Talagrand pseudocompacts and Rainwater sets

Manuel López Pellicer (IUMPA, UPV)

and S. López Alfonso (UPV)

Cáceres, March 09 - 11, 2017
XIII Encuentro Red de Análisis Funcional y Aplicaciones

Project Prometeo I/2013/058

UNIVERSITAT POLITÈCNICA DE VALÈNCIA

GENERALITAT VALENCIANA
CONSELLERIA D'EDUCACIÓ, CULTURA I ESPORT

Outline

Outline

Notations and well known facts on υX and βX

X is a completely regular (Hausdorff) space.
$C^b(X)$, and $C(X)$ for X pseudocompact, have the $\|\cdot\|_\infty$.
$f \mapsto f^\beta$ is a linear isometry from $C^b(X)$ onto $C(\beta X)$.
We identify X and βX with $\{\delta_x : x \in X\}$ and $\overline{\{\delta_x : x \in X\}}^{\text{weak}^*}$ in $B_{C^b(X)^*}$ (weak*).
Let $x \in \beta X$. Then $x \in \upsilon X$, the Hewitt realcompactification of X,

- iff each $f \in C(X)$ admits a continuous extension to $X \cup \{x\}$, hence $f \mapsto f^\nu$ is a biyection from $C(X)$ onto $C(\nu X)$,
- iff $V \cap X \neq \emptyset$ for each βX-zero V containing x, whence X is G_δ-dense in υX.

X pseudocompact means
$C(X) = C^b(X) \Longleftrightarrow \upsilon X = \beta X \Longleftrightarrow X$ is G_δ-dense in βX.

A sequentially continuous map

If $Y \subset C^b(X)^*$ separates functions in $C^b(X)$, then σ_Y is the pointwise convergence topology on $C^b(X)$ (on $C(X)$).
$C_p(X) := (C(X), \sigma_X)$.

Claim

Let Y be G_δ-dense in Z. $(C(Z), \sigma_Y)$ and $C_p(Z)$ have the same convergent sequences ((rel) sequentially compact subsets). If f_0 is σ_Y adherent to (f_n) then f_0 is also adherent in $C_p(Z)$. If moreover $Z \subset \beta Y$, i.e., $Z \subset \upsilon Y$, then $f \mapsto f^\nu|_Z$ is a linear isomorphism from $C(Y)$ onto $C(Z)$.

Proof.

If $Z_n := \{z \in Z : f_n(z) = f_n(x)\}$ then $\exists\, z_x \in Y \cap \bigcap_{n=0}^{\infty} Z_n$.
$C(Y)$ is embedds in $C(\upsilon Y)$. □

Rainwater subsets in the dual unit ball of a Banach space E

Definition

A subset X of B_{E^*} is a *Rainwater set* if in B_E the topologies σ_X and $\sigma_{B_{E^*}}$ have the same convergent sequences.

This means that each *bounded* sequence of E that converges pointwise on X converges weakly*.

- From Choquet's integral representation it theorem follows that *the set of extreme points of the closed dual unit ball is a Rainwater set for E* (Rainwater's theorem).
- From Simons lemma it follows that *each James boundary is a Rainwater set for E* (a subset J of B_{E^*} is a *James boundary* if each element of E attains its maximum in B_{E^*} in J).

Rainwater theorem for $C(X)$

Theorem (Rainwater's theorem for $C(X)$)

Each compact X is a Rainwater set for $C(X)$.

Proof.

By Arens-Kelly theorem, $\operatorname{Ext} B_{C(X)^*} = \{\pm\delta_x : x \in X\}$.
Hence $\{\pm\delta_x : x \in X\}$, and also X, is a $C(X)$-Rainwater set □

Other proof.

By Riesz representation theorem, $C(X)^* = rca(\mathcal{B}(X))$.
If $\{f_n\}_{n=1}^{\infty}$ is $C(X)$ bounded and $f_n(x) \to f(x)$, $\forall x \in X$, then, by Lebesgue dominated convergence theorem, $\langle f_n, \mu\rangle \to \langle f, \mu\rangle$ for every $\mu \in C(X)^*$, i.e., $f_n \to f$ weakly. □

Outline

2. Some topological properties of Rainwater sets for $C^b(X)$
   - Pseudocompactness
   - G_δ-density

Pseudocompact $\Longleftrightarrow$ Rainwater set ($\Longrightarrow$)

Proposition

X is a Rainwater set for $C^b(X) \Longleftrightarrow X$ is pseudocompact.

Proof.

(Known) If X is pseudocompact, $\{f_n\}_{n=1}^{\infty}$ is bounded in $C(X)$ and $(f_n)_n \to f$ in $C_p(X)$ then for each $y \in \upsilon X = \beta X$

$$f_n^\beta(y) = f_n^\upsilon(y) \to \text{(by Claim) } f^\upsilon(y) = f^\beta(y).$$

By Rainwater theorem for $C(\beta X)$

$$\left\langle f_n^\beta, \mu \right\rangle \to \left\langle f^\beta, \mu \right\rangle \text{ for each } \mu \in C(\beta X)^* = C^b(X)^*.$$

Thus X is a Rainwater set for $C^b(X)$. □

Pseudocompact $\Longleftrightarrow$ Rainwater set ($\not\Longrightarrow$)

Proof, conversely.

Suppose that X is not pseudocompact. For each $y \in \beta X \setminus \upsilon X$ there exists $h_y \in C(\beta X)$, $h_y(y) = 0$, $h_y(\upsilon X) \subset]0,1]$. Now define

1. $g_y \in C(\beta X)$ by $g_y(z) = (1 + h_y(z))^{-1}$, $z \in \beta X$.
2. $f_n := (g_y)^n$.

Clearly $g_y(\upsilon X) \subset]0,1[$ and $g_y(y) = 1$, hence from

- $\|f_n|_X\|_\infty \leq 1$,
- $\lim_n f_n(x) = \lim_n g_y(x)^n = 0$, $\forall x \in X$,
- $\langle f_n|_X, \delta_y \rangle = f_n(y) = g_y(y)^n = 1$, $(f_n)_n \not\to 0$ (weak*),

we conclude that X is not a Rainwater set for $C^b(X)$. □

Pseudocompact subsets Y of $B_{C^b(X)^*}$ with $X \subset Y$

Proposition

$(Y, \mathrm{weak}^*_{|Y})$ *pseudocompact* $\Longrightarrow$ *Y is a $C^b(X)$-Rainwater set.*

Proof.

Let T the product of the isometry-inmersions

$$C^b(X) \longrightarrow C(B_{C^b(X)^*}) \longrightarrow C(Y).$$

$T^*(C(Y)^*) = C^b(X)^*$.
Let $\{u_n\}_{n=0}^{\infty}$ be $C^b(X)$-bounded and $\lim_n u_n = u_0$ in σ_Y.
Then $\{Tu_n\}_{n=0}^{\infty}$ is $C(Y)$-bounded and $\lim_n Tu_n = Tu_0$ in $C_p(Y)$.
By pseudocompactness, for each $T^*\mu \in C^b(X)^*$ $(\mu \in C(Y)^*)$

$$\lim_n \langle u_n, T^*\mu\rangle = \lim_n \langle Tu_n, \mu\rangle = \langle Tu_0, \mu\rangle = \langle u_0, T^*\mu\rangle$$

Examples

Example

An infinite discrete space I is not Rainwater for $C^b(I) = \ell_\infty(I)$.

Example

If D is a dense subset of $\beta N \setminus N$ of cardinality $|D| \leq c$, then $N \cup D$ is a Rainwater set for ℓ_∞. ($N \cup D$ is pseudocompact).

Example (Kalenda)

A Valdivia compact X which is not Corson's compact contains a non compact Y which is a Rainwater set for $C(X)$.

Exists a countably compact $Y \subsetneq X$ with $\nu Y = \beta Y = X$
Y -homeomorphic to a bounded closed subset of some $\Sigma(\Gamma)$- is non compact and Rainwater set of $C^b(Y) = C(\beta Y) = C(X)$.

Rainwater subsets of a pseudocompact (Rw$\Longrightarrow$ G_δ-dense)

Proposition

Let Y be a subset of a pseudocompact X. Y is a Rainwater set for $C(X)$ if and only if Y is G_δ-dense in X.

Proof.

If Y is not G_δ-dense in X there exists $x_0 \in \bigcap_{n=1}^{\infty} U_n \downarrow$, non void G_δ subset of X which that does not meet Y.
Let $g_n \in C(X)$ with $0 \leq g_n \leq 1$, $g_n(X \setminus U_n) = \{0\}$ and $g_n(x_0) = 1$, $\forall n \in \mathbb{N}$. From

$$\lim_n g_n = 0, \text{ in } \sigma_Y, \text{ and } \quad \lim_n \delta_{x_0}(g_n) = 1$$

we get that Y is not a Rainwater set for $C(X)$.

Rainwater subsets of a pseudocompact (G_δ-dense$\Longrightarrow$Rw)

Proposition

Let X' be a Rainwater set for $C^b(X)$ and let Y be a G_δ-dense in $(X', \mathrm{weak}^*|_{X'})$. Then Y is a Rainwater set for $C^b(X)$.
Hence if υZ is homeomorphic to a Rainwater set for $C^b(X)$, then Z is also homeomorphic to a Rainwater set for $C^b(X)$.

Proof.

If $(f)_{nn=0}^{\infty}$ is bounded in $C^b(X)$ and $\lim_n f_n = f_0$ in σ_Y, then:

- By G_δ-density $\lim_n f_n = f_0$ in $\sigma_{X'}$,
- By Rainwater condition $\lim_n f_n = f_0$ $(C^b(X), \mathrm{weak})$.

Hence Y is a Rainwater set for $C^b(X)$.
Particular case: Z is G_δ-dense in υZ. □

Rainwater subset of a pseudocompact $\not\Longrightarrow$ pseudoc.

Example

There exists pseudocompact X that contains a non pseudocompact subset Y that it is Rainwater set for $C^b(X)$.

Proof.

Let G pseudocompact such that $G \times G$ not pseudocompact.
$G \times G$ is G_δ-dense in $G \times \beta G$.
$G \times \beta G$ is pseudocompact (pseudocompact $\times$ compact).
Hence $G \times G$ is a Rainwater set for $C(G \times \beta G)$.

Remark

Let $X = Y \cup \{\infty\}$ be the Alexandroff compactification of a discrete space Y of nonmeasurable infinite cardinality. Clearly Y is not pseudocompact G_δ-dense in Eberlein compact X.

Outline

3. Rainwater sets and weak K-analyticity in $C^b(X)$
   - Some equivalences
   - Applications

Two lemmas

$X = T(\mathbb{N}^{\mathbb{N}})$ is *K-analytic* if T is u.s.c.c. valued (ordered).

Lemma

$C^b(X)$ weakly K-analytic $\Longrightarrow$ X pseudocompact.

Proof.

$C^b(X) = C(\beta X) \Longrightarrow \beta X$ Talagrand compact $\Longrightarrow \beta X$ F.-U.
$(y_n \in \upsilon X)_n \to y_0 \notin \upsilon X \Longrightarrow |\beta\mathbb{N}| \leqslant |\mathbb{N}|$!!!!. Hence $\beta X = \upsilon X$. □

Lemma

Y Rainwater for $C^b(X) \Longrightarrow Y$ separates functions of $C^b(X)$.

Proof.

$f(y) = g(y) \Longrightarrow \{f, g, f, g, \ldots\}$ weak converges. □

Weak K-analyticity in $C(X)$

Theorem

Let X be completely regular. The following are equivalent:

1. *X is pseudocompact and $C_p(X)$ is K-analytic.*
2. *There exists a Rainwater set Y for $C^b(X)$ such that $(C^b(X), \sigma_Y)$ is K-analytic and $C_p(Y)$ is angelic.*
3. *There exists a Rainwater set Y for $C^b(X)$ such that $(C^b(X), \sigma_Y)$ is both K-analytic and angelic.*
4. *$C^b(X)$ is weakly K-analytic.*

Proof of 1 $\Rightarrow$ 2 and 2 $\Longrightarrow$ 3.

1 $\Rightarrow$ 2 Take $Y := X$ (pseudocompact) $\Longrightarrow C_p(Y)$ angelic.
2 $\Longrightarrow$ 3 $Y \subset B_{C^b(X)^*} \Longrightarrow (C^b(X), \sigma_Y)$ embeds in $C_p(Y)$. □

Weak K-analyticity in $C^b(X)$

Proof: 3($\exists$Rw Y : $(C^b(X), \sigma_Y)$ K-analytic angelic) $\Longrightarrow$ 4($C^b(X)$ weak K-analytic)$\Longrightarrow$ 1(X psdcom, $C_p(X)$ K-analytic).

$3 \Rightarrow 4$. By 3, $W := \left(B_{C^b(X)}, \sigma_Y|_{B_{C^b(X)}}\right)$ is angelic and has an an ordered K-analytic representation $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$.
W and $Z := \left(B_{C^b(X)}, \mathrm{weak}|_{B_{C^b(X)}}\right)$ has the same convergent sequences (Y is Rainwater).
Hence $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a compact resolution of the angelic space Z.
Then $\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a K-analytic representation of Z.
Whence $C^b(X) = \bigcup_{n=1}^{\infty} nB_{C^b(X)}$ is weakly K-analytic.
$4 \Rightarrow 1$. $(C^b(X), \mathrm{weak})$ K-analytic $\Longrightarrow$ X pseudocompact (Lemma) and $C_p(X)$ K-analytic. □

Another characterization of Talagrand compact sets

Theorem

Let Y be a G_δ-dense subset of a pseudocompact X. $C_p(X)$ is K-analytic $\Longleftrightarrow$ $(C(X), \sigma_Y)$ is K-analytic.

Proof.

$\Longrightarrow$ obvious. ($\Longleftarrow$) By the Claim if M is σ_Y-r.c.c. then M is r.c.c. in the angelic space $C_p(X)$.
$\overline{M}^{C_p(X)}$ is compact and F-U $\Longrightarrow$ $\overline{M}^{\sigma(Y)}$ compact and F-U
$\Longrightarrow$ $(C(X), \sigma_Y)$ angelic.
Y Rainwater $C(X)$. Hence we may apply Th 3 $\Longrightarrow$ 1. □

Corollary

Let Y be a G_δ-dense subset of a compact X. X is Talagrand compact if and only if $(C(X), \sigma_Y)$ is K-analytic.

The last example

Remark

If x_0 is a non-isolated point of a pseudocompact X, $Y := X \setminus \{x_0\}$ and if $\{x_0\}$ is not a G_δ-subset of X, then each nonempty G_δ-subset G of X intersects Y (otherwise $G = \{x_0\}$) hence $C_p(X)$ is K-analytic $\Longleftrightarrow$ $(C(X), \sigma_Y)$ is K-analytic.

Example

$\{\omega_1\}$ is not G_δ-subset of $[0, \omega_1]$ and $C_p([0, \omega_1])$ is not K-analytic, because $[0, \omega_1]$ is not Fréchet-Urysohn.
Hence the pseudocompact space $Y := [0, \omega_1)$ verifies that $C_p([0, \omega_1))$ is not K-analytic.

Ferrando, Kąkol and L-P study the case "x_0 G_δ-subset of X" in *On spaces $C_b(X)$ weakly K-analytic.*

The last slide

¡MUCHAS GRACIAS!

References I

- Arens, R. F. and Kelley, J. L., *Characterizations of the space of continuous functions over a compact Hausdorff space*, Trans. Amer. Math. Soc. **62** (1947), 499-508.
- Ferrando, J. C., Kąkol, J. and M. López-Pellicer, M., *On spaces $C_b(X)$ weakly K-analytic*, submitted.
- García-Ferreira, S. and García-Máynez, A., *On weakly pseudocompact spaces*, Houston J. Math. **20** (1994), 145-159.
- Kalenda, O., *A characterization of Valdivia compact spaces*, Collect. Math. **51** (2000), 59-81.

References II

- King, D. M. and Morris, S. A., *The Stone-Čech compactification and weakly Fréchet spaces*, Bull. Austral. Math. Soc. **42** (1990), 349-352.
- Moll, S. and Sánchez Ruiz, L. M., *A note on a theorem of Talagrand*, Topology Appl. **153** (2006), 2905-2907.
- Nygaard, O., *A remark on Rainwater's theorem*, Ann. Math. Inform. **32** (2005), 125-127.
- Rainwater, J., *Weak convergence of bounded sequences*, Proc. Amer. Math. Soc. **14** (1963), 999-999.
- Simons, S., *A convergence theorem with boundary*, Pacific J. Math. **40** (1972), 703-708.

References III

- Simons, S., *An eigenvector proof of Fatou's lemma for continuous functions*, Math. Intelligencer 17 (3) (1995), 67–70.
- Talagrand, M., *Espaces de Banach faiblement K-analytiques*, Annals of Math. **110** (1979), 407-438.