Set-algebras with uniform boundedness deciding property

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Leicester, August 2-5, 2016
31st Summer Conference on Topology and its Applications

Project Prometeo I/2013/058

Outline

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Notations

Let $\mathcal{B}$ a subset of a set-algebra $\mathcal{A}$ of subsets of Ω.
$L(\mathcal{B})$ means:

- span$\{e_C : C \in \mathcal{B}\}$
- endowed with the supremum norm $\|\cdot\|$.

Recall that

- The dual of $(L(\mathcal{A}), \|\cdot\|)$ with the dual norm is isometric to
- the Banach space $ba(\mathcal{A})$ of bounded variation finitely additive measures on A with the variation norm $|\cdot| := |\cdot|(\Omega)$.

It is well known that

$$|\cdot| \text{ is equivalent to } \mathcal{A}\text{-supremum norm}$$

defined by

$$|\mu| := \sup\{|\mu(A)| : A \in \mathcal{A}\}$$

Nikodym property of $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition (Schachermayer)

$\mathcal{B}$ has property *N if each $\mathcal{B}$-pointwise bounded subset M of ba*$(\mathcal{A})$ *is bounded in ba*(A), *i.e.,*

$$\sup_{\mu \in M} |\mu(A)| < \infty, \forall A \in \mathcal{B} \Longrightarrow \sup_{A \in \mathcal{A}, \mu \in M} |\mu(A)| < \infty$$

Then $\{e_C : C \in \mathcal{B}\}^{\circ\circ}$ neighborhood of 0 $\Longrightarrow \overline{L(\mathcal{B})} = L(\mathcal{A})$.

Definition (Valdivia)

$\mathcal{B}$ *has property sN if* $\mathcal{B} = \cup_m \mathcal{B}_m \uparrow \Longrightarrow \exists\, \mathcal{B}_n$ *which has property N.*

Web Nikodym property of $\mathcal{B} \subset \mathcal{A}$ (set-algebra)

Definition

A family $\mathcal{W} := \{A_{m_1 m_2 \ldots m_p} : (m_1, m_2, \ldots, m_p) \in \cup_{s \in \mathbb{N}} \mathbb{N}^s\}$ of subsets of A is *increasing web* if

1. $A = \cup_{m_1} A_{m_1}$
2. $A_{m_1 m_2 \ldots m_p} = \cup_{m_{p+1}} A_{m_1, m_2, \ldots, m_p m_{p+1}}, \forall p, m_1, .., m_{p+1} \in \mathbb{N}$.

Each sequence $(A_{m_1 m_2 \ldots m_p})_p$ is called a *strand* in $\mathcal{W}$.

Definition (Kakol-LP)

$\mathcal{B}$ has property *wN* if *each increasing web* $\{\mathcal{B}_t : t \in \cup_s \mathbb{N}^s\}$ *in $\mathcal{B}$ has a N-strand* (*=its sets have property N*).

Properties *N*, *sN* and *wN* in σ-algebras

Theorem

For a σ-algebra $\mathcal{S}$ of subsets of a set Ω it was shown:

1. *$\mathcal{S}$ has property N (Nikodym-Dieudonné-Grothendieck).*
2. *$\mathcal{S}$ has property sN (Valdivia).*
3. *$\mathcal{S}$ has property wN (Kakol-LP).*

Properties *N*, *sN* and *wN* in algebras

Example

The algebra of finite and co-finite subsets of $\mathbb{N}$ fails to have property *N*.

Example (Schachermayer, 1982)

The algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has property *N*.

Example (Valdivia, 2013)

The algebra $\mathcal{J}(K)$ of Jordan measurable subsets of $K := \Pi_{1 \leqslant i \leqslant k}[a_i, b_i]$ has property *sN*.

First we show that $\mathcal{J}(K)$ has property *wN*.

Outline

2 Web Nikodym property in $\mathcal{J}(K)$
- Deep unboundedness and property N
- Deep unboundedness in $ba(\mathcal{J}(K))$
- Web Nikodym property in $\mathcal{J}(K)$

Notations in $\cup_s \mathbb{N}^s$

$\{t = (t_1, t_2, \ldots, t_p), u = (u_1, u_2, \ldots, u_q)\} \subset \cup_s \mathbb{N}^s$.
p is the *length of* t.
$t(i) := (t_1, t_2, \ldots, t_i)$ is the *section of length* i of t, $1 \leqslant i \leqslant p$.
$t(i) := \emptyset$ if $i > p$. If $t = (t_1)$ then $t = t_1$.
$t \times u := (t_1, t_2, \ldots, t_p, t_{p+1}, t_{p+2}, \ldots, t_{p+q})$, with $t_{p+j} := u_j$.

Definition

$(t^n = (t_1^n, t_2^n, \ldots, t_n^n, \ldots) \in \cup_s \mathbb{N}^s)_n$ is an *infinite chain* if for each $n \in \mathbb{N}$

$$t^{n+1}(n) = t^n(n) \neq \emptyset.$$

Notations in $\cup_s \mathbb{N}^s$

$t \times s$ ($t \times s \in U \subset \cup_s \mathbb{N}^s$) is *an extension of* t (of t *in* U).

Definition

U is increasing at $t = (t_1, t_2, \ldots, t_p) \in \cup_s \mathbb{N}^s$ if there exists

- $t^1 = (t_1^1, t_2^1, \ldots\ldots) \in U$, with $t_1 < t_1^1$ and
- $t^i = t(i-1) \times (t_i^i, t_{i+1}^i, \ldots\ldots) \in U$, with $t_i < t_i^i$, for each $1 < i \leqslant p$.

Definition

U is increasing (increasing respect to $V \subset \cup_s \mathbb{N}^s$) if U is increasing at each $t \in U$ (at each $t \in V$).

U is increasing if $\forall\ t = (t_1, t_2, \ldots, t_p) \in U$,

$$|U(1)| = |\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty,\ 1 \leqslant i < p$$

NV-tree

Definition

An NV-tree T is an increasing subset of $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ without infinite chains such that each $t = (t_1, t_2, \ldots, t_p) \in T$ verifies that

- the length of each extension of $t(p-1)$ in T is p
- and $\{t(i) : 1 \leqslant i \leqslant p\} \cap T = \{t\}$.

Example

The infinite subsets of $\mathbb{N}$ are NV-trees, named trivial NV-trees.

Example

$\mathbb{N}^i$, $i \in \mathbb{N}\backslash\{1\}$, and $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are non trivial NV-trees.

Example

The product $T \times U$ of NV-trees is an NV-tree

Elementary properties of a NV-tree T

Proposition

Each increasing subset S of T is an NV-tree. Whence if $S_n \subset T$ and S_{n+1} is increasing respect to S_n then $\cup_n S_n$ is an NV-tree.

Proposition

If U $(\subset T)$ does not contain a NV-tree then $T \backslash U$ contains an NV-tree.

Proposition

$\{B_u : u \in \cup_s \mathbb{N}^s\} \uparrow$ *web in* $B \Longrightarrow B = \cup\{B_t : t \in T\}$.

Proof.

This equality follows from the following trivial facts:

1. $B = \cup_{u \in T} B_{u(1)} \uparrow$, $B_{u(i)} = \cup_{u(i) \times n \in T(i+1)} B_{u(i) \times n}$
2. $b \in B \Longrightarrow \exists t \in T : b \in B_t$, ($T$ does not contain infinite chains).

Deep B-unbounded sets

Definition

Let $B \in \mathcal{A}$. $M \subset ba(\mathcal{A})$ is deep B-unbounded if for each finite subset $\mathcal{Q}$ of $\{e_A : A \in \mathcal{A}\}$

$$\sup\{|\mu(C)| : \mu \in M \cap \mathcal{Q}^\circ,\ C \in \mathcal{A},\ C \subset B\} = \infty.$$

Proposition

Let $M(\subset ba(\mathcal{A}))$ be deep B-unbounded and $\{B_i \in \mathcal{A} : 1 \leqslant i \leqslant q\}$ a partition of B. There exists j, $1 \leqslant j \leqslant q$, such that M is deep B_j-unbounded.

Proof.

If there exist finite $\mathcal{Q}^i \subset \{e_A : A \in \mathcal{A}\}$ with $\sup\{|\mu|(C_i) : \mu \in M \cap (\mathcal{Q}^i)^\circ\} < H_i$, $1 \leqslant i \leqslant q$, then for $\mathcal{Q} = \cup_{1 \leqslant i \leqslant q} \mathcal{Q}^i$ we get $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} < \Sigma_{1 \leqslant i \leqslant q} H_i$. □

Deep Ω-unbounded sets and sequences

Proposition

M is deep Ω-unbounded if and only if M° is not neighborhood of zero in its linear hull or if its $\overline{\text{span}\{M^\circ\}}$ has infinite codimension in $L(\mathcal{A})$. In particular, if $\overline{\text{span}\{M^\circ\}} = L(\mathcal{A})$ then M is deep Ω-unbounded if and only M is unbounded.

Proposition (Increasing sequences without N property)

Let $\mathcal{A}$ be an algebra of subsets of Ω and let $(\mathcal{B}_m)_m$ be an increasing sequence of subsets of $\mathcal{A}$. If each $\mathcal{B}_m$ does not have N-property and $\overline{\text{span}\{e_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$ then $\exists\ n_0 \in \mathbb{N}$ such that for each $m \geqslant n_0$ there exists a deep Ω-unbounded subset M_m of $ba(\mathcal{A})$ which is pointwise bounded in $\mathcal{B}_m$.

In particular, this proposition holds if $\cup_m \mathcal{B}_m = \mathcal{A}$ or if $\cup_m \mathcal{B}_m$ has N-property.

Webs without N-strands and NV-trees

Proposition

Let $\mathcal{B} := \{\mathcal{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in $\mathcal{A}$ without N-strands.

1. *Then there exists an NV-tree T' such that for each $t = (t_1, t_2, \ldots, t_q) \in T'$ the set $\mathcal{B}_t$ does not have property N and if $p > 1$ then $\mathcal{B}_{t(i)}$ has property N if $i < p$.*
2. *There exists an NV-tree $T \subset T'$ such that for each $t \in T$ there exists a deep Ω-unbounded subset M_t of $ba(\mathcal{S})$ which is $\mathcal{B}_t$-pointwise bounded.*

$\mathcal{J}(K)$, $K := \Pi_{1\leqslant i\leqslant k}[a_i, b_i] \subset \mathbb{R}^k$

Definition

A bounded $B \subset \mathbb{R}^k$ is Jordan measurable whenever the Lebesgue measure of $\overline{B}\backslash\mathring{B}$ is zero.

$\{B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$, $M \subset ba(\mathcal{J}(K))$, deep B-unbounded.

Proposition

If $\alpha > 0$ and $\epsilon > 0$ $\exists C_1, C_1' \in \mathcal{J}(K)$, disjoints contained in B and $\mu \in M$ such that

1. $|\mu(C_1)| > \alpha$, $|\mu(C_1')| > \alpha$, $\Sigma_{1\leqslant j\leqslant r}\mu(Q_j) \leqslant 1$,
2. $C_1 \cup C_1' \subset \Pi_{1\leqslant i\leqslant k}[c_i, d_i] \subset K$, $\Pi_{1\leqslant i\leqslant k}(d_i - c_i) < \epsilon$
3. *M is deep C_1'-unbounded.*

$B \subset K := \Pi_{1 \leqslant i \leqslant k}[a_i, b_i] \subset \mathbb{R}^k$

Proof.

There exists $B_n \subset B$, M deep B_n-unbounded and $B_n \subset k$-compact interval of measure $< \epsilon$.
Let $\mathcal{Q} := \{B_n, Q_1, \ldots, Q_r\}$. As

$$\sup\{|\mu(C)| : \mu \in rM \cap \mathcal{Q}^\circ, C \subset B_n, C \in \mathcal{A}\} = \infty,$$

$\exists D_1 \subset B_n$, $D_1 \in \mathcal{J}(K)$, $\exists \lambda \in rM \cap \mathcal{Q}^\circ : |\lambda(D_1)| > r(1+\alpha)$.
Hence

$$\mu = r^{-1}\lambda \in M, |\mu(B_n)| \leqslant r^{-1} \leqslant 1 \text{ and } \Sigma_{1 \leqslant j \leqslant r} |\mu(Q_j)| \leqslant r^{-1}r = 1$$

and we get $|\mu(B_n \backslash D_1)| \geqslant |\mu(D_1)| - |\mu(B_n)| > 1 + \alpha - 1 = \alpha$.
M is deep D_1-unbounded or $B_n \backslash D_1$-unbounded. □

$\{B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$, $M \subset ba(\mathcal{J}(K))$

Corollary

If M deep B-unbounded, $q \in \mathbb{N}$, $\alpha > 0$, $\epsilon > 0$, $\exists$ in B pairwise disjoint Jordan measurable subsets C_1, C_2,. . ., C_q and μ_1, μ_2,. . ., $\mu_q \in M$ such that:

1. $|\mu_i(C_i)| > \alpha$, $\Sigma_{1 \leqslant j \leqslant r} \mu_i(Q_j) \leqslant 1$, $i = 1, 2, \ldots, q$,
2. *M is deep C_q-unbounded*
3. $\cup_{1 \leqslant i \leqslant q} C_i \subset \Pi_{1 \leqslant i \leqslant k}[c_i, d_i] \subset K$, $\Pi_{1 \leqslant i \leqslant k}(d_i - c_i) < \epsilon$.

Proof.

There exists two disjoint subsets C_1, $C_1' \in \mathcal{J}(K)$, in B, and $\mu_1 \in M$:

1. $|\mu_1(C_1)| > \alpha$, $|\mu_1(C_1')| > \alpha$, $\Sigma_{1 \leqslant j \leqslant r} \mu_1(Q_j) \leqslant 1$,
2. M is deep C_1'-unbounded
3. $C_1 \cup C_1' \subset \Pi_{1 \leqslant i \leqslant k}[c_i, d_i] \subset K$, $\Pi_{1 \leqslant i \leqslant k}(d_i - c_i) < \epsilon$.....

□

$\{B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$, $M_t \subset ba(\mathcal{J}(K))$, $t \in T$

Proposition

If T is an NV-tree, each M_t is deep B-unbounded, $\alpha > 0$, $\epsilon > 0$ and $\{t^j : 1 \leqslant j \leqslant k\} \subset T$, then

- *there exists in B two disjoints subsets $B_1, B_1' \in \mathcal{J}(K)$,*
- *and there exists $\mu_1 \in M_{t^1}$, an NV-tree T_1 and k intervals $\Pi_{1 \leqslant i \leqslant k}[c_i^s, d_i^s] \subset K$, $1 \leqslant s \leqslant q$,*

such that:

1. $\{t^j : 1 \leqslant j \leqslant k\} \subset T_1 \subset T$ *and M_t is deep B_1'-unbounded,* $\forall t \in T_1$.
2. $|\mu_1(B_1)| > \alpha$ *and* $\Sigma\{|\mu_1(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$.
3. $B_1 \cup B_1' \subset \cup_{1 \leqslant s \leqslant q} \Pi_{1 \leqslant i \leqslant k}[c_i^s, d_i^s] \subset K$, $\Sigma_{1 \leqslant s \leqslant q}(\Pi_{1 \leqslant i \leqslant k}(d_i^s - c_i^s)) < \epsilon$.

$t^j := (t_1^j, t_2^j, \ldots, t_{p_j}^j)$, $1 \leqslant j \leqslant k$

Proof.

Let $q := 2 + \Sigma_{1 \leqslant j \leqslant k} p_j$, and let $\{B_1, B_2, \ldots, B_m\}$ be a $\mathcal{J}(K)$ partition of B with each $B_i \neq \emptyset$ and contained in a k-interval of measure $< \epsilon/q$. Exists

$$B_n,\ 1 \leqslant n \leqslant m, \text{ such that } M_{t^1} \text{ is deep } B_n\text{-unbounded.}$$

By Corollary (with $\{B_n, Q_1, \ldots, Q_r\}$, M_{t^1}, α and ϵ/q) exits a $\mathcal{J}(K)$ partition $\{C_1, C_2, \cdots, C_q\}$ of B_n and $\{\lambda_1, \lambda_2, \cdots, \lambda_q\} \subset M_{t^1}$ such that:

$$|\lambda_k(C_k)| > \alpha \quad \text{and} \quad \Sigma_{1 \leqslant i \leqslant r} |\lambda_k(Q_i)| \leqslant 1, \quad \text{for} \quad k = 1, 2, \ldots, q,$$

□

$\{D_i : 1 \leqslant i \leqslant q'\} := \{B_i : i \neq n\} \cup \{C_i : 1 \leqslant i \leqslant q\}$

Proof.

- $j \in \{1, 2, \ldots, k\} \implies \exists i^j \in \{1, 2, \ldots, q'\}$ such that

$$M_{t^j} \text{ is deep } D_{i^j}\text{-unbounded.}$$

- $t^j = (t_1^j, t_2^j, \ldots, t_{p_j}^j)$, $1 \leqslant j \leqslant k$, and $2 \leqslant m \leqslant p_j$ then $M_{(t_1^j, t_2^j, \ldots, t_{m-1}^j) \times w}$ is deep B-unbounded if $t^j(m-1) \times v \in T$. Whence $\exists i_m^j \in \{1, 2, \ldots, q'\}$ and an NV-tree V_m^j $(t^j(m-1) \times V_m^j \subset T)$ such that

$$M_{(t_1^j, t_2^j, \ldots, t_{m-1}^j) \times w} \text{is deep } D_{i_m^j}\text{-unbounded } \forall w \in V_m^j.$$

- $\exists i_0 \in \{1, 2, \ldots, q'\}$ and an NV-tree $T_{i_0} \subset \{t \in T : t_1^j < t(1),$ $1 \leqslant j \leqslant k\}$ such that

$$M_t \text{ is deep } D_{i_0}\text{-unbounded } \forall t \in T_{i_0}.$$

B_1 and μ_1

Proof.

Let D be the union of D_{i_0} and all D_{i^j} and $D_{i^j_m}$.
D is a union of less that q sets, whence there exists C_h such that

$$C_h : D \subset B \backslash C_h.$$

The union T_1 of T_{i_0} and all t^j and $(t^j_1, t^j_2, \ldots, t^j_{m-1}) \times V^j_m$ has the increasing property.
As M_t is deep D-unbounded for each $t \in T_1$ we get the proof with

$$B_1 := C_h, \quad B_1' := D, \quad \text{and} \quad \mu_1 := \lambda_h,$$

because by construction

1. $|\mu_1(B_1)| > \alpha$, $\Sigma_{1 \leqslant i \leqslant r} |\mu_1(Q_i)| \leqslant 1$, and
2. $B_1 \cup B_1' \subset \cup_{s \in J} \{\Pi_{1 \leqslant i \leqslant k}[c^s_i, d^s_i]\}$, $|J| \leqslant q$, whence $\Sigma_{s \in J}(\Pi_{1 \leqslant i \leqslant k}(d^s_i - c^s_i)) < (\epsilon/q)q = \epsilon$.

$\{B, Q_1, \ldots, Q_r\} \subset \mathcal{J}(K)$, $M_t \subset ba(\mathcal{J}(K))$, $t \in T$

Corollary

If T is an NV-tree, each M_t is deep B-unbounded, $\alpha > 0$, $\epsilon > 0$ and $\{t^j : 1 \leqslant j \leqslant k\} \subset T$, then there exists

- *a subset $\{B_1, B_2, \ldots, B_k, B'\} \subset \mathcal{J}(K)$ of pairwise disjoint subsets of B,*
- *k measures $\mu_j \in M_{t^j}$, $1 \leqslant j \leqslant k$, and an NV-tree T^**

such that:

1. *$\{t^j : 1 \leqslant j \leqslant k\} \subset T^* \subset T$ and M_t is deep B'-unbounded $\forall t \in T^*$.*
2. *$|\mu_j(B_j)| > \alpha$, $\Sigma\{|\mu_j(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$, for $1 \leqslant j \leqslant k$.*
3. *B' contained in a finite union of k-intervals of measure $< \epsilon$.*

Apply Proposition.
Apply Proposition with $\{t^2, t^3, \ldots, t^k, t^1\}$ and $B := B'_1$...

$\mathcal{J}(K)$ has property wN

Theorem

The algebra $\mathcal{J}(K)$ has property wN.

Proof.

If $\mathcal{J}(K)$ does not have property wN there exists in $\mathcal{J}(K)$ an increasing web

$$\{\mathcal{B}_t : t \in \cup_s \mathbb{N}^s\}$$

without N-strands.
Whence there exists an NV-tree T and a family

$$\{M_t : t \in T\}$$

of deep K-unbounded subsets of $ba(\mathcal{J}(K))$ which are $\mathcal{B}_t$-pointwise bounded. □

The first induction

Proof.

By induction we determine

- an NV-tree $\{t^i : i \in \mathbb{N}\} \subset T$ and $(k_j \in \mathbb{N})_j \uparrow$

such that for each $(i,j) \in \mathbb{N}^2$ with $i \leqslant k_j$

- there exists a set $B_{ij} \in \mathcal{J}(K)$ and $\mu_{ij} \in M_{t^i}$

that verify

1. $\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v, 1 \leqslant v < j\}) < 1$,
2. $|\mu_{ij}(B_{ij})| > j$,
3. $B_{ij} \cap B_{i'j'} = \emptyset$ if $(i,j) \neq (i',j')$ and
4. $\cup\{B_{sv} : s \leqslant k_v, j < v\} \subset \cup_{1\leqslant s \leqslant q_j}(\Pi_{1\leqslant i \leqslant k}[c^s_{ij}, d^s_{ij}]) \subset K$
5. $\Sigma_{1\leqslant s\leqslant q_j}(\Pi_{1\leqslant i\leqslant k}(d^s_{ij} - c^s_{ij})) < 2^{-j}$, for each $j \in \mathbb{N}$.

□

First induction. Step I

Proof.

Select $t^1 \in T$.
By Corollary with

$$B := \Omega, \quad \alpha = 1 \quad \text{and} \quad \epsilon = 2^{-1}$$

there exists

$$B_{11},\ B_1' \in \mathcal{J}(K), \quad \mu_{11} \in M_{t^1} \quad \text{and an} \quad NV\text{-tree } T_1$$

such that

- $|\mu_{11}(B_{11})| > 1$, $t^1 \in T_1 \subset T$,
- M_t is deep B_1'-unbounded for each $t \in T_1$ and
- $B_1' \subset \cup_{1 \leqslant s \leqslant q_1} \{\Pi_{1 \leqslant i \leqslant k}[c_{i1}^s, d_{i1}^s]\} \subset K$, such that
- $\Sigma_{1 \leqslant s \leqslant q_1}(\Pi_{1 \leqslant i \leqslant k}(d_{i1}^s - c_{i1}^s)) < 2^{-1}$.

Define $k_1 := 1$ and $S^1 := \{t^1\}$. □

First induction. Induction hypothesis

Proof.

Let us suppose that we have obtained for each $1 < j \leqslant n$:

- The natural numbers $k_1 < k_2 < k_3 < \ldots < k_n$,
- the NV-trees $T_1 \supset T_2 \supset T_3 \supset \ldots \supset T_n$,
- $S^j := \{t^i : i \leqslant k_j\} \subset T_j$, S^j increasing respect S^{j-1},
- $\{\mu_{iv} \in M_{t^i} : i \leqslant k_v, 1 \leqslant v \leqslant j\}$
- $\{B'_j, B_{iv} : i \leqslant k_v, 1 \leqslant v \leqslant j\} \subset \mathcal{J}(K)$, pairwise disjoint,
- $|\mu_{ij}(B_{ij})| > j$ and $\Sigma_{s \leqslant k_v, 1 \leqslant v < j} |\mu_{ij}(B_{sv})| < 1$, $i \leqslant k_j$,
- $t \in T_j \Longrightarrow M_t$ is deep B'_j-unbounded,
- $\cup_{s \leqslant k_v, j < v \leqslant n} B_{sv} \subset B'_j \subset \cup_{1 \leqslant s \leqslant q_j} \Pi_{1 \leqslant i \leqslant k} [c^s_{ij}, d^s_{ij}] \subset K$,
- $\Sigma_{1 \leqslant s \leqslant q_j} (\Pi_{1 \leqslant i \leqslant k} (d^s_{ij} - c^s_{ij}]) < 2^{-j}$.

□

First induction. End

Proof.

Select $S_{n+1} := \{t^{k_n+1}, \ldots, t^{k_{n+1}}\} \subset T_n \backslash \{t^i : i \leqslant k_n\}$ increasing respect to S^n and apply Corollary with

$$\{B'_n, B_{sv} : s \leqslant k_v, 1 \leqslant v \leqslant n\}, \ \ T_n, \ \ \alpha = n+1 \ \ \epsilon = 2^{-n-1}$$

and $S^{n+1} := \{t^i : i \leqslant k_{n+1}\} \subset T_n$ to get for each $1 \leqslant i \leqslant k_{n+1}$:

- The $\mathcal{J}(K)$-measurable pairwise disjoint subsets B'_{n+1}, $B_{1,n+1}, \ldots, B_{k_{n+1},n+1}$ of B'_n,
- $\mu_{in+1} \in M_{t^i}$ and the increasing tree T_{n+1} such that

1. $|\mu_{in+1}(B_{in+1})| > n+1$, $\Sigma_{s \leqslant k_v, 1 \leqslant v \leqslant n} |\mu_{in+1}(B_{sv})| < 1$,
2. $S^{n+1} \subset T_{n+1} \subset T_n$, M_t deep B'_{n+1}-unbdounded $\forall t \in T_{n+1}$,
3. $B'_{n+1} \subset \cup_{1 \leqslant s \leqslant q_{n+1}} \{\Pi_{1 \leqslant h \leqslant k}[c^s_{h,n+1}, d^s_{h,n+1}]\} \subset K$ and
4. $\Sigma_{1 \leqslant s \leqslant q_{n+1}} (\Pi_{1 \leqslant i \leqslant k}(d^s_{i,n+1} - c^s_{i,n+1}]) < 2^{-n-1}$.

Induction result

$$
\begin{array}{cccc}
B_{11}, \mu_{11} \in M_1 & B_{12}, \mu_{12} \in M_1 & B_{13}, \mu_{13} \in M_1 & \dots \\
 & B_{22}, \mu_{22} \in M_2 & B_{23}, \mu_{23} \in M_2 & \dots \\
 & \vdots & \vdots & \\
 & B_{k_2 2}, \mu_{k_2 2} \in M_{k_2} & B_{k_2 3}, \mu_{k_2 3} \in M_{k_2} & \dots \\
 & & \vdots & \dots \\
 & B_{k_2 2}, \mu_{k_2 2} \in M_{k_2} & B_{k_2 3}, \mu_{k_2 3} \in M_2 & \dots \\
 & & \vdots & \\
 & & B_{k_3 3}, \mu_{k_3 3} \in M_{k_3} & \dots
\end{array}
$$

Some Jordan measurable sets

Proof.

For each $H_j \subset \{1, 2, \dots, k_j\}$, $j \in \mathbb{N}$, the set

$$B := \cup\{B_{ij} : \ i \in H_j, \ j \in \mathbb{N}\}$$

is Jordan measurable, because for each $j_0 \in \mathbb{N}$

$$\overline{B} \backslash \mathring{B} \subset \{\cup_{i \in H_j, \, j \leqslant j_0} (\overline{B_{ij}} \backslash \mathring{B}_{ij})\} \cup \overline{B'_{j_0}}$$

$$\overline{B'_{j_0}} \subset \cup_{1 \leqslant s \leqslant q_{j_0}} \{\Pi_{1 \leqslant i \leqslant k} [c^s_{ij_0}, d^s_{ij_0}]\},$$

the Lebesgue measure of each $\overline{B_{ij}} \backslash \mathring{B}_{ij}$ is 0 and $\overline{B'_{j_0}}$ is contained in a Lebesgue measurable set of measure less or equal than 2^{-j_0}. □

Second induction: Basic idea

Proof.

For $(i_n)_n := (1, 1, 2, 1, 2, 3, \ldots)$ there exists a $(j_n)_n \uparrow$ in $\mathbb{N}$ such that

$$|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})| < 1, \text{ for each } n \in \mathbb{N}.$$

In fact, if $|\mu_{i_n j_n}|(\Omega) < s_n \in \mathbb{N}$, J_n is an infinite subset of $\mathbb{N} \backslash \{1, 2, \ldots, j_n\}$, $\{L_u, 1 \leqslant u \leqslant s_n\}$ is a partition of J_n in infinite subsets and $B_u := \cup_{s \leqslant k_v, v \in L_u} B_{sv}$, then as

$$\Sigma\{|\mu_{i_n j_n}|(B_u) : 1 \leqslant u \leqslant s_n\} \leqslant \mu(\Omega) < s_n$$

there exists u', with $1 \leqslant u' \leqslant s_n$, such that

$$|\mu_{i_n j_n}|(B_{u'}) < 1.$$

To finish the induction let j_{n+1} be the first element of $L_{u'}$ and $J_{n+1} := L_{u'} \backslash \{j_{n+1}\}$.
From $\cup_{m>n} B_{i_m j_m} \subset B_{u'}$ it follows that $|\mu_{i_n j_n}(\cup_{m>n} B_{i_m j_m})| < 1$. □

The contradiction

Proof.

S^{n+1} increasing respect $S^n \implies \{t^i : i \in \mathbb{N}\}$ is an NV-tree.

$$H := \cup_{s\in\mathbb{N}} B_{i_s j_s} \in \mathcal{J}(K) = \cup\{\mathcal{B}_{t^i} : i \in \mathbb{N}\} \implies \exists r : H \in \mathcal{B}_{t^r}.$$

Let $(n_p)_p \uparrow$, $i_{n_p} = r$ then $\{\mu_{i_{n_p} j_{n_p}} : p \subset \mathbb{N}\} \subset M_{t^r}$, which is $\mathcal{B}_{t^r}$-pointwise bounded. The inequality

$$\sup\{\left|\mu_{i_{n_p} j_{n_p}}(H)\right| : p \in \mathbb{N}\} < \infty$$

contradicts:

$$\left|\mu_{i_{n_p} j_{n_p}}(\cup_{m<n_p} B_{i_m j_m})\right| < 1$$

$$\mu_{i_{n_p} j_{n_p}}(B_{i_{n_p} j_{n_p}}) > j_{n_p} > n_p \quad \text{and}$$

$$\left|\mu_{i_{n_p} j_{n_p}}(\cup_{n_p<m} B_{i_m j_m})\right| < 1$$

□

Set-algebras with *wN*-property

Proposition

Let $(\mathcal{A}_n)_n$ be a decreasing sequence of subsets of an algebra $\mathcal{A}$ of subsets of Ω such that for each $n \in \mathbb{N}$,

1. *Ω is covered by a finite subset of $\mathcal{A}_n$.*
2. $\{A \cap B : A \in \mathcal{A}_n, B \in \mathcal{A}\} \subset \mathcal{A}_n$.
3. $\{C \cup D : C, D \in \mathcal{A}_{n+1}\} \subset \mathcal{A}_n$.

Then $\mathcal{A}$ has wN-property when the following condition hold:

- *If a pairwise disjoint sequence $(A_n)_n$, with each $A_n \in \mathcal{A}_n$ and such that for each n there exists $B_n \in \mathcal{A}_n$ that verify*

$$\cup_{n<m} A_m \subset B_n$$

  for each n, then $\cup_n A_n \in \mathcal{A}$.

Follow the scheme to prove *wN*-property in $J(K)$ to get the proof. Clearly $J(K)$ and each σ-algebra verify this proposition.

Outline

3. Open question and application
   - Uniform bounded deciding sets
   - On bounded vector measures in $\mathcal{J}(K)$

Uniform bounded deciding sets

The fact that a set algebra $\mathcal{A}$ has *N*-property if each $\mathcal{A}$-poinwise bounded subset *M* of $ba(\mathcal{A})$ is uniformly bounded in the unit ball of $(L(\mathcal{A}), \|\cdot\|)$, motivates the *ubd* definition (Shapiro and others).

Definition

A subset *A* of a normed space $(E, \|\cdot\|)$ is uniform bounded deciding (*ubd*) if each *A*-poinwise bounded subset *M* of E' is uniformly bounded in its unit ball.

- A is *ubd* $\iff \{\|x\|^{-1} x : x \in A\backslash\{0\}\}$ is *ubd*,
- E is *ubd* $\iff (E, \|\cdot\|)$ is barrelled and
- $\mathcal{A}$ has *N*-property $\iff \{e(A) : A \in \mathcal{A}\}$ is *ubd* subset of $(L(\mathcal{A}), \|\cdot\|)$.

Characterization of *ubd* subsets of $(E, \|\cdot\|)$

1. If B be a non-*ubd* subset of $(E, \|\cdot\|)$,
2. there exist an unbounded B-pointwise bounded subset M in the dual Banach space.
3. Then $B = \cup_n B_n$, with $B_n := \{x \in B : |x^*(x)| \leqslant n, x^* \in M\}$ non-norming.
4. span $B =$ span $\{\cup_n n^{-1} B_n\}$, and $A := \cup_n n^{-1} B_n$ non-norming ($M \subset A^\circ$).

For each $m \in \mathbb{N}$ there exists $x'_m \in (m(\text{absco } A))^\circ$ with $\|x'_m\| > m$, whence

5. span $B = \cup_m m(\text{absco } A)$ is a non-*ubd* (unbounded set $\{x'_m : m \in \mathbb{N}\}$ is pointwise bounded in span B).
6. B be a non-*ubd* subset of $(E, \|\cdot\|)$.

Valdivia *sN* open question

Open question (Valdivia 2013). *Let $\mathcal{A}$ be a set-algebra that has property N. Has the set-algebra $\mathcal{A}$ the property sN*?
This question motivated the following analogous problem:
Open question. Assume that an *ubd* subset B of a normed space is written as a countable, increasing union $B := \cup_n B_n$. Does there exists an index m so that B_m is a *ubd* set?

A $\mathcal{J}(K)$ property

A *bounded finitely additive vector measure*, or simply a *bounded vector measure*, μ defined in an algebra $\mathcal{A}$ of subsets of Ω with values in a topological vector space $E(\tau)$ is a map $\mu\colon \mathcal{A} \to E$ such that $\mu(\mathcal{A})$ is a bounded subset of E and $\mu(B \cup C) = \mu(B) + \mu(C)$, for each pairwise disjoint subsets B, $C \in \mathcal{A}$.
The set $\mu(\mathcal{A})$ is a bounded subset of $E(\tau)$ if and only if the E-valued linear map $\mu : L(\mathcal{A}) \to E$ defined by $\mu(e_B) := \mu(B)$, for each $B \in \mathcal{A}$, is continuous.

Proposition

Let $(\mathcal{B}_{m_1})_{m_1}$ be an increasing covering of the algebra $\mathcal{J}(K)$. There exits $n_1 \in \mathbb{N}$ such that $\mathcal{B}_{n_1}$ has property wN.

Proof.

If $\{\mathcal{B}_t^{m_1} : t \in \cup_s \mathbb{N}^s\}$ is an increasing web in $\mathcal{B}_{m_1}$ without N-strands, for each $m_1 \in \mathbb{N}$, then $\{\mathcal{B}_t^{m_1} : (m_1, t) \in \mathbb{N} \times \cup_s \mathbb{N}^s\}$ is an increasing web without N-strands, contradiction. □

A localization property

Proposition

Let μ be a bounded vector measure defined in $\mathcal{J}(K)$ with values in $E(\tau)$, inductive limit $(E_{m_1}(\tau_{m_1}))_{m_1}$ of (LF)-spaces. There exists E_{m_1} such that for each defining sequence $(E_{n_1 m_2}(\tau_{n_1 m_2}))_{m_2}$ there exists $E_{n_1 n_2}$ such that $\mu(\mathcal{S})$ is $\tau_{n_1 n_2}$-bounded.

Proof.

$(\mathcal{B}_{m_1} := \mu^{-1}(E_{m_1}))_{m_1} \uparrow$ covers $\mathcal{J}(K)$. Exists $\mathcal{B}_{n_1}$ with property *wN*. If $(E_{n_1 m_2}(\tau_{n_1 m_2}))_{m_2}$ defines $E_{n_1}(\tau_{n_1})$ then $(\mathcal{B}_{n_1 m_2} := \mu^{-1}(E_{n_1 m_2}))_{m_2} \uparrow$ covers $\mathcal{B}_{n_1}$. If $\mathcal{B}_{n_1 n_2}$ has property *N*, the closed map

$$\mu|_{L(\mathcal{B}_{n_1 n_2})} : (L(\mathcal{B}_{n_1 n_2}), \|\cdot\|) \to E_{n_1 n_2}(\tau_{n_1 n_2})$$

has a continuous extension υ to $(L(\mathcal{J}(K)), \|\cdot\|)$ in $E_{n_1 n_2}(\tau_{n_1 n_2})$.
$\upsilon(A) = \mu(A)$, for each $A \in \mathcal{J}(K)$ (by continuity of $\mu : L(\mathcal{J}(K)) \to E(\tau)) \Longrightarrow \mu(\mathcal{J}(K))$ is a $\tau_{n_1 n_2}$-bounded. □

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