Strong norming sets and the separable quotient problem

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Outline

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Mazur open *SQ* problem.

Problem (S. Mazur, 1932)

Does any infinite-dimensional Banach space X have a separable infinite dimensional quotient (SQ for short)?

Here Banach space means also infinite dimensional.
X has $SQ \iff X$ is mapped onto a separable Banach space under a continuous linear map.
Every separable Banach space admits a quotient with a Schauder basis (Johnson and Rosenthal) then:

Problem (*SQ* problem reformulated)

Does any infinite-dimensional Banach space admits SQ with a Schauder basis?

Saxon-Wilansky's *SQ* equivalent conditions (1977).

Theorem (Saxon-Wilansky)

For a Banach space X t.f.a.e.:

1. *X contains a dense non-barrelled linear subspace.*
2. *X admits a strictly increasing $(X_n)_n$ of closed subspaces with $\cup_n X_n$ dense in X.*
3. *X^* admits a strictly decreasing sequence of weak*-closed subspaces whose intersection consists only of the zero element.*
4. *X has a separable quotient.*

We will prove that E has *SQ* iff there exists in S_E a dense non strong norming subset.

Outline

2. Strong norming sets.

Norming sets.

Definition

A subset C of a normed space $(E, \|\cdot\|)$ is norming iff C verifies one of the following conditions:

1. $C^{oo} = \overline{\mathrm{abcx}\, C}$ is a bounded neighborhood of 0 in $(E, \|\cdot\|)$.
2. C and C^{o} are bounded (or C^{oo} and C^{o} are neighborhood of 0).
3. The Minkowski functional of C^{oo} is a norm equivalent to $\|\cdot\|$.

Clearly C norming iff C^{o} is norming iff $\{\|x\|^{-1} x : x \in C\backslash\{0\}\} \subset S_E$ is a norming subset.

Strong norming sets of normed space $(E, \|\cdot\|)$.

$C \subset E$ is s-norming if $C = \cup_m C_m \uparrow \Rightarrow \exists$ norming C_n.

Proposition

C is s-norming iff C is bounded and each C-pointwise bounded subset M of E' is norm-bounded.

If $M\,(\subset E^*)$ is C-pointwise bounded and

$$C_m := \{x \in C : |f(x)| \leqslant m, \ \forall f \in M\},$$

then $C = \cup_m C_m \uparrow$. Hence C_n norming $\Rightarrow M$ $\|\cdot\|$-bounded.
Conversely: If $C = \cup_m C_m \uparrow$ and $f_m \in C_m^{\circ}$, $\|f_m\| > m$, $m \in \mathbb{N}$, then $M := \{f_m : m \in \mathbb{N}\}$ is $\|\cdot\|$-unbounded and pointwise bounded on C.

Examples of strong norming sets.

Example

The unit sphere S_E of a Banach space $(E, \|\cdot\|)$ is s- norming. If $(E, \|\cdot\|)$ is a reflexive Banach space then the set of exposed points of S_E is s-norming.

Let $\mathcal{A}$ be an algebra of subsets of a set Ω. The dual of $(L(\mathcal{A}), \|\cdot\|) = (\mathrm{span}\{e_A : A \in \mathcal{A}\}, \sup \mathrm{norm})$ is isomorphically isometric to $(ba(\mathcal{A}), \|\cdot\|)$ by $\mu(e_A) := \mu(A)$.
$\mathcal{B} \subset \mathcal{A}$ has Nikodým property if $\mathcal{B}$-pointwise boundedness imply $\|\cdot\|$-boundedness.

Example

$\mathcal{B}$ has Nikodým property iff $\{e_A : A \in \mathcal{B}\}$ is a s-norming subset of $S_{L(\mathcal{A})}$.

Strands of norming sets.

By Nikodým-Grothendieck and Valdivia theorems for a σ-algebra $\mathcal{A}$ we have:
If $\mathcal{A} = \cup_{n_1} \mathcal{A}_{n_1} \uparrow$, there exists m_1 such that $\mathcal{A}_{m_1}$ has Nikodým property ($\mathcal{A}$ has strong Nikodým property or *sN* property).
This is a particular result of the following (Kakol-LP)
If $\{\mathcal{A}_{n_1,n_2,\cdots,n_p} : n_i \in \mathbb{N}, i \in \mathbb{N}\}$ is an increasing web of a σ-álgebra $\mathcal{A}$, there exists a sequence $\{m_q : q \in \mathbb{N}\}$ such that each $\mathcal{A}_{m_1,m_2,\cdots,m_q}$, $q \in \mathbb{N}$, has Nikodým property. Hence with the notation considered:

Example

Each $\{e_A : A \in \mathcal{A}_{m_1,m_2,\cdots,m_q}\}$, $q \in \mathbb{N}$, is a *s*-norming subset of $S_{L(\mathcal{A})}$.

Examples of non *s*-norming sets in $(E, \|\cdot\|)$.

For each $\cup_{n\in\mathbb{N}} F_n \uparrow$, closed subspaces there exists $(f_n)_n$ in E^* with $\|f_n\| = n$ and $f_n(F_n) = \{0\}$. Hence $S_E \cap [\cup_{n\in\mathbb{N}} F_n]$ is not *s*-norming.
In particular, if F is a closed subspace of $(E, \|\cdot\|)$ and $A := \{x_n : n \in \mathbb{N}\}$ is a sequence of linear independent vectors $\mathrm{mod} F$, then $S_E \cap [F + \mathrm{span}\, A]$ is not *s*-norming.
Moreover, when $(E/F, \|\cdot\|_{E/F})$ is separable infinite dimensional, $\exists A : \overline{F + \mathrm{span}\, A} = E$. Then $G := S_E \cap [F + \mathrm{span}\, A]$ is a non strong norming dense in S_E.

Outline

3 Strong norming sets and *SQP*.

A Lemma related with Baire's Category Theorem.

Lemma

Let B be a closed absolutely convex subset of $(E, \|\cdot\|)$ which is not neigborhood of 0. If $\overline{\operatorname{span} B} = E$ then the codimension of $\operatorname{span} B$ is not countable.

Proof.

If $\{x_n : n \in \mathbb{N}\}$ is a cobase of $\operatorname{span} B$ in E, then there exists m such that the absolutely convex closed hull of

$$m[B \cup \{x_n : n \leq m\}]$$

is a neighborhood of 0. Then B is a neighborhood of 0 in $\operatorname{span} B$ and, by density, B is a neighborhood of 0 in E, contradiction. □

One more equivalence.

Theorem

For an infinite dimensional Banach space $(E, \|\cdot\|)$ the following properties are equivalent:

1. S_E *contains a dense subset G non strong norming.*
2. $(E, \|\cdot\|)$ *admits an infinite dimensional separable quotient.*

Proof.

$2 \Rightarrow 1$ Proved.
$1 \Rightarrow 2$ Let H be unbounded in $(E^*, \|\cdot\|)$ and G-pointwise bounded.
$B := H^{o}$ is not 0-neighborhood in $(E, \|\cdot\|)$ and $\overline{\operatorname{span} B} = E$ $(G \subset \operatorname{span} B)$. Hence cod $\operatorname{span} B$ in $(E, \|\cdot\|)$ is not countable. □

The dense direct sum $\operatorname{span}\{x_i, i \in \mathbb{N}\} \oplus F$.

Proof.

$\exists\, x_n \in S_E$ and $f_n \in E^*$, $n \in \mathbb{N}$, such that:

1. $f_n(x_n) = 1$, $x_1 \in S_E \backslash \operatorname{span} B$ and $x_n \in (S_E \cap f_1^{\perp} \cap \cdots \cap f_{n-1}^{\perp}) \backslash \operatorname{span}(B \cup \{x_1, \cdots, x_{n-1}\})$, for $n > 1$,
2. and if $x \in B$ there exists $(a_n)_n$, $|a_n| \leqslant 2^{-n}$, $\forall n \in \mathbb{N}$, such that

$$f_n(x + a_1 x_1 + \cdots + a_n x_n) = 0.$$

□

The induction.

Proof.

$x_1 \in S_E \setminus \operatorname{span} B$. By H-B $\exists f_1 \in E^* : f_1(x_1) = 1$, $|f_1(v)| \leqslant 1$, $\forall v \in 2B$. Hence $\exists\, |a_1| \leqslant 2^{-1}$:

$$f_1\left(x + a_1 x_1\right) = 0.$$

$x_n \in (S_E \cap f_1^{\perp} \cap f_2^{\perp} \cap \cdots \cap f_{n-1}^{\perp}) \setminus \operatorname{span}\left(B \cup \{x_1, x_2, \cdots, x_{n-1}\}\right)$.
By H-B $\exists f_n \in E^*$, $f_n(x_n) = 1$, $|f_n(v)| \leqslant 1$,
$\forall v \in 2^n \{B + b_1 x_1 + \ldots + b_{n-1} x_{n-1} : |b_i| \leqslant 1, 1 \leqslant i < n\}$.
Hence $\exists\, |a_n| \leqslant 2^{-n}$:

$$f_n(x + a_1 x_1 + \cdots + a_{n-1} x_{n-1} + a_n x_n) = 0.$$

□

The density.

Proof.

$f_n(x + a_1 x_1 + \cdots + a_n x_n) = 0$ and $x_{n+i} \in f_n^{\perp}$, $\forall n, i \in N$, imply

$$x + \Sigma_{i=1}^{\infty} a_i x_i \in F := \cap\{f_n^{\perp} : n \in \mathbb{N}\} \Rightarrow B \subset \overline{\mathrm{span}\,\{x_i, i \in \mathbb{N}\}} + F$$

Hence $\mathrm{span}\,\{x_i, i \in \mathbb{N}\} \oplus F$ verifies that

$$E = \overline{\mathrm{span} B} \subset \overline{\mathrm{span}\,\{x_i, i \in \mathbb{N}\} \oplus F} \subset E \Rightarrow 2.$$

□

Argyros, Dodos and Kanellopoulos (2008).

With Ramsey Theory, Argyros, Dodos and Kanellopoulos (2008) proved that if X^* is non-separable then X^{**} contains an unconditional family of size $|X^{**}|$. As application they proved

Theorem (Argyros-Dodos-Kanellopoulos, 2008)

Every dual Banach space has a SQ.

Corollary

The space $\mathcal{L}(X, Y)$ of bounded linear operators between Banach spaces X and Y equipped with the operator norm has a separable quotient provided $Y \neq \{\mathbf{0}\}$.

Proof.

X^* is complemented in $\mathcal{L}(X, Y)$. □

Question related to Argyros - D - K theorem.

Every infinite dimensional dual Banach space $(E^*, \|\cdot\|)$ has an infinite dimensional separable quotient (S.A. Argyros, P. Dodos and V. Kanellopoulos, *Unconditional families in Banach spaces*. Math. Ann. **341** (2008) 15-38).
Hence, the unit sphere S_{E^*} contains a non strong norming dense subset. This result motivates the following problem.

Problem

Determine a dense and non strong norming subset in the unit sphere S_{E^*} of the dual of a Banach space $(E, \|\cdot\|)$.

Aplication to WCG Banach spaces.

Corollary

Every WCG Banach space $(E, \|\cdot\|)$ *has SQ.*

Proof.

If $(E, \|\cdot\|)$ is reflexive apply A-D-K.
If a WCG $(E, \|\cdot\|)$ is not reflexive $\exists B$ weakly compact and absol. convex subset such that $\overline{\text{span}\, B} = X$, with B not 0-neighborhood in $(E, \|\cdot\|)$.
Since $G := S_E \cap [\text{span}\, B]$ is a non norming dense subset of S_E we get that $(E, \|\cdot\|)$ has SQ. □

An equivalent condition.

S_E contains a dense subset G non strong norming is equivalent to the condition that there exists an unbounded sequence $(y_n)_n$ in E^* which is pointwise bounded in a dense subset of E.

Definition (Śliwa)

A sequence $(y_n)_n$ is pseudobounded if it is point-wise bounded on dense subspace of E and $\sup_n \|y_n\| = \infty$.

Hence an infinite dimensional Banach space E has a *SQ* if and only if E^* has a pseudobounded sequence.

Outline

4. Strongly normal sequences and *SQP*.
   - *SQ* for small Banach spaces

Śliva strongly normal sequences and *SQP*.

Definition

$(y_n)_n \subset S(E^*)$ is normal if in E^* if $\lim_n y_n(x) = 0, \forall x \in E$.

This happens iff $\overline{\{x \in E : y_n(x) \to 0\}} = E$.

Definition

It is strongly normal if $\overline{\{x \in E : \sum_{n=1}^{\infty} |y_n(x)| < \infty\}} = E$.

Theorem (Josefson-Nissenzweig)

If $\dim E = \infty$, E^* *contains a* $(y_n)_n$ *normal.*

Schauder basis.

Definition

$(x_n)_n$ in a lcs F is a Schauder basis for F if for each element x of F there is a unique sequence $(\alpha_n)_n$ of scalars such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ and the coefficient functionals x_n^*, $n \in \mathbb{N}$, defined by $x_n^*(x) = \alpha_n$ are continuous on F.

Definition

A Schauder basic sequence is a Schader basis on its closed linear span X in F.

Johnson and Rosenthal proved that any normal (y_n) in the dual E^* of a separable Banach space E has a Schauder basic subsequence $(y_{k(n)})$ in E^*_σ. Then each separable Banach space has a quotient with Schader basis.

Śliva theorem (2012) for ∞ dim. Banach spaces.

Theorem (Śliwa)

Any strongly normal sequence $(y_n)_n$ in E^* contains a Schauder basic subsequence $(y_{k(n)})_n$ in E^*_σ.

Proof (First step).

Let $\varphi : E \to E^{**}$ be the cannonical embedding map.
For every finite dimensional subspace $Y \subset E^*$ and every $\epsilon \in (0, 1/2)$ there exists a finite $H \subset S(E)$ such that $\forall f \in S(Y^*)$ there is $x \in H$ with

$$\|f - \varphi(x)|_Y\| < 2\epsilon.$$

□

Śliva theorem (2012) for ∞ dim. Banach spaces.

Proof (Second step).

By induction we select $(y_{k(n)})_n$ such that $\forall n$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$

$$\left\|\sum_{i=1}^{n} \alpha_i y_{k(i)}\right\| \leq \left(1 + 2^{1-n}\right) \left\|\sum_{i=1}^{n+1} \alpha_i y_{k(i)}\right\|.$$

Then $(y_{k(n)})_n$ is a Schauder basic sequence in E^* and $\|P_n\| \leq \prod_{k=n}^{\infty} \left(1 + 2^{1-k}\right) < 1 + 2^{4-n}$, where

$$P_n : Y \to Y, \quad P_n\left(\sum_{i=1}^{\infty} \alpha_i y_{k(i)}\right) = \sum_{i=1}^{n} \alpha_i y_{k(i)}$$

and $Y = \operatorname{span}\{y_{k(n)} : n \in \mathbb{N}\}$. □

Śliva theorem (2012) for ∞ dim. Banach spaces.

Proof (Third step).

The coefficient functionals $(f_n)_n$ of $(y_{k(n)})_n$ is a basic Schauder sequence of $F = \overline{\mathrm{span}\{f_n : n \in \mathbb{N}\}}^{Y^*}$. Hence the sequence of coefficient functionals $(g_n)_n$ of $(f_n)_n$ is basic Schauder sequence of F_σ^*.
For $T := \varphi|_Y$, $T(E) = F$, hence $T^* : F_\sigma^* \to E_\sigma^*$ is and isomorphism onto its closed image and then $(T^* g_n)_n$ is a Schauder basis sequence in E_σ^*.
As $T^* g_n(x) = g_n(Tx) = g_n\left(\Sigma_{1=1}^{\infty} y_{k(i)}(x) f_i\right) = y_{k(n)}(x)$ we get

$$(T^* g_n)_n = ((y_{k(n)})_n.$$

$\square$

Śliwa characterization *SQ* (2012) for ∞ dim. Banach spaces.

Definition

A sequence $\{x_n^*\}$ in X^* is pseudobounded if it is pointwise bounded on a dense subspace of X and $\sup_n \|x_n^*\| = \infty$.

Theorem (Śliwa)

For a Banach space X tfae:

1. *X has a separable quotient.*
2. *X^* has a strongly normal sequence.*
3. *X^* has a basic Schauder sequence in the weak* topology.*
4. *X^* has a pseudobounded sequence.*

Śliwa characterization *SQ* (2012) for ∞ dim. Banach spaces.

Proof.

$2 \Rightarrow 3$ (donne). $3 \Rightarrow 2$. $y = \sum_{i=1}^{\infty} x_n(y) y_n$, in E^*_σ, for $y \in Y = \overline{\operatorname{span}\{y_n : n \in \mathbb{N}\}}^{E^*_\sigma}$.
The linear span $X := \operatorname{span}\{x_n : n \in \mathbb{N}\}$ verifies

$$\left(X + {}^{\perp}Y\right)^{\perp} = \left(X \cup {}^{\perp}Y\right)^{\perp} = X^{\perp} \cap Y = \{0\},$$

hence the density of $X + {}^{\perp}Y$ implies

$$\overline{\{x \in E : \sum_{n=1}^{\infty} |y_n(x)| < \infty\}} = E.$$

Śliwa characterization *SQ* (2012) for ∞ dim. Banach spaces.

Proof.

$4 \Rightarrow 2$. (z_n) pointwise bounded in D, $\overline{D} = E$, with $\left\|z_{k(n)}\right\| > n^2$ $\Rightarrow \sum_{n=1}^{\infty} \left\|z_{k(n)}\right\|^{-1} |z_n(x)| < \infty$.

$3 \Rightarrow 4$. In the proof $3 \Rightarrow 2$ we may suppose that $\|y_n\| > n$ and we get 4. □

More than strongly normal sequence.

Proposition

ℓ_1 is isomorphic to E/F iff $\exists (y_n^* \in S(E^*))_n$ such that $\{x \in X : \sum_{n=1}^{\infty} |y_n^* x| < \infty\} = E$ iff E has a complemented copy of ℓ_1.

Proof.

If $Q : E \to \ell_1$ defines the quotient and $Q^*(e_n) = x_n^*$ then

- $\sum_{n=1}^{\infty} x_n^*$ is weakly* unconditionally Cauchy and
- $\inf_{n\in\mathbb{N}} \|x_n^*\| > 0$. Hence $(y_n^* := \|x_n^*\|^{-1} x_n^*)_n$ is as required.

Conversely, the sequence $\{y_n^*\}$ defines a weak* Cauchy series $\sum_{n=1}^{\infty} y_n^*$ which does not converge in E^*.
Then $\ell_\infty \hookrightarrow E^*$, hence E has a complemented copy of ℓ_1. □

S-SR theorem for 'small' Banach spaces.

Definition

Density character is the smallest cardinal of the dense subsets.

Definition

The *bounding cardinal* $\mathfrak{b}$ is the minimum of cardinals of unbounded subsets of $\left(\mathbb{N}^{\mathbb{N}}, \leq^*\right)$.

$\leq^*$, *eventual dominance preorder*, i.e., $\alpha \leq^* \beta$ if the set $\{n \in \mathbb{N} : \alpha(n) > \beta(n)\}$ is finite. Hence

$$\mathfrak{b} := \inf\{|F| : F \subseteq \mathbb{N}^{\mathbb{N}}, \forall \alpha \in \mathbb{N}^{\mathbb{N}} \; \exists \beta \in F \;\; \textit{with} \; \alpha <^* \beta\}.$$

$\mathfrak{b}$ is a regular cardinal and $\aleph_0 < \mathfrak{b} \leq \mathfrak{c}$.
Martin's Axiom $\Rightarrow \mathfrak{b} = \mathfrak{c}$.

S-SR theorem for 'small' Banach spaces.

Corollary (Saxon–Sánchez Ruiz)

If $\aleph_0 \leq d(X) < \mathfrak{b}$ *then* E *has a separable quotient.*

Proof using strongly normal sequences.

Let $\overline{D} = E$ and $|D| < \mathfrak{b}$. and let $(y_n^*)_n$ be a normal sequence in X^*.

For $x \in D$ choose $\alpha_x = (\alpha_x(n))_n \in \mathbb{N}^{\mathbb{N}} : |y_k^* x| < 2^{-n}$, $\forall k \geq \alpha_x(n)$, for each $n \in \mathbb{N}$.

Then $\sum_n |y^*_{\beta(n)} x| < \infty$ if $\alpha_x \leq^* \beta$.

$\{\alpha_x : x \in D\}$ is not unbounded. $\exists \gamma \in \mathbb{N}^{\mathbb{N}}$ with $\alpha_x \leq^* \gamma$, $\forall x \in D$.

Then

$$X = \overline{D} \subset \overline{\{x \in X : \sum_{n=1}^{\infty} |y_n^* x| < \infty\}} \subset X.$$

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