sequences of natural numbers

$$(n_i)_{i\in\mathbb{N}} \quad \text{and } (k_i)_{i\in\mathbb{N}}, \quad \text{such that } \ \lim_i x_{n_i,k_i} = x.$$

Metrizability in Fréchet-Urysohn topological groups

Theorem

Each Fréchet-Urysohn topological group G with a long Σ-base $\{U_\alpha : \alpha \in \Sigma\}$ is metrizable.

Proof.

If $\exists \alpha \in \Sigma$ such that $D_k(\alpha)$ is not a e-neighborhood, $\forall k \in \mathbb{N}$, then

$$e \in \overline{G \setminus D_k(\alpha)} \Longrightarrow \exists \{x_{n,k} \in G \setminus D_k(\alpha)\}_{n \in \mathbb{N}} \text{ converging to } e,$$

hence there exists $(n_i)_{i\in\mathbb{N}} \uparrow$ and $(k_i)_{i\in\mathbb{N}} \uparrow$ such that $\lim_i x_{n_i,k_i} = e$.

$$x_{n_i,k_i} \notin D_{k_i}(\alpha) \Longrightarrow \exists \beta_{k_i} \in \Sigma,\ \beta_{k_i}(k_i) = \alpha(k_i),\ x_{n_i,k_i} \notin U_{\beta_{k_i}}.$$

If $\beta_{k_i} \leqslant \gamma \in \Sigma$, $i \in \mathbb{N}$ then each $x_{n_i,k_i} \notin U_\gamma$ (it contradicts $\lim_i x_{n_i,k_i} = e$). A countable base of neighborhoods of e is

$$\{D_{k_\alpha}(\alpha) : \alpha \in \Sigma\}, \quad k_\alpha \text{ minimal with } D_{k_\alpha}(\alpha) \text{ } e\text{-neighborhood}.$$

Long Σ-bases in products

Corollary

Let $\{G_t)_{t\in T}$ be a family of metrizable topological groups.

$$G := \prod_{t\in T} G_t \text{ has a long } \Sigma\text{-base} \iff |T| \leq \aleph_0 \ (i.e., G \text{ metrizable})$$

Proof.

(Only nontrivial part) Let e_t be the neutral in G_t for $t \in T$. By Noble, the G-dense $G_0 := \{x = (x_t) \in G : |t \in T : x_t \neq e_t| \leq \aleph_0\}$ is F-U.

$$G \text{ long } \Sigma\text{-base} \implies G_0 \text{ long } \Sigma\text{-base} \implies G_0 \text{ metrizable} \implies G \text{ metrizable.}$$

□

Corollary

The space $C_p(X)$ has a long Σ-base if and only if X is countable. $(\mathbb{R}^X = \overline{C_p(X)})$

The dominating cardinal

In $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$

$$a \leqslant^* b, \quad \text{it means } \exists m : a_n \leq b_n, \text{ when } m \leqslant n$$

and

$$a <^* b, \quad \text{it means } \exists m : a_n < b_n, \text{ when } m \leqslant n.$$

The ordinal ω_1 is the first ordinal with $|\omega_1|$ uncountable and

$$\aleph_1 := |\omega_1|$$

Definition

The *dominating cardinal* $\mathfrak{d}$ is the least cardinality for cofinal subsets of the preordered space $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.

One has

$$\aleph_1 \leqslant \mathfrak{d} \leqslant \mathfrak{c}.$$

A well known lemma in $\mathbb{N}^{\mathbb{N}}$

- If $a(n) := (a_m(n))_m \in \mathbb{N}^{\mathbb{N}}$, $n \in \mathbb{N}$, $b_m := 1 + \sup_{n \leqslant m} a_m(n)$ and $b = (b_m)_m$, with $m \in \mathbb{N}$, then

$$a(n) <^* b, \quad \forall n \in \mathbb{N}$$

- If $\aleph_1 = \mathfrak{d}$ there exists a biyection between $[0, \omega_1)$ and a cofinal ω_1-sequence $(a(\alpha) : \alpha \in [0, \omega_1))$ in $\left(\mathbb{N}^{\mathbb{N}}, \leqslant^*\right)$.

With an easy transfinite induction we get:

Lemma

If $\aleph_1 = \mathfrak{d}$ there exists a cofinal ω_1-sequence $\Gamma := (b(\alpha) : \alpha \in [0, \omega_1))$ in $\left(\mathbb{N}^{\mathbb{N}}, \leqslant^*\right)$ such that

$$\leqslant^*_{|\Gamma} = \ <^*_{|\Gamma}$$

and the biyection

$$i : \left([0, \omega_1), <\right) \longrightarrow \left(\Gamma, <^*_{|\Gamma}\right)$$

defined by $i(\alpha) := b(\alpha)$, $\alpha \in [0, \omega_1)$, preserves the order.

Example and open question

Example

If $\aleph_1 = \mathfrak{d} < \mathfrak{c}$ the space $([0,\omega_1), \tau_<)$ has an ordered compact covering $\{K_b : b \in \Gamma\}$, with Γ unbounded, directed and boundedly complete proper subset Γ of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact sets of X.

If $(b_m)_m = b(\alpha) \in \Gamma$, then $K_{(b_m)_m} = [0,\alpha]$.

Corollary (In any ZFC model for which $\aleph_1 = \mathfrak{d} < \mathfrak{c}$)

There exists a long Σ-base of 0-neigh in $C_c([0,\omega_1))$ not a $\mathfrak{G}$-base.

Problem

Let X be a separable metric space admitting a compact ordered covering of X indexed by an unbounded and boundedly complete proper subset of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact sets of X. Is then X a Polish space?

Outline

3. Long Σ-bases and strict angelicity

Boundedly complete subsets of $\mathbb{N}^{\mathbb{N}}$ and strong domination

Proposition

Let $\{K_a : a \in \Sigma\}$ be an ordered compact covering in X swallowing compact subsets with Σ boundedly complete. X is strongly dominated by a second countable space.

Proof.

Let $T : \Sigma \to \mathcal{K}(X)$ defined by $T(a) = K_a$ and let $C \in \mathcal{K}(\Sigma)$.

$$\begin{aligned} \sup\{a_m : a \in C\} &< \infty, \forall\, m \in \mathbb{N} \Longrightarrow \exists b \in \Sigma \text{ such that } a \leq b, \forall a \in C. \\ T(C) &= \cup_{a \in C} K_a \subseteq K_b \Longrightarrow K_C := \overline{T(C)} \text{ is compact} \end{aligned}$$

$\mathcal{B} := \{K_C : C \in \mathcal{K}(\Sigma)\}$ swallows the compact sets, because if $P \in \mathcal{K}(X)$ $\exists d \in \Sigma$ with $P \subset K_d = K_{\{d\}}$.
Hence X is strongly Σ-dominated and Σ is separable metric. □

Compact metrizability in groups with long Σ-base

Let K be a compact set and $\Delta := \{(x,x) : x \in K\}$.

Proposition (Cascales, Orihuela, Tkachuk, 2011)

K is metrizable if and only if $(K \times K)\setminus\Delta$ is strongly dominated by a second countable space.

Theorem

Let $\{U_a : a \in \Sigma\}$ be a long Σ-base in a top group G. Every compact subset K of G is metrizable.

Proof.

Let $Q \in \mathcal{K}[(K \times K)\setminus\Delta]$.

$$\begin{aligned} e &\notin \{xy^{-1} : (x,y) \in Q\} \Longrightarrow \exists U_\alpha \cap \{xy^{-1} : (x,y) \in Q\} = \emptyset \\ Q &\subset W_\alpha := \{(x,y) \in (K \times K)\setminus\Delta : xy^{-1} \notin U_\alpha\} \end{aligned}$$

The $(K \times K)\setminus\Delta$-c c $\{W_\alpha : \alpha \in \Sigma\}$ s c subsets.K metrizable (COK) □

Strict angelicity in $C_c(X)$ with long Σ-base

Lemma

Let $\{K_a : a \in \Sigma\} \subset \mathcal{K}(X)$ be an ordered family swallowing compact subsets of $Y := \cup\{K_a : a \in \Sigma\}$, with Σ unbounded, boundedly complete and $\overline{Y} = X$. Each compact subset of $C_c(X)$ is metrizable.

Proof.

There exists k with $\sup_{a\in\Sigma} a_k = \infty$. If

$$U_\alpha := \{f \in C(Y) : \sup_{y\in A_\alpha} |f(y)| \leq a_k^{-1}\}$$

then $\{U_\alpha : \alpha \in \Sigma\}$ is a long Σ-base for a lc topology τ on $C(Y)$.
We have proved that each compact subset K of $(C(Y), \tau)$ is metrizable.
Each $K \in \mathcal{K}(C_c(X))$ is also metrizable, because

- the restriction map $S : C_c(X) \to (C(Y), \tau)$ is continuous and
- $S_{|K}$ is an homeomorphism).

Strict angelicity in $C_c(X)$ with long Σ-base

Theorem

If $C_c(X)$ has a long Σ-base then $C_c(X)$ is strict angelic.

Proof.

Lat $\{U_\alpha : \alpha \in \Sigma\}$ be a long Σ-base.
The ordered family $\{K_\alpha : \alpha \in \Sigma\} \subset \mathcal{K}(X)$ corresponding to the long Σ-base is

- a compact covering of X swallowing compact subsets.
- Σ is unbounded and boundedly complete.

X is web-compact, so $C_p(X)$ is angelic (by 1987 Orihuela's angelic theorem).
$C_c(X)$ is also angelic by angelic lemma.
And the compact subsets of $C_c(X)$ are metrizables. □

Comments around a recent Ferrando paper

Theorem (Small remark on Orihuela's angelic theorem)

*[X web compact $\Longrightarrow C_p(X)$ angelic] $\Longleftrightarrow$
[X contains a dense Lindelöf Σ-space $\Longrightarrow C_p(X)$ angelic].*

Recall, X is web compact if there exists a map $T : \Sigma(\subset \mathbb{N}^{\mathbb{N}}) \longrightarrow \mathcal{P}(X)$ such that $\overline{\cup_{a\in\Sigma} T(a)} = X$ and $a(n) \longrightarrow a$ in Σ and $x_n \in T(a(n))$ implies that $(x_n)_n$ has an adherent point in X.
X is Lindelöf Σ if there exists a map $T : \Sigma(\subset \mathbb{N}^{\mathbb{N}}) \longrightarrow \mathcal{K}(X)$ such that $\cup_{a\in\Sigma} T(a) = X$ and $a(n) \longrightarrow a$ in Σ and $x_n \in T(a(n))$ implies that $(x_n)_n$ has an adherent point in $T(a)$ ($\Longleftrightarrow T$ is upper semi-continuous).
Ferrando gets easily the following deep Baturov result: If *X contains a dense Lindelöf space then every compact subset K of $C_p(X)$* is monolithic (its separable subspaces are metrizable) and Fréchet-Urysohn ($y \in \overline{Y} \subset K \Longrightarrow y = \lim y_n$, $y_n \in Y$), because he proves that K is Gul'ko compact, i.e., $C_p(K)$ is Lindelöf Σ-space.

Last comment

THANK YOU VERY MUCH!

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