On the separable quotient problem for Banach spaces and spaces $C(X)$

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1. **Introduction; a bit of the history.**
2. **Separable Quotient Problem for Banach spaces; general approach.**
3. **Separable Quotient Problem for spaces $C(X)$.**
4. **GM-spaces and Separable Quotient Problem for lcs**
5. **Open questions.**

Introduction. We consider only infinite-dimensional tvs

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Problem 1 (Mazur 1933)

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1. Reflexive Banach spaces have separable quotient. Direct consequence of (Schaefer): If F closed in Banach E, then $E'_\beta/F^\perp \approx F'_\beta$.
2. Every WCG Banach space has a separable quotient.
3. The same holds for any WCG locally convex space. Hence the weak*-dual of a metrizable lcs has a separable quotient.

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3. If E has a subspace $\approx c_0$, E has a complemented subspace $\approx \ell^1$ (Bessaga, Pełczyński).

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3. If E has a subspace $\approx c_0$, E has a complemented subspace $\approx \ell^1$ (Bessaga, Pełczyński).
4. If the dual ball B^* is not weakly*-sequentially compact, then either E has c_0 as a quotient or E contains ℓ^1 (consequently E has ℓ^2 as a quotient) (Hagler, Johnson). Hence, every Banach space E containing a copy of $\ell_1(\mathbb{R})$ has a separable quotient. B^* is weakly*-sequentially compact for WCG Banach spaces (Amir, Lindestrauss).

1. Hence we have a large class of Banach spaces not being WCG but admitting a separable quotient.
2. Every separable Banach space has a quotient with a Schauder basis (Johnson, Rosenthal).
3. Last theorem shows that Mazur and Pełczyński problems are equivalent.

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Theorem 3 (Rosenthal)

A Banach space E has a separable infinite-dimensional quasi-complemented subspace iff the space E has a separable quotient.

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2. Every (LF)-space, i.e. the inductive limit of a strictly increasing sequence of Fréchet lcs, has a separable quotient (Saxon, Narayanaswami).

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3. For every atomless finite measure space (Ω, Σ, μ) with dim $L^1(\Omega) > c$ there exists $\Omega_1 \subset \Omega$ with $\mu(\Omega_1) > 0$ such that the space $L^p(\Omega_1)$, $0 < p < 1$, has no separable quotient (Popov (1984)). This answered a question of Drewnowski.

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4. If E is a dual-separating metrizable and complete tvs which is WCG, then E has a separable quotient (Kąkol-Śliwa).

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Theorem 4 (Wilansky-Saxon (1977))

For a Banach space E the following are equivalent:

(i) *E has a separable quotient.*

(ii) *E contains a dense non barrelled vector subspace.*

(iii) *E contains a dense s_σ-subspace F, i.e. F is a union of a strictly increasing sequence of closed linear subspaces.*

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Theorem 5 (Argyros, Dodos, Kanellopoulos (2008))

The Banach dual of any Banach space E has a separable quotient.

Theorem 6 (Śliwa (2012))

For a Banach space E the following are equivalent.

(i) *E has a separable quotient.*

(ii) *E' has a strongly normal sequence.*

(iii) *The weak*-dual $(E', \sigma(E', E))$ has a Schauder basic sequence.*

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A sequence (y_n) in the unit sphere $S(E')$ of E' is *normal* (*strongly normal*) in E' if $y_n(x) \to 0$ for all $x \in E$ ($\{x \in E : \sum_n |y_n(x)| < \infty\}$ is dense in E). The Josefson-Nissenzweig theorem states that the dual of any infinite-dimensional Banach space contains a normal sequence.

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2. These are also the closed ideals of $C_c(X)$.
3. An algebra quotient of $C_c(X)$ or $C_p(X)$ is one by a closed ideal, thus preserving vector multiplication.

Theorem 8 (Kąkol-Saxon-Todd (2015))

The following are equivalent.

(1) *X is not pseudocompact.*

(2) *$C_c(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$.*

(3) *$C_p(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$.*

(4) *$C_c(X)$ admits a quotient isomorphic to $\mathbb{R}^{\mathbb{N}}$.*

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1. If X is pseudocompact and $C_c(X)$ is barrelled, then $\{f \in C(X) : |f(x)| \leq 1, x \in X\}$ is a neighb. of zero; equivalently X is compact and $C_c(X)$ is a Banach space.

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2. X is not pseudocompact iff $C_c(X)$ contains a copy (in fact complemented) of $\mathbb{R}^{\mathbb{N}}$.

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Theorem 9 (Kąkol-Saxon-Todd (2014))

If $C_c(X)$ is barrelled, it admits a separable quotient. Indeed, if X is compact, this is Rosenthal's result. If X is not compact (so X is not pseudocompact), $C_c(X)$ contains a complemented copy of $\mathbb{R}^{\mathbb{N}}$.

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1. X is of pointwise countable type (Arkhangell'ski) if every point in X is contained in some compact set K which has a countable base. All first countable, as well as, Čech-complete spaces are of this type.

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2. Must arbitrary $C_c(X)$ have separable quotients?
3. Rosenthal's theorem and previous facts leave only the case where X is countably compact and not compact.

X- compact with a sequence $(x_n)_n$ of distinct points converging to x_0. $T : C_c(X) \to c$, $f \mapsto (f(x_n))_n$ is a continuous linear surjection. The quotient $C_c(X)/T^{-1}(0)$ is an algebra quotient, as $\ker T$ is $\mathfrak{J}(A)$ with $A := \{x_0, x_1, x_2, \dots\}$.

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Theorem 11 (Kąkol-Saxon (2015))

The following three statements are equivalent for any X.

(1) *X admits a closed denumerable set.*

(2) *$C_c(X)$ admits a separable algebra quotient.*

(3) *$C_p(X)$ admits a separable algebra quotient.*

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Since $\beta\mathbb{N}$ lacks a closed (equivalently, compact) denumerable set, $C_c(\beta\mathbb{N})$ lacks separable algebra quotients, even though $C_c(\beta\mathbb{N})$ has separable quotient. Also, $C_c(\beta\mathbb{N})$ contains a copy of c but c is not an algebra quotient of $C_c(\beta\mathbb{N})$.

Extension of Argyros-Dodos-Kanellopoulos theorem for $C_c(X)$.

Theorem 12 (Kąkol-Saxon-Todd (2014))

Both the strong and weak duals of $C_c(X)$ admit separable quotients. In fact:

(i) *If X is Warner bounded, $C_c(X)$ contains c_0; otherwise it contains a dense subspace of $\mathbb{R}^{\mathbb{N}}$.*

(ii) *If $C_c(X)'_\beta$ is normed, it contains a complemented copy of ℓ^1; otherwise it contains a complemented copy of ϕ, i.e. an $\aleph_0$-dimensional vector space with the finest locally convex topology.*

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X is Warner bounded, if for every disjoint sequence $(U_n)_n$ of nonempty open sets in X, there is a compact set in X that meets infinitely many of the U_n.

Theorem 13 (Kąkol-Saxon-Todd (2015))

The following are equivalent.

(i) X *is Warner bounded.*

(ii) $C_c(X)$ *does not contain a dense subspace of* $\mathbb{R}^{\mathbb{N}}$.

(iii) $C_c(X)$ *contains* c_0 *and no dense subspace of* $\mathbb{R}^{\mathbb{N}}$

(iv) $C_c(X)'_\beta$ *does not contain* ϕ.

(v) $C_c(X)'_\beta$ *is normed.*

(vi) $C_c(X)'_\beta$ *is normed and contains* ℓ_1 *complemented.*

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2. Neither $C_p(X)$ nor $C_c(X)$ is a GM-space. Indeed, assume $C_c(X)$ is a GM-space. Then barrelled $C_c(X)$ does not contain an infinite-dimensional bounded set. We may assume that X is not pseudocompact, otherwise X would be compact (as $C_c(X)$ is barrelled) and then $C_c(X)$ would be a Banach space, a contradiction. As X is not pseudocompact, $C_c(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$ which has an infinite-dimensional bounded set, also a contradiction. If $C_p(X)$ is a GM-space, $C_p(X)$ is barrelled, so $C_p(X) = C_c(X)$ and the previous case applies.

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A lcs E in which every dense subspace is barrelled (for example a GM-space) admits a separable quotient iff E is not quasi-Baire.

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Problem 15

Does there exist compact X such that $C_p(X)$ does not have separable quotients?