Journal of Convex Analysis
Volume 22 (2015), No. 4, 905–915

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Characterizing P-spaces X in Terms of $C_p(X)$*

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Received: March 3, 2014
Revised manuscript received: July 7, 2014
Accepted: September 3, 2014

Dual weak barrelledness led us to prove *X is a P-space if and only if every pointwise eventually zero sequence in $C_p(X)$ is summable*, and other better known characterizations. Novel ones recall utility functions from economics and Arkhangel'skii's (strict) τ-continuity. Mackey $\aleph_0$-barrelled duality leads us to prove *X is discrete if and only if every bounded σ-compact set in $C_p(X)$ is relatively compact.* We relax the σ-compact hypothesis of Velichko and the σ-countably compact hypothesis of Tkachuk/Shakhmatov: *X is a P-space if and only if $C_p(X)$ is σ-relatively sequentially complete.*

Keywords: P-spaces, $C_p(X)$, relatively compact, weak barrelledness

2010 Mathematics Subject Classification: 54C35, 46A08

1. Introduction

A completely regular Hausdorff (Tichonov) space X is a *P-space* if every countable intersection of open sets is open. $C_p(X)$ is the space $C(X)$ of continuous real-valued functions on X endowed with the topology of pointwise convergence. A sequence $\{f_n\}_n$ in $C_p(X)$ is *pointwise eventually zero* if for each point x in X, the set $\{n \in \mathbb{N} : f_n(x) \neq 0\}$ is finite. If we note that a locally convex space E is primitive if and only if each pointwise eventually zero sequence in $(E', \sigma(E', E))$

*Ferrando and Kąkol supported by the Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2013/058; Kąkol, also by the National Center of Science, Poland, grant no. N N201 605340.

ISSN 0944-6532 / $ 2.50

is summable, our [5, Theorem 6] translates into a natural scale of characterizations (*1*)–(*6*), to which (*7*) is addended:

Theorem 1.1. *The following seven statements are equivalent.*

(*1*) *X is a P-space.*
(*2*) *Every bounded sequence in* $C_p(X)$ *is relatively compact.*
(*3*) *Every bounded set in* $C_p(X)$ *is relatively countably compact.*
(*4*) $C_p(X)$ *is sequentially complete.* (Buchwalter/Schmets)
(*5*) $C_p(X)$ *is locally complete.*
(*6*) *Every pointwise eventually zero sequence in* $C_p(X)$ *is summable.*
(*7*) *Every uniformly bounded eventually zero sequence in* $C_p(X)$ *is summable.*

Proof. (*1*) ⇒ (*2*). Let $\{f_n\}_n$ be a bounded sequence in $C_p(X)$ with closure K in the product space $\mathbb{R}^X$. By the Tichonov theorem, K is compact; we must show that $K \subset C(X)$. For each $x \in X$ we set

$$N(x) = \bigcap_{m,n\in\mathbb{N}} \{z \in X : |f_n(z) - f_n(x)| < 1/m\},$$

a neighborhood of x, by (*1*), on which each f_n has the constant value $f_n(x)$. If $h \in \mathbb{R}^X$ and $h(y) \neq h(x)$ for some y in $N(x)$, then, clearly, x, y and $\varepsilon := |h(y) - h(x)|/2$ determine an open neighborhood of h in $\mathbb{R}^X$ that misses $\{f_n\}_n$. Therefore every h in K is constant on each $N(x)$, and thus continuous at x; i.e., $K \subset C(X)$.

(*2*) ⇒ (*3*) ⇒ (*4*) ⇒ (*5*). Trivially.

(*5*) ⇒ (*6*). If $\{f_n\}_n$ is a pointwise eventually zero sequence in $C_p(X)$, then $\{2^n f_n\}_n$ is also, and thus is bounded. By (*5*), the series $\sum_n 2^{-n}(2^n f_n) = \sum_n f_n$ converges in $C_p(X)$ [14, 2.1].

(*6*) ⇒ (*7*). Trivially.

(*7*) ⇒ (*1*). We must show that the intersection U of any decreasing sequence $\{U_n\}_n$ of neighborhoods of a point y in X is, itself, a neighborhood of y. We can inductively choose a decreasing sequence $\{V_n\}_n$ of neighborhoods of y and a sequence $\{h_n\}_n \subset C_p(X)$ such that, for all $n \in \mathbb{N}$, we have $|h_n| \leq 1$, $V_n \subset U_n$, $h_n(y) = 1$, h_n vanishes off V_n, and $|h_k(x) - 1| < 1/n$ for all $x \in V_{n+1}$ when $k \leq n$.

Clearly, the pointwise limit h of $\{h_n\}_n$ is identically 1 on $V := \bigcap_k V_k$ and vanishes elsewhere. Moreover, each $h_n - h_{n+1}$ vanishes on V, and if x is not in some V_k, then $(h_n - h_{n+1})(x) = 0 - 0 = 0$ for all $n > k$. Thus $\{h_n - h_{n+1}\}_n$ is eventually zero at all points of X, and $|h_n - h_{n+1}| \leq 2$ for each n. The pointwise sum $h_1 - h$ of the telescoping series $\sum_n (h_n - h_{n+1})$ is in $C(X)$ by (*7*), and so, then, is h. Therefore $h^{-1}((1/2, 3/2)) = V$ is a neighborhood of y contained in U. □

Besides [(*1*) ⇔ (*4*)], Buchwalter/Schmets proved $C_p(X)$ *is quasicomplete if and only if X is discrete.* Since non-discrete P-spaces exist, between (*1*) and (*2*) we <u>cannot</u> insert *(1⁺)* *Every bounded set in $C_p(X)$ is relatively compact.* Two acceptable conditions:

(1) Every sequence in $C(X)$ is equicontinuous (Buchwalter/Schmets);
(2) The closure in $\mathbb{R}^X$ of each countable set in $C(X)$ is contained in $C(X)$.

We labor in weak barrelledness conditions mirrored by *(2)*–*(6)*, from ℓ^∞-*barrelled* to *primitive* (see [5, Theorem 6], [12, Theorem 2]). With an arbitrary locally convex space E replacing $C_p(X)$ and with *(6)* suitably interpreted, condition (n) is strictly stronger than condition $(n+1)$ for $n = 2, \dots, 5$, as [15] shows, making *(2)* very much stronger than *(6)*. Therefore to us, the equivalence of *(2)* and *(6)* is a rare and wonderful thing, dual to the (infrequent) equivalence of ℓ^∞-*barrelled* and *primitive*, a serendipitous union of topology and analysis.

However, it turns out that the equivalence of *(1)*–*(7)* is not really new. Indeed, reading [2, 2.2] more carefully, one sees that Buchwalter and Schmets, with older terminology, explicitly state and prove the equivalence of *(1)*, (1), *(4)*, and *(5)*. Moreover, their *Preuve* shows [*(1)* ⇒ (1) ⇒ *(4)* ⇒ *(5)* ⇒ *(6 ∧ 7)* ⇒ *(1′)*], where *(1′)* is the zero-set condition [7] famously equivalent to *(1)*. Thanks to Profs. Gruenhage and Tkachuk, we found [*(1)* ⇔ (2)] and a paraphrase of [*(1)* ⇔ *(3)*] in [16, S. 355, S. 397] and [11, p. 102]. But [(1) ⇒ (2)] is elemental, corollary to [10, 8.12]; Tichonov's theorem makes [(2) ⇒ *(2)*] immediate; and [*(2)* ⇒ *(3)* ⇒ *(4)*] is completely trivial.

Thus we must cede Theorem 1.1 to Buchwalter and Schmets. Nevertheless, our proof of [*(7)* ⇒ *(1)*] is decidedly simpler and more direct than their proof of [*(5)* ⇒ *(1′)*], about which they wrote *C'est là le point délicat* [2, 2.2].

2. More weak barrelledness motivations

Ferrando and Sánchez Ruiz [6] articulated what is now known as the class of inductive locally convex spaces, and showed that a Mackey space is inductive if (and only if) it is primitive. This result, the definitions of *inductive* and *primitive* (omitted here), and [5, Theorem 6] motivate characterizations, below, previously unknown to us.

A Tichonov space X is the *inductive limit* of an increasing sequence $\{X_n\}_n$ of subsets covering X if subsets S must be open in X whenever each $S \bigcap X_n$ is relatively open in X_n. In the context of utility functions, mathematical economists call X the *inclusive* (or *inclusion*) *inductive limit* of $\{X_n\}_n$ [3].

If $\mathcal{M} = \{X_\alpha\}_{\alpha \in A}$ is a family of subsets covering X, we say that a real-valued function f on X is [*$\mathcal{M}$-continuous*] ⟨*strictly $\mathcal{M}$-continuous*⟩ if, for each $\alpha \in A$, [the restriction $f|_{X_\alpha}$ is continuous] ⟨there is some $g_\alpha \in C(X)$ with $f|_{X_\alpha} = g_\alpha|_{X_\alpha}$⟩. Our definition of ⟨strict⟩ $\mathcal{M}$-continuity coincides with Arkhangel'skii's ⟨strict⟩ τ-continuity (τ an infinite cardinal) [1] when $\mathcal{M}$ consists of all subsets of X of cardinality τ.

In [5, Theorem 6] we proved [$(a) \Leftrightarrow (d)$] of Theorem 2.1, below. Half with some effort. The other half with ease: $[(a) \Rightarrow (1) \Rightarrow (4) \Rightarrow (d)]$ is clear. Part (c) had eluded us altogether. The whole follows very easily given $[(6) \Rightarrow (1)]$, *le point délicat.*

Theorem 2.1. *The following four statements are equivalent.*

(a) *X is a P-space.*
(b) *X is the inductive limit of each increasing sequence $\mathcal{M} = \{X_n\}_n$ of subsets covering X.*
(c) *For each $\mathcal{M}$ as in (b), every $\mathcal{M}$-continuous function is continuous.*
(d) *For each $\mathcal{M}$ as in (b), every strictly $\mathcal{M}$-continuous function is continuous.*

Proof. $(a) \Rightarrow (b)$. Suppose each $S \bigcap X_n$ is relatively open and $x \in S \bigcap X_m$ for some m. For each $k \in \mathbb{N}$ let $\mathcal{O}_k$ be an open set in X such that $\mathcal{O}_k \bigcap X_{m+k} = S \bigcap X_{m+k}$. By (a), $\bigcap_k \mathcal{O}_k$ is a neighborhood of x, clearly contained in S. Therefore S is open in X.

$(b) \Rightarrow (c)$. If f is $\mathcal{M}$-continuous and U is open in $\mathbb{R}$, then $f^{-1}(U) \bigcap X_n = (f|_{X_n})^{-1}(U)$ is open in X_n for each n. Hence (b) implies $f^{-1}(U)$ is open in X.

$(c) \Rightarrow (d)$. Trivially.

$(d) \Rightarrow (a)$. We demonstrate (6). Let $\{f_k\}_k$ be a pointwise eventually zero sequence in $C_p(X)$. Set $X_n = \bigcap_{k>n} f_k^{-1}(0)$ for each $n \in \mathbb{N}$, and let f be the pointwise limit of the series $\sum_k f_k$. Clearly, $\mathcal{M} := \{X_n\}_n$ is as in (b), and $f|_{X_n} = \left(\sum_{k \leq n} f_k\right)|_{X_n}$ for each n, so that f is strictly $\mathcal{M}$-continuous. By (d), f is continuous, and (6) holds. □

Corollary 2.2. *X is a P-space if and only if for every increasing sequence $\mathcal{M} = \{X_n\}_n$ of subsets covering X, the canonical embedding of $C_p(X)$ into the product $\prod_{n=1}^{\infty} C_p(X_n)$ has closed range.*

Proof. *Necessity:* Given $M = \{X_n\}_n$ as above, the canonical embedding T is defined by $T(f) = (f|_{X_1}, f|_{X_2}, \dots)$ for each $f \in C(X)$. Assume X is a P-space, so that (c) holds. Let $u = (g_1, g_2, \dots)$ be a point in the product space. If there exist $\varepsilon > 0$, $n < m$ and $x \in X_n$ such that $|g_n(x) - g_m(x)| = \varepsilon$, then an open neighborhood of u misses each $T(f)$, since $|g_n(x) - f(x)|, |g_m(x) - f(x)| < \varepsilon/2$ would imply $|g_n(x) - g_m(x)| < \varepsilon$. Thus if u is in the closure of $T(C_p(X))$, then $g_m|_{X_n} = g_n$ for all $n < m$. We assume this holds, so that a function f on X is defined by positing $f|_{X_n} = g_n$ for all n. Now (c) implies f is continuous and $u = T(f)$, as desired.

Sufficiency: We show the closed range condition implies (d). If $\mathcal{M} = \{X_n\}_n$ determines a canonical embedding T and f is strictly $\mathcal{M}$-continuous, there exists $\{h_n\}_n \subset C(X)$ such that each $f|_{X_n} = h_n|_{X_n}$. Because $\{X_n\}_n$ is increasing,

$$T(h_n) = \left(f|_{X_1}, \dots, f|_{X_n}, h_n|_{X_{n+1}}, h_n|_{X_{n+2}}, \dots\right).$$

As $n \to \infty$, the images converge in the product space to $(f|_{X_1}, \dots, f|_{X_k}, \dots)$, which equals $T(g)$ for some g in $C(X)$ since the range of T is assumed closed. Therefore $f|_{X_k} = g|_{X_k}$ for all k, so that $f = g$ is continuous, and (d) holds. $\square$

The $C_p(X)$ view of weak barrelledness [9] partitions the sixteen distinct properties listed in [13] into just two equivalence classes represented by *barrelled* and *inductive*. The dual of $C_p(X)$ is denoted by $L(X)$, and $L_m(X)$ denotes $L(X)$ given its Mackey topology $\mu(L(X), C(X))$. Following Buchwalter and Schmets, one easily sees, as in [5], that $L_m(X)$ *is Baire-like if and only if* X *is finite; is barrelled if and only if* X *is discrete; is* ℓ^∞*-barrelled if and only if* X *is a P-space; and has none of the sixteen properties if* X *is not a P-space.* In [13], Saxon completed the $L_m(X)$ view of weak barrelledness: $L_m(X)$ *spaces trichotomize the sixteen properties into three equivalence classes represented by* Baire-like, barrelled, *and* ℓ^∞-barrelled, answering the question in [5] by proving that $L_m(X)$ *is barrelled if (and only if) it is* $\aleph_0$*-barrelled.* His statement and proof involve convexity and the Saxon-Tweddle splitting theorem for $\aleph_0$-barrelled Mackey spaces, unlike the direct topological approach below.

Theorem 2.3. *The following three statements are equivalent.*

- (i) *X is discrete.*
- (ii) *Every bounded set in* $C_p(X)$ *is relatively compact* (Buchwalter/Schmets).
- (iii) *Every bounded* σ*-compact set in* $C_p(X)$ *is relatively compact.*

Proof. If (i) holds, then $C_p(X) = \mathbb{R}^X$ and (ii) holds by the Tichonov theorem. Obviously, (ii) implies (iii).

Now assume (iii) holds and X is not discrete. To complete the proof, we argue to a contradiction. Since singleton sets are compact, $(iii) \Rightarrow (2) \Rightarrow (1)$, so X is a P-space and therefore its topology has a base of open-and-closed sets [7, 4K 8]; i.e., a base of (open) sets U whose characteristic functions $\mathbf{1}_U$ are continuous. Let y be a non-isolated point in X. Zorn's lemma produces a maximal family $\mathcal{F}$ of non-empty pairwise disjoint open sets U which miss y and have continuous characteristic functions $\mathbf{1}_U$ $(U \in \mathcal{F})$, where $\mathbf{1}_U$ is defined to be identically 1 on U and 0 on $X \backslash U$. Since $\mathcal{F}$ is maximal and y is not isolated, every neighborhood of y meets $\bigcup \mathcal{F}$. Thus the characteristic function χ of $\bigcup \mathcal{F}$ is not continuous at y.

For each $n \in \mathbb{N}$ let

$$K_n = \{\mathbf{1}_W : W \text{ is the union of } \leq n \text{ members of } \mathcal{F}\}.$$

Each K_n is in $C_p(X)$, closed and bounded in $\mathbb{R}^X$, hence compact in $C_p(X)$, so the pointwise bounded countable union $\bigcup_n K_n$ is relatively compact in $C_p(X)$ by (iii). Therefore the closure K in $\mathbb{R}^X$ of the union coincides with the closure in $C_p(X)$. But, transparently, χ is in K, thus is continuous, a contradiction. $\square$

Remarkably, all these results hide the weak barrelledness that led us to them.

3. Applications

A sequence $\{f_n\}_n$ in $C(X)$ is *pointwise eventually constant* if for each $x \in X$ there exists a constant $f(x)$ such that $f_n(x) = f(x)$ for all but finitely many n. Now $\{f_n\}_n$ is pointwise eventually [zero] ⟨constant⟩ if and only if [the sequence $\{\sum_{k\leq n} f_k\}_n$ of partial sums is pointwise eventually constant] ⟨the sequence $\{f_n - f_{n+1}\}_n$ is eventually zero⟩, and thus (*6*) is equivalent to

(*6′*) *Every pointwise eventually constant sequence converges in* $C_p(X)$.

Similarly, (*7*) is equivalent to

(*7′*) *Every uniformly bounded pointwise eventually constant sequence converges in* $C_p(X)$.

Conditions (*6′*), (*7′*) appear to dramatically relax (*4*). One may also observe that a function f in $\mathbb{R}^X$ is strictly $\mathcal{M}$-continuous for some $\mathcal{M}$ as in (*b*) if and only if f is the limit (in $\mathbb{R}^X$) of a pointwise eventually constant sequence in $C_p(X)$, and thus $[(a) \Leftrightarrow (d)]$ follows immediately from $[(1) \Leftrightarrow (6')]$. A subset A of $C_p(X)$ is *eventually complete* if every pointwise eventually constant sequence in A has a (pointwise) limit in A. A subset A of a topological vector space E is *relatively sequentially complete* if every Cauchy sequence in A converges in E; thus every relatively countably compact set is relatively sequentially complete. If E is a countable union of relatively sequentially complete sets, it is *σ-relatively sequentially complete.* Let $C^b(X)$ denote the linear subspace of functions in $C(X)$ having bounded range. With the uniform norm, $C^b(X)$ becomes a Banach space, denoted $C^b_u(X)$, having unit ball $\{f \in C(X) : |f(x)| \leq 1 \text{ for all } x \in X\}$.

Theorem 3.1. *The following three statements are equivalent.*

(*A*) *X is a P-space.*
(*B*) *$C_p(X)$ is σ-relatively sequentially complete.*
(*C*) *$C_p(X)$ is a countable union of closed eventually complete sets.*

Proof. From $[(1) \Rightarrow (4), (6')]$ immediately follows $[(A) \Rightarrow (B), (C)]$.

$(B) \Rightarrow (A)$. We demonstrate (*7′*), assuming $C_p(X)$ is covered by relatively sequentially complete subsets K_n $(n \in \mathbb{N})$. Let $\{f_n\}_n$ be a uniformly bounded eventually constant sequence in $C(X)$ with limit f in $\mathbb{R}^X$. The Baire category theorem provides $q \in \mathbb{N}$ such that, in the Banach space $C^b_u(X)$, the closure B of $A := K_q \bigcap C^b_u(X)$ has an interior point h. Thus there is some $\varepsilon > 0$ such that

$$h + \varepsilon D \subset B,$$

where D is the unit ball in $C^b_u(X)$. Since $\{f_n\}_n$ is uniformly bounded, there is some $\delta > 0$ such that $\delta f_n \in \varepsilon D$ for all n. Therefore $\{h + \delta f_n\}_n$ is a pointwise eventually constant sequence in B whose limit in $\mathbb{R}^X$ is $h + \delta f$. Since A is norm-dense in B, for each n there is some $g_n \in K_q$ with $|g_n(x) - (h + \delta f_n)(x)| < 1/n$ for all $x \in X$. One readily sees that $\{g_n\}_n$ converges pointwisely to $h + \delta f$, which must be in $C(X)$ since K_q is relatively sequentially complete. Therefore f is in $C(X)$; i.e., $(7')$ holds.

$(C) \Rightarrow (A)$. To demonstrate $(7')$ again under the assumption that the covering sets K_n are closed and eventually complete, we repeat the first part of the above proof, quickly obtaining $A = B$ and $h + \delta f \in K_q$ without need of the intermediate g_n. □

[Velichko] ⟨Tkachuk and Shakhmatov⟩ proved that if $C_p(X)$ is [σ-compact] ⟨σ-countably compact⟩, then X is finite. (The converses are trivial.) Via angelicity, Ferrando/Kąkol equated both hypotheses to the formally weaker condition that $C_p(X)$ be σ-relatively countably compact [4], [8, Proposition 9.6, Theorem 9.13]. We relax the hypothesis yet further, requiring only that $C_p(X)$ be the countable union of (pointwise) bounded relatively sequentially complete sets K_n ($n \in \mathbb{N}$). The key step: X is a P-space [$(B) \Rightarrow (A)$]. If we now assume $\{x_n\}_n$ is a sequence of distinct points in X then, as K_n is pointwise bounded, there exists a positive number M_n such that for each $n \in \mathbb{N}$,

$$|f(x_n)| < M_n \text{ for every } f \in K_n.$$

One well knows that, since X is a P-space, there exists $h \in C(X)$ with $h(x_n) = M_n$ for each n [8, Lemma 9.5]. But this means that $h \notin \bigcup_n K_n = C(X)$, a contradiction, and we must conclude that X is finite. We have proved

Corollary 3.2. *If $C_p(X)$ is a countable union of bounded relatively sequentially complete sets, then X is finite.*

Again, the converse is trivial. Unlike ours, Arkhangel'skii's more complicated treatment centers on a beautiful result that, as he notes, does not have a converse, to wit: *If Y is a dense subset of X and $C_p(Y|X)$ is σ-countably compact, then Y is a P-space and X is pseudocompact* [1, I.2.2]. (In a slight abuse of notation, $C_p(Y|X)$ is that subspace of $C_p(Y)$ consisting of all functions in $C(Y)$ that can be extended to a function in $C(X)$.) We generalize this result *a la* Theorem 3.1:

Theorem 3.3. *Let Y be a nonempty subset of the Tichonov space X. Then Y is a P-space if $C_p(Y|X)$ is a countable union of sets K_n that are (α) relatively sequentially complete, or (β) closed and eventually complete. In either case (α) or (β), if Y is dense in X and each K_n is bounded, then X is pseudocompact.*

Proof. The last statement follows from the fact that, in any case, if Y is dense and each K_n is bounded, there cannot exist a discrete sequence $\{U_n\}_n$ of nonempty open sets in X. Otherwise, we could choose $x_n \in U_n \bigcap Y$ and $M_n > 0$ and $h \in C(X)$ as above with $h|_Y \notin \bigcup_n K_n = C_p(Y|X)$, a contradiction. This is just Arkhangel'skii's argument in the first few lines of the proof of [1, I.2.2].

Suppose $y \in Y$ and U is the intersection of any decreasing sequence $\{U_n\}_n$ of neighborhoods in X of y. We must show that $U \bigcap Y$ is a relative neighborhood of y in both cases (α) and (β). As in Theorem 1.1, we inductively choose a decreasing sequence $\{V_n\}_n$ of neighborhoods of y in X and a sequence $\{h_n\}_n \subset C(X)$ such that, for all $n \in \mathbb{N}$, we have $|h_n| \leq 1$, $V_n \subset U_n$, $h_n(y) = 1$, h_n vanishes off V_n, and $|h_k(x) - 1| < 1/n$ for all $x \in V_{n+1}$ when $k \leq n$.

Now $\{h_n\}_n$ is a uniformly bounded pointwise eventually constant sequence in $C(X)$ whose limit h in $\mathbb{R}^X$ is identically 1 on $V := \bigcap_k V_k$ and vanishes elsewhere. For each n let $J_n = \{f \in C(X) : f|_Y \in K_n\}$. Since $\{K_n\}_n$ covers $C_p(Y|X)$, it is clear that $\{J_n\}_n$ covers $C(X)$. The Baire category theorem provides $q \in \mathbb{N}$ such that, in the Banach space $C_u^b(X)$, the closure B of $A := J_q \bigcap C^b(X)$ has an interior point g. Thus there exists a fixed $\delta > 0$ such that each $g + \delta h_n \in B$.

In case (β), continuity of the restriction map $R : C_p(X) \to C_p(Y|X)$ ensures that $R^{-1}(K_q) = J_q$ is closed in $C_p(X)$, so that $B = A$ and

$$K_q = R(J_q) \supset R(A) = R(B) \supset \{(g + \delta h_n)|_Y\}_n .$$

The sequence $\{(g + \delta h_n)|_Y\}_n$ must converge in the eventually complete K_q, necessarily to $(g + \delta h)|_Y = g|_Y + \delta h|_Y$, so that $h|_Y \in C_p(Y|X)$. By the definition of $C_p(Y|X)$ there exists $f \in C(X)$ such that $f|_Y = h|_Y$, and

$$V \bigcap Y = h^{-1}((1/2, 3/2)) \bigcap Y = f^{-1}((1/2, 3/2)) \bigcap Y$$

is a relative neighborhood of y contained in $U \bigcap Y$, as desired.

Assume (α) holds. The proof of Theorem 3.1 shows that $(g + \delta h)|_Y$ is the limit in $\mathbb{R}^Y$ of a sequence in K_q and so is in $C_p(Y|X)$. Thus $h|_Y \in C_p(Y|X)$, etc. □

Corollary 3.2 follows immediately from Theorem 3.3: With $Y = X$, invoke the standard fact that pseudocompact P-spaces are finite.

Although case (α) relaxes Arkhangel'skii's hypothesis, neither (α) nor (β) admits the converse.

Example 3.4. Let X be the one-point compactification of $\mathbb{N}$; i.e., $X = \mathbb{N} \bigcup \{\omega_0\}$, $Y := \mathbb{N}$ is discrete, and $\{X \setminus \{1, \ldots, n\} : n \in \mathbb{N}\}$ is a base of neighborhoods of ω_0. Then Y is a P-space dense in X, but neither case (α) nor (β) holds.

Indeed, $C_p(Y|X)$ is just the Banach space c of convergent real sequences with the coarser topology of coordinate-wise convergence. Thus $C_p(Y|X)$ contains, for each $n \in \mathbb{N}$, the point f_n, where

$$f_n(i) = \begin{cases} 1 & \text{if } i \text{ is odd and } 1 \leq i \leq n; \\ 0 & \text{if } i \text{ is even and } 1 \leq i \leq n; \\ 0 & \text{if } i > n. \end{cases}$$

Each f_n is in the sup-norm unit ball of c. Assuming case (β), Baire provides $g \in c$ and $\delta > 0$ such that the eventually constant sequence $\{g + \delta f_n\}_n$ converges in $C_p(Y|X)$, necessarily to the point u whose i^{th} coordinate is $g(i) + \delta$ for i odd, $g(i)$ for i even. Since $g \in c$, it is clear that u is not in $c = C_p(Y|X)$, a contradiction. If we assume case (α), we obtain, as above, a possibly different sequence which still must converge in $C_p(Y|X)$ to some u of the previous form, still a contradiction. $\diamondsuit$

However, we may obtain converses by considering subsets between Y and X. Trivially, *X is a P-space if and only if every subset containing Y is a P-space.* Less trivially,

Theorem 3.5. *Let Y denote a fixed dense subset of the Tichonov space X and let Z denote sets of the form $Y \bigcup \{x\}$, where x is an arbitrary point in X. The following seven statements are equivalent.*

I. *X is a P-space.*
II. *Each Z (of the above form) is a P-space.*
III. *Each $C_p(Z|X)$ is sequentially complete.*
IV. *Each $C_p(Z|X)$ is σ-relatively sequentially complete.*
V. *Each $C_p(Z|X)$ is eventually complete.*
VI. *Each $C_p(Z|X)$ is a countable union of closed eventually complete sets.*
VII. *$C_p(Y|X)$ is eventually complete and each sequence in $C(X)$ which is eventually zero at the points of Y is also eventually zero at the points of X.*

Proof. *I* $\Rightarrow$ *II*. Trivially; every subset of a P-space is a P-space.

II $\Rightarrow$ *I*. Let f be the limit in $\mathbb{R}^X$ of a sequence in $C(X)$. From [(4) $\Leftrightarrow$ (1)], and assuming *II*, we have each $f|_Z \in C(Z)$, and it suffices to show that $f \in C(X)$. Let $x_0 \in X$ and $\varepsilon > 0$ be given. Set $Z_0 = Y \bigcup \{x_0\}$. Since $f|_{Z_0} \in C(Z_0)$, in X there exists an open neighborhood U of x_0 such that $|f(y) - f(x_0)| < \varepsilon/2$ for all $y \in U \bigcap Y$. Let $u_1 \in U$ and set $Z_1 = Y \bigcup \{u_1\}$. Since $f|_{Z_1} \in C(Z_1)$, in X there is an open neighborhood V of u_1 such that $|f(y) - f(u_1)| < \varepsilon/2$ for all $y \in V \bigcap Y$. By density there exists $y_0 \in U \bigcap V \bigcap Y$. Therefore

$$|f(u_1) - f(x_0)| \leq |f(u_1) - f(y_0)| + |f(y_0) - f(x_0)| < \varepsilon.$$

Since $x_0 \in X$ and $u_1 \in U$ were arbitrarily chosen, we have $f \in C(X)$.

I $\Rightarrow$ *III*. Given Z and a sequence $\{f_n\}_n \subset C(X)$ such that $\{f_n(z)\}_n$ is a real Cauchy sequence for every $z \in Z$, it suffices to show that $\{f_n(x)\}_n$ is Cauchy for each $x \in X$, since $C_p(X)$ is sequentially complete by *I*. Suppose to the contrary, that $\{f_n(x_0)\}_n$ is not Cauchy for some $x_0 \in X$. Then there exists $\varepsilon > 0$ and a subsequence $\{g_n\}_n$ of $\{f_n\}_n$ such that $|g_n(x_0) - g_{n+1}(x_0)| \geq \varepsilon$ for all $n \in \mathbb{N}$. Set

$$h_n = \frac{g_n - g_{n+1}}{g_n(x_0) - g_{n+1}(x_0)}.$$

Clearly, $h_n(x_0) = 1$ for all n, and $\lim_n h_n(y) = 0$ for all $y \in Y \subset Z$. Since X is a P-space, so is the subset $Z_0 := Y \bigcup \{x_0\}$. By $[(1) \Rightarrow (4)]$, the Cauchy sequence $\{h_n|_{Z_0}\}_n$ must converge in $C_p(Z_0)$ to the function h, where $h(x_0) = 1$ and $h(Y) = \{0\}$, contradicting continuity of h.

Obviously, *III* $\Rightarrow$ *IV*, and Theorem 3.3 ensures *IV* $\Rightarrow$ *II*. Thus *I–IV* are equivalent. Also, *III* $\Rightarrow$ *V* $\Rightarrow$ *VI*, and Theorem 3.3 ensures *VI* $\Rightarrow$ *II*. Thus *I–VI* are equivalent. We conclude the proof by showing *V* $\Leftrightarrow$ *VII*.

VII $\Rightarrow$ *V*. Let $\{f_n\}_n$ be a sequence in $C(X)$ which is eventually constant at the points of a given Z. Now $\{f_n - f_{n+1}\}_n$ is eventually zero at the points of Z and thus, by *VII*, at the points of X, so that $\{f_n\}_n$ is eventually constant at the points of X and has a limit f in $\mathbb{R}^X$. It is enough to show that $f \in C(X)$. Since $C_p(Y|X)$ is assumed to be eventually complete, there exists $g \in C(X)$ with $g|_Y = f|_Y$. In fact, $\{g - f_n\}_n$ is eventually zero at the points of Y, hence at the points of X. Therefore $f = g \in C(X)$; i.e., V holds.

V $\Rightarrow$ *VII*. The first part of *VII* follows trivially. Suppose $\{f_n\}_n$ is a sequence in $C(X)$ which is eventually zero at the points of Y but not at some point $x \in X$. There is a subsequence $\{h_n\}_n$ such that $h_n(x) \neq 0$ for each $n \in \mathbb{N}$. If $g_n = h_n / h_n(x)$ and $Z = Y \bigcup \{x\}$, then $\{g_n|_Z\}_n$ is a pointwise eventually constant sequence in $C_p(Z|X)$ whose pointwise limit g is in $C(Z)$ by V. But $g(x) = 1$ while g vanishes on the dense subset Y of Z, contradicting continuity; *VII* follows. □

In our Example, the fact that $X \backslash Y$ is a singleton now seems prototypical. Also, we have the characterization that Arkhangel'skii may have sought:

Corollary 3.6. *X is a P-space if and only if $C_p(Y|X)$ is σ-relatively sequentially complete for each dense subset Y of X.*

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