Rainwater sets and weak K-analyticity in $C^b(X)$

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Outline

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Notations and well known facts

X is a completely regular (Hausdorff) space, $C^b(X)$ the linear subspace of $C(X)$ consisting of all those *bounded* functions.
We identify X and βX with its copies $\{\delta_x : x \in X\}$ and $\overline{\{\delta_x : x \in X\}}$ in $B_{C^b(X)^*}$ (weak*).
$C^b(X)$, and $C(X)$ for X pseudocompact, have the $\|\cdot\|_\infty$.
$f \mapsto f^\beta$ is a linear isometry from $C^b(X)$ onto $C(\beta X)$.

Let $x \in \beta X$. Then $x \in \upsilon X$, the Hewitt realcompactification of X,
• iff each $f \in C(X)$ admits a continuous extension to $X \cup \{x\}$,
• iff $V \cap X \neq \emptyset$ for each βX-zero V containing x.
Hence the map $f \mapsto f^\nu$ is a biyection from $C(X)$ onto $C(\nu X)$ and X is G_δ-dense in υX,
• i. e., each υX-zero (or nonempty G_δ-set of υX) meets X•
X *pseudocompact* means
$C(X) = C^b(X) \iff \upsilon X = \beta X \iff X$ is G_δ-dense in βX.

A sequentially continuous map

If $Y \subset C^b(X)^*$ separates the functions of $C^b(X)$, then σ_Y is the topology of the pointwise convergence on Y on $C^b(X)$ (or in $C(X)$) .
If $Y = X$, then $\sigma_Y = \tau_p$, and $(C(X), \sigma_X) = C_p(X)$.

Claim

If Y is a G_δ-dense subset of Z then the immersion $(C(Z), \sigma_Y)$ onto $C_p(Z)$ is sequentially continuous
(They have the same (rel) sequentially compact subsets).
If moreover $Z \subset \beta Y$, i.e., $Z \subset \upsilon Y$, then $f \mapsto f^\nu|_Z$ is a linear isomorphism from $C(Y)$ onto $C(Z)$.

Proof.

If $Z_n := \{z \in Z : f_n(z) = f_n(x)\}$ then $\exists\, z_x \in Y \cap \bigcap_{n=0}^{\infty} Z_n$.
Each $C(Y)$ is embedded in $C(\nu Y)$. □

Rainwater sets of a Banach space E

Definition

A subset X of B_{E^*} is a *Rainwater set* if in E every *bounded* sequence of E that converges pointwise on X converges pointwise on B_{E^*} (i.e., converges weakly).

- *The set of extreme points of the closed dual unit ball is a Rainwater set for E.*
  This classic Rainwater's theorem follows from Choquet's integral representation theorem.
- A subset J of B_{E^*} such that each element of E attains its maximum in B_{E^*} in X is a *James boundary*.
  From Simons lemma follows that *each James boundary is a Rainwater set for E*.

The case $C(X)$, with X compact

Theorem (Rainwater's theorem for $C(X)$)

Each compact X is a Rainwater set for $C(X)$.

Proof.

By Arens-Kelly theorem, $\mathrm{Ext}\, B_{C(X)^*} = \{\pm\delta_x : x \in X\}$.
Hence $\{\pm\delta_x : x \in X\}$, and also X, is a $C(X)$-Rainwater set □

Other proof.

By Riesz representation theorem, $C(X)^* = rca(\mathcal{B}(X))$.
If $\{f_n\}_{n=1}^{\infty}$ is $C(X)$ bounded and $f_n(x) \to f(x)$, $\forall x \in X$,
then by Lebesgue dominated convergence theorem,
$\langle f_n, \mu\rangle \to \langle f, \mu\rangle$ for every $\mu \in C(X)^*$, i.e., $f_n \to f$ weakly. □

Outline

2. Some topological properties of Rainwater sets for $C^b(X)$
   - Pseudocompactness
   - G_δ-density

Pseudocompactness and Rainwater sets

Proposition

X is a Rainwater set for $C^b(X) \iff X$ is pseudocompact.

Proof.

(Known) If X is pseudocompact, $\{f_n\}_{n=1}^{\infty}$ is bounded in $C(X)$ and $f_n(x) \to f(x)$ for every $x \in X$ then for each $y \in \upsilon X = \beta X$

$$f_n^{\beta}(y) = f_n^{\upsilon}(y) \to \text{(by Claim) } f^{\upsilon}(y) = f^{\beta}(y).$$

$\{f_n^{\beta}\}_{n=1}^{\infty}$ is bounded in $C(\beta X)$. By Rainwater Th. for $C(\beta X)$

$$\left\langle f_n^{\beta}, \mu \right\rangle \to \left\langle f^{\beta}, \mu \right\rangle \text{ for each } \mu \in C(\beta X)^* = C^b(X)^*.$$

Thus X is a Rainwater set for $C^b(X)$. □

Second part of the proof

Proof, conversely.

Suppose that X is not pseudocompact. For each $y \in \beta X \setminus \upsilon X$ there exists $h_y \in C(\beta X)$, $h_y(y) = 0$, $h_y(\upsilon X) \subset]0,1]$. Now define

1. $g_y \in C(\beta X)$ by $g_y(z) = (1 + h_y(z))^{-1}$, $z \in \beta X$.
2. $f_n := (g_y)^n$.

Clearly $g_y(\upsilon X) \subset]0,1[$ and $g_y(y) = 1$, hence from

- $\|f_n|_X\|_\infty \leq 1$,
- $f_n(x) = g_y(x)^n \to 0$, $\forall x \in X$, and as $\forall n \in \mathbb{N}$
- $\langle f_n|_X, \delta_y \rangle = f_n(y) = g_y(y)^n = 1$, $(f_n)_n \not\to 0$ (weak),

we conclude that X is not a Rainwater set for $C^b(X)$. □

Pseudocompactness and Rainwater sets II

Proposition

If Y is a pseudocompact subset of $B_{C^b(X)^*}$ (weak*) that contains X, then Y is a Rainwater set for $C^b(X)$.

Proof.

Let $\{u_n\}_{n=0}^{\infty}$ be $C^b(X)$-bounded and $\langle u_n, y\rangle \to \langle u_0, y\rangle$, $\forall y \in Y$. If $T : C^b(X) \to C(Y)$ is defined by $(Tu)(y) :=< u, y >$, then

- $\{Tu_n\}_{n=0}^{\infty}$ is $C(Y)$-bounded and $T^*(C(Y)^*) = C^b(X)^*$,
- $(Tu_n)(y) \to (Tu_0)(y)$, $\forall y \in Y$, hence (pseudocompactness)

$$\langle Tu_n, \mu\rangle \to \langle Tu_0, \mu\rangle, \forall \mu \in C(Y)^*,$$

hence $\langle u_n, T^*\mu\rangle \to \langle u_0, T^*\mu\rangle$, $\forall T^*\mu \in T^*(C(Y)^*) = C^b(X)^*$. Hence $u_n \to u_0$ weakly, i.e., Y is a Rainwater set for $C^b(X)$. □

Examples

Example

An infinite discrete space I is not Rainwater for $C^b(I) = \ell_\infty(I)$.

Example

If D is a dense subset of $\beta N \setminus N$ of cardinality $|D| \leq c$, then $N \cup D$ is a Rainwater set for ℓ_∞. ($N \cup D$ is pseudocompact).

Example

A Valdivia compact X which is not Corson's compact contains a non compact Y which is a Rainwater set for $C(X)$.

Exists a countably compact $Y \subsetneq X$ with $\nu Y = \beta Y = X$
Y -homeomorphic to a bounded closed subset of some $\Sigma(\Gamma)$- is non compact and Rainwater set of $C^b(Y) = C(\beta Y) = C(X)$.

G_δ-density and Rainwater sets

Proposition

Let X be a Rainwater set for $C^b(X) = C(X)$. A subset $Y \subset X$ is a Rainwater set for $C(X)$ if and only if Y is G_δ-dense in X.

Proof.

Suppose that Y is not G_δ-dense in X.
Then there exists $x_0 \in \bigcap_{n=1}^{\infty} U_n \downarrow$
-non void G_δ-subset of X which that does not meet Y-.
Let $g_n \in C(X)$ with $0 \leq g_n \leq 1$, $g_n(X \setminus U_n) = \{0\}$ and $g_n(x_0) = 1$, $\forall n \in \mathbb{N}$.
Clearly $g_n \to 0$ in σ_Y and $\delta_{x_0}(g_n) = 1$, $\forall n \in \mathbb{N}$.
Hence Y is not a Rainwater set for $C(X)$.
The converse is a particular case of the next Proposition. □

G_δ-density and Rainwater sets

Proposition

Let X' be a Rainwater set for $C^b(X)$ and let Y be a G_δ-dense in $(X', \mathrm{weak}^*|_{X'})$. Then Y is a Rainwater set for $C^b(X)$.
Hence if υZ is homeomorphic to a Rainwater set for $C^b(X)$, then Z is also homeomorphic to a Rainwater set for $C^b(X)$.

Proof.

Let $\{f_n\}_{n=0}^{\infty}$ be a uniformly bounded sequence in $C^b(X)$ such that $f_n(y) \to f_0(y)$ for every Y.
By sequent. cont. Claim $f_n(x) \to f_0(x)$, for each $x \in X'$.
Then, as X' is Rainwater, $f_n \to f_0$ in $(C^b(X), \mathrm{weak})$.
Hence Y is a Rainwater set for $C^b(X)$.
Particular case: As Z is dense and C-embedded in υZ, then Z is G_δ-dense in υZ. □

$Y \subset X$, Y Rainwater for $C^b(X) \implies Y$ pseudocomp.?

Example

There exists X that contains a non pseudocompact subset that it is Rainwater set for $C^b(X)$.

Proof. Let G pseudocomp. such that $G \times G$ not pseudocomp.

If a bounded $\{f_n\}_{n=0}^{\infty} \subset C^b(G \times \upsilon G)$ verifies that

$$\lim_{n\to\infty} f_n(t,s) = f_0(t,s), \forall (t,s) \in G \times G,$$

then $\{f_n\}_{n=1}^{\infty}$ pointwise converges to f_0 in $G \times \upsilon G$
$-$ if $(t,s) \in G \times \upsilon G$, $\exists s_t \in G : f_n(t,s) = f_n(t,s_t)$, $n \in \mathbb{N}_0$$-$.
$X = G \times \upsilon G$ pseudocompact $\implies f_n \to f$ in $(C(X), \mathrm{weak})$.
$G \times G$ (non-pseudocompact) is Rainwater for $C(G \times \upsilon G)$. □

Outline

3 Rainwater sets and weak K-analyticity in $C^b(X)$
- Some equivalences
- Applications

Two lemmas

$X = T(\mathbb{N}^{\mathbb{N}})$ is *K-analytic* if T is u.s.c.c. valued (ordered).

Lemma

$C^b(X)$ weakly K-analytic $\Longrightarrow X$ Rainwater set for $C^b(X)$.

Proof.

$C^b(X) = C(\beta X) \Longrightarrow \beta X$ Talagrand compact $\Longrightarrow \beta X$ F.-U.
$(y_n \in \upsilon X)_n \to y_0 \notin \upsilon X \Longrightarrow |\beta\mathbb{N}| \leqslant |\mathbb{N}|$!!!!. Hence $\beta X = \upsilon X$. □

Lemma

Y Rainwater for $C^b(X) \Longrightarrow Y$ separates functions of $C^b(X)$.

Proof.

$f(y) = g(y) \Longrightarrow \{f, g, f, g, \ldots\}$ weak converges. □

K-analyticity in some $C(X)$

Theorem

Let X be completely regular. The following are equivalent:

1. *X is pseudocompact and $C_p(X)$ is K-analytic.*
2. *There exists a Rainwater set Y for $C^b(X)$ such that $(C^b(X), \sigma_Y)$ is K-analytic and $C_p(Y)$ is angelic.*
3. *There exists a Rainwater set Y for $C^b(X)$ such that $(C^b(X), \sigma_Y)$ is both K-analytic and angelic.*
4. *$C^b(X)$ is weakly K-analytic.*

Proof of $1 \Rightarrow 2$ and $2 \Longrightarrow 3$.

$1 \Rightarrow 2$ Let $Y := X$ (pseudocompact) $\Longrightarrow C_p(Y)$ angelic.
$2 \Longrightarrow 3$ Y Rainwater$\Longrightarrow (C^b(X), \sigma_Y)$ embeds in $C_p(Y)$. □

Weakly K-analyticity in some $C^b(X)$

Proof: 3($\exists$Rw $Y : (C^b(X), \sigma_Y)$ K-analytic angelic) $\Longrightarrow$ 4($C^b(X)$ weak K-analytic)$\Longrightarrow$ 1(X psd com $C_p(X)$ K-analytic).

$3 \Rightarrow 4$. Let $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be an ordered K-analytic representation of $\left(B_{C^b(X)}, \sigma_Y|_{B_{C^b(X)}}\right)$.
Each sequence $\{f_n\}_{n=1}^{\infty}$ in K_α contains $\{f_{n_i}\}_{i=1}^{\infty} \to f$ in the angelic $(K_\alpha, \sigma_Y|_{K_\alpha})$.
$\|f_n\|_\infty \leq 1$, $\forall n \in \mathbb{N} \Longrightarrow$($Y$ is Rainwater) $f_{n_i} \to f$ in $(K_\alpha, \mathrm{weak})$.
By angelicity (Eberlein-Šmulian!) $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a K-analytic representation of $\left(B_{C^b(X)}, \mathrm{weak}|_{B_{C^b(X)}}\right)$.
$C^b(X) = \bigcup_{n=1}^{\infty} nB_{C^b(X)}$ is also weakly K-analytic.
$4 \Rightarrow 1$. By Lemma X is pseudocompact and $(C^b(X), \mathrm{weak})$ K-analytic $\Longrightarrow C_p(X)$ K-analytic. $\square$

Another characterization of Talagrand compact sets

Corollary (X pseudocompact, recall $\Longrightarrow$ ($C_p(X)$ *angelic*).)

$C_p(X)$ K-analytic $\Longleftrightarrow$ $(C(X), \text{weak})$ K-analytic.

Theorem (Let X be compact and Y G_δ-dense in X.)

X is Talagrand (i.e., $C_p(X)$ K-anal) iff $(C(X), \sigma_Y)$ is K-analytic.

Proof.

$C_p(X)$ K-analytic $\Longrightarrow$ $(C(X), \sigma_Y)$ K-analytic.
($\Longleftarrow$) M r.c.c.-$(C(X), \sigma_Y)$ + Y G_δ-*dense* $\Longrightarrow$ M is r.c.c. in the angelic space $C_p(X)$.
$\overline{M}^{C_p(X)}$ is compact and F-U$\Longrightarrow$ $\overline{M}^{\sigma(Y)}$ compact and F-U
$\Longrightarrow$ $(C(X), \sigma_Y)$ angelic.
As Y is G_δ-dense in the compact X $\Longrightarrow$ Y Rainwater $C(X)$.
(Th 3 $\Longrightarrow$ 1) $(C(X), \sigma_Y)$ K-analytic $\Longrightarrow$ $C_p(X)$ K-analytic. □

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