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A Counterexample about the Regularity of the Solutions of the Hyperbolic Heat Equation

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Abstract. We give an example of a fundamental solution of the Neumann problem for the hyperbolic heat equation in a disk D which is not in $L^2(D\times]0,T[)$ whatever be $T>0$.

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1. Introduction

Due to the importance of the hyperbolic heat equation in modern engineering processes of application of short laser pulses of high intensity, we have made in [6] a detailed mathematical study of the regularity of solutions of this equation. In particular, it was shown in $\big[$[6], theorem 13 and corollary 15, part b$\big]$ that, given a bounded open set $D\subset\mathbb{R}^n$, $n\in\mathbb{N}$ locally lying on the same side of its $\mathcal{C}^\infty$-class boundary ∂D and $(\mathbf{x}_0,t_0)\in D\times\mathbb{R}$, there is a unique function $\widetilde{G}(\mathbf{x},t)$ such that

$$\widetilde{\mathcal{A}}(\widetilde{G}):=\Delta\widetilde{G}-\frac{\partial^2\widetilde{G}}{\partial t^2}-2\frac{\partial\widetilde{G}}{\partial t}=-\delta(\mathbf{x}-\mathbf{x}_0)\delta(t-t_0)\quad\text{in }\ D\times\mathbb{R},\tag{1.1}$$

$$\nabla\widetilde{G}(\mathbf{x},t)\ =\ 0\quad\text{in}\quad\partial D\times\mathbb{R},\qquad\exists\, v\in\mathbb{R}\ /\ \widetilde{G}(\mathbf{x},t)=0\ \ \text{in}\ \ D\times]-\infty,v]\tag{1.2}$$

and $\widetilde{G}\in H^\eta\big(]-T,T[,H^{-(\varepsilon+2\xi)}(D)\big)$ for every $T>0,\varepsilon>0$ and $0<\eta<\xi<\frac{1}{2}$. This result means that, almost everywhere in $]-T,T[$, the function $\widetilde{G}(\mathbf{x},t)$ of $\mathbf{x}\in D$ is very close to be a function in $L^2(D)$.

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The main purpose of this paper is to show that this result cannot be improved, *showing that, if* D *is the open circle* $\{\mathbf{x} = (x, y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$, $\mathbf{x}_0 = (x_0, y_0) = (0, 0)$ *and* $T > 0$, *the solution* $\widetilde{G}(\mathbf{x}, t) := \widetilde{G}(x, y, t|0, 0, t_0)$ *of the* Eqs. (1.1) *and* (1.2) *does not belong to* $L^2(D\times]-T, T[)$ *because the "spatial" function* $\widetilde{G}(\mathbf{x}, t) \notin L^2(D)$ *for every* $t \in]0, T[$ *except in a numerable set of values of* t. To this task, topological considerations will be fundamental to set our key theoretical results before to deal with merely computational aspects.

The notation is standard in general. i will always denote the imaginary unit and $H(t)$ will be the Heaviside function. Given a Banach space X, we shall write B_X for its open unit ball. We refer the reader to [5] and [9] for the interpolation spaces theory and its applications to the definitions and properties of Sobolev spaces $H^s(\Sigma), H_0^s(\Sigma)$ and $H^{-s}(\Sigma), s \geq 0$. In particular, if $\Sigma \subset \mathbf{R}^2$ is an open bounded set with the cone property or $\mathbb{R}^2$ itself and $0 < \varepsilon \neq \frac{1}{2} < 1$, by $\big[$[9], section 4.3.1, theorem 2, section 2.4.2, theorem 2 and remark 4 and section 4.2.3, remark 5$\big]$ and $\big[$[5], chapter 1, theorem 11.6$\big]$, one has the isomorphisms

$$H^{-(1+\varepsilon)}(\Sigma) \approx \left[H^{-1}(\Sigma), H^{-2}(\Sigma)\right]_\varepsilon, \quad \left(H^{1+\varepsilon}(\Sigma)\right)' \approx \left[\left(H^1(\Sigma)\right)', \left(H^2(\Sigma)\right)'\right]_\varepsilon. \tag{1.3}$$

For this reason, ε will have the same meaning in all the paper, and to apply $\big[$[6], theorem 13 and corollary 15, part b$\big]$, η and ξ will be always real numbers such that $0 < \eta < \xi < \frac{1}{2}$. $\widetilde{Q} : (H^{1+\varepsilon}(\Sigma))' \longrightarrow H^{-(1+\varepsilon)}(\Sigma) = (H^{1+\varepsilon}(\Sigma))' / (H_0^{1+\varepsilon}(\Sigma))^\perp$ will denote the canonical quotient map.

Occasionally, we shall need the space

$$\Xi^r(\Sigma) = \left\{ f \in L^2(\Sigma) \;\middle|\; \|f\|_{\Xi^r(\Sigma)} = \left(\sum_{|\boldsymbol{\alpha}| \leq r} \left\| d(\mathbf{x})^{|\boldsymbol{\alpha}|} \, \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}) \right\|^2_{L^2(\Sigma)} \right)^{\frac{1}{2}} < \infty \right\}$$

where $d(\mathbf{x})$ denotes the distance from $\mathbf{x} := (x, y) \in \Sigma$ up to the boundary $\partial\Sigma$ and $r \in \mathbb{N} \cup \{0\}$. This definition is extended to the case $r \in]0, \infty[$ by complex interpolation setting $\Xi^r(\Sigma) = \left[\Xi^{k+1}(\Sigma), \Xi^k(\Sigma)\right]_{k+1-r}$ for $k < r < k+1, k \in \mathbb{N} \cup \{0\}$. One has $H^r(\Omega) \subset \Xi^r(\Omega)$ and $\mathcal{D}(\Sigma)$ is dense in $\Xi^r(\Sigma)$, $r \geq 0$ ([5], chapter 2, (6.21)). Then, defining $\Xi^{-r}(\Sigma) := (\Xi^r(\Sigma))'$ the inclusion $\Xi^{-r}(\Sigma) \subset \left(H^r(\Sigma)\right)'$ holds.

We shall identify every $f(u, v) \in \mathcal{D}(\Sigma)$ with its extension by 0 to $\mathbb{R}^2$. An easy consequence from (1.3) and the definition of the complex interpolation method is that, given $f(u, v) \in \mathcal{D}(\Sigma)$, there is $C_\Sigma > 0$ independent of $f(u, v)$ such that

$$\left\|e^{u+v} f(u, v)\right\|_{H_0^{1+\varepsilon}(\mathbb{R}^2)} \leq C_\Sigma \left\|f(u, v)\right\|_{H_0^{1+\varepsilon}(\mathbb{R}^2)}. \tag{1.4}$$

The Fourier transform of a distribution $T(u) \in \mathcal{S}'(\mathbb{R})$ will be denoted by $\mathcal{F}_u\big[T\big](\nu)$ and, analogously, by $\mathcal{F}_{u,v}\big[T\big](\nu_1, \nu_2)$ if $T(u, v) \in \mathcal{S}'(\mathbb{R}^2)$.

The use of polar coordinates $x = \rho \cos\omega, y = \rho \sin\omega$ will be central in the paper. If $\Omega :=]0, 1[\times]0, 2\,\pi[$, by change in variables, functions defined on D are naturally related to functions defined on Ω. In the sequel, this relation will be expressed writing f for functions defined on Ω and $\widetilde{f}$ for the corresponding function defined on D. More formally, every function $\widetilde{f}(x,y,t)$ defined for $(x,y,t)\in D\times\mathbb{R}$ generates a function $f(\rho,\omega,t):=\widetilde{f}(\rho\cos\omega,\rho\sin\omega,t)$ defined for $(\rho,\omega,t)\in\Omega\times\mathbb{R}$.

This relation (or change in variables) can be extended to distributions setting that every $\widetilde{T}\in\mathcal{D}'(D)$ generates $T\in\mathcal{D}'(\Omega)$ defined by the rule

$$\forall\varphi(\rho,\omega)\in\mathcal{D}(\Omega)\quad\big\langle T(\rho,\omega),\varphi(\rho,\omega)\big\rangle=\left\langle\widetilde{T}(x,y),\frac{\widetilde{\varphi}(x,y)}{\sqrt{x^2+y^2}}\right\rangle. \tag{1.5}$$

By the theorem of change in variables in a double integral, it follows that, if $\widetilde{T}_{\widetilde{f}}\in\mathcal{D}'(D)$ is the distribution associated to a function $\widetilde{f}$, then $T_{\widetilde{f}}(\rho,\omega)=f(\rho,\omega)$. In the same way,

$$\widetilde{T}(x,y)=\delta(x-x_0)\delta(y-y_0)\in\mathcal{D}'(D)\Longrightarrow T(\rho,\omega)=\frac{\delta(\rho-\rho_0)}{\rho}\delta(\omega-\omega_0), \tag{1.6}$$

where $\rho_0=\sqrt{x_0^2+y_0^2}$ and $\omega_0=Arg(x_0+i\,y_0)$. Moreover, it is straightforward to check that the expression in polar coordinates of the Laplacian $\Delta\widetilde{T}(x,y)$ has the same formal representation as if $\widetilde{T}$ would be an ordinary function.

For Bessel functions, [10] and the handbook [1] are our references. We use the standard notation $J_\nu(z),Y_\nu(z)$ for the ordinary Bessel functions of first and second kind and $I_\nu(z),K_\nu(z)$ for the modified Bessel functions of first an second kind and order $\nu\in[0,\infty[$ in each case. Using Bachmann and Landau's symbols, by inspection of $\big[$[1], 9.6.10 and 9.6.11$\big]$, it is clear that

$$\forall\,n\in\mathbb{N}\cup\{0\}\quad I_n(z)=\left(\frac{z}{2}\right)^n\left(\frac{1}{n!}+\mathfrak{A}_n(z)\right)\quad\text{with}\ \ \mathfrak{A}_n(z)=o(1)\quad\text{if}\ \ z\to 0, \tag{1.7}$$

$$K_0(z)=-I_0(z)\ \log\frac{z}{2}+\mathfrak{K}_0(z)\ \ \text{with}\ \ \mathfrak{K}_0(z)=1+o(1)\quad z\in\mathbb{C}\backslash]-\infty,0],\ \ z\to 0 \tag{1.8}$$

and

$$\forall\,n\in\mathbb{N}\quad K_n(z)=(-1)^{n+1}I_n(z)\log\frac{z}{2}+\mathfrak{K}_n(z) \tag{1.9}$$

with

$$\mathfrak{K}_n(z)=\frac{P_{2n-2}(z)}{z^n}+\frac{z^n}{2^n}O(1)\quad z\in\mathbb{C}\backslash]-\infty,0],\ \ z\to 0 \tag{1.10}$$

where P_{2n-2} is a polynomial in z of degree $2n-2$ such that $P_{2n-2}(0)=2^{n-1}$ and the functions $\mathfrak{A}_n$ and $\mathfrak{K}_n(z)$, $n\in\mathbb{N}\cup\{0\}$ are *holomorphic functions* on $\mathbb{C}$ and $\mathbb{C}\backslash\{0\}$, respectively.

The paper is structured as follows: Sect. 2 (very long) is devoted to the computation of $\widetilde{G}(x,y,t|0,0,t_0)$ by inversion of its Schwartz–Laplace transformation $\mathfrak{L}\big[\widetilde{G}(x,y,t|0,0,t_0)\big](x,y,s)$ with respect to the temporal variable t. In the final Sect. 3, we show that this function meets our goal.

Section 2 is subdivided into five subsections. In Sect. 2.1, of theoretical functional analytic character, we will show that $\mathfrak{L}\big[\widetilde{G}(x,y,t|0,0,t_0)\big](x,y,s)$ can be obtained as the limit of $\mathfrak{L}\big[\widetilde{G}(x,y,t|x_0,y_0,t_0)\big](x,y,s)$ in the topology of the space $H^\eta\big(]-T,T[,H^{-(\varepsilon+2\xi)}(D)\big)$ when $(x_0,y_0)\in D':=D\backslash\big\{(x,0)\in\mathbb{R}^2 \;\big|\; x\in[0,1]\big\}$ approaches to $(0,0)$. As a consequence, we need first to find $\mathfrak{L}\big[\widetilde{G}(x,y,t|x_0,y_0,t_0)\big](x,y,s)$, $(x_0,y_0)\in D'$. This task has two parts. The first one, Sect. 2.2, is essentially of topological theoretical character too. After switching to polar coordinates, we show that the cartesian expression of the Fourier expansion of $-e^{-s\,t_0}\frac{\delta(\rho-\rho_0)}{\rho}\delta(\omega-\omega_0)$ with respect to ω is convergent in $\big(H^{1+\varepsilon}(D)\big)'$. This allows us to apply the theoretical results of [6] to write $\mathfrak{L}\big[\widetilde{G}(x,y,t|x_0,y_0,t_0)\big](x,y,s)$ as the sum of certain convergent series $\sum_{n=1}^{\infty}\widetilde{V}_n(s,x,y)$ in the topology of $H^{-(\varepsilon+2\xi)}(D)$. The explicit expression of $V_n(s,\rho,\omega)$ is found in Sect. 2.3. Section 2.4 is devoted to find $\mathfrak{L}\big[\widetilde{G}(x,y,t|0,0,t_0)\big](x,y,s)$, and the inversion is made in Sect. 2.5.

2. Finding $\widetilde{G}(x,y,t|0,0,t_0)$

2.1. Determining $\mathfrak{L}[\widetilde{G}(x,y,t\big|0,0,t_0)]$

Taking the Schwartz–Laplace transform $\mathfrak{L}[\widetilde{G}](s,x,y)$ with respect to the t-variable in Eqs. (1.1) and (1.2), we obtain the equations

$$\widetilde{\mathcal{X}}_s\big(\mathfrak{L}[\widetilde{G}]\big):=\frac{\partial^2\mathfrak{L}[\widetilde{G}]}{\partial x^2}+\frac{\partial^2\mathfrak{L}[\widetilde{G}]}{\partial y^2}-(s^2+2\,s)\mathfrak{L}[\widetilde{G}]=-e^{-s\,t_0}\delta(x-x_0)\delta(y-y_0) \tag{2.1}$$

$$\nabla\mathfrak{L}[\widetilde{G}](s,x,y)=0 \quad \text{in } \partial D. \tag{2.2}$$

Lemma 2.1. $\lim_{(x_0,y_0)\to(0,0)}\delta(x-x_0)\delta(y-y_0)=\delta(x)\delta(y)$ *in* $H^{-(1+\varepsilon)}(D)$.

Proof. Defining $\Delta(x_0):=\delta(x-x_0)-\delta(x)$ and $\Delta(y_0):=\delta(y-y_0)-\delta(y)$, we have

$$\begin{aligned}&\big\|\big(\delta(x-x_0)\delta(y-y_0)-\delta(x)\delta(y)\big)\big\|_{H^{-(1+\varepsilon)}(D)}\\ &\quad\le\big\|\Delta(x_0)\delta(y-y_0)\big\|_{H^{-(1+\varepsilon)}(D)}+\big\|\delta(x)\Delta(y_0)\big\|_{H^{-(1+\varepsilon)}(D)}.\end{aligned} \tag{2.3}$$

Let $\chi(x)$ (resp. $\chi(y)$) be the characteristic function of the x-interval $]-1,1[$, (resp. of the y-interval $]-1,1[$). Given $\beta>0$, there is $\widetilde{\varphi}(x,y)\in\mathcal{D}(D)\cap B_{H_0^{1+\varepsilon}(D)}$ such that

$$\begin{aligned}&\big\|\Delta(x_0)\delta(y-y_0)\big\|_{H^{-(1+\varepsilon)}(D)}-\beta\le\big|\big\langle\Delta(x_0)\delta(y-y_0),\widetilde{\varphi}(x,y)\big\rangle_D\big|\\ &\quad=\big|\big\langle\chi(x)\chi(y)\;\Delta(x_0)\delta(y-y_0)e^{-x}e^{-y},e^{x+y}\widetilde{\varphi}(x,y)\big\rangle_{\mathbb{R}^2}\big|\\ &\quad\le\Big\|\chi(x)\;e^{-x}\;\Delta(x_0)\chi(y)\;e^{-y}\;\delta(y-y_0)\Big\|_{H^{-(1+\varepsilon)}(\mathbb{R}^2)}\Big\|\;e^{x+y}\;\widetilde{\varphi}(x,y)\Big\|_{H_0^{1+\varepsilon}(\mathbb{R}^2)}\end{aligned}$$

and using (1.4) and $\big[$[2], theorem 2.6.2$\big]$

$$\le C_D \left\| \chi(x)\, e^{-x} \Delta(x_0)\, \chi(y)\, e^{-y}\, \delta(y-y_0) \right\|_{H^{-(1+\varepsilon)}(\mathbb{R}^2)}$$

$$= C_D \left\| \frac{\mathcal{F}_{x,y}\big[\chi(x)\, \Delta(x_0)\, e^{-x}\, \chi(y)\, e^{-y}\, \delta(y-y_0)\big](\nu_1,\nu_2)}{(1+\nu_1^2+\nu_2^2)^{\frac{1+\varepsilon}{2}}} \right\|_{L^2(\mathbb{R}^2)}$$

$$= C_D \left\| \frac{\mathcal{F}_x\Big[\chi(x)\, \Delta(x_0)\, e^{-x}\Big](\nu_1)\, \mathcal{F}_y\Big[\chi(y)\, e^{-y}\, \delta(y-y_0)\Big](\nu_2)}{(1+\nu_1^2+\nu_2^2)^{\frac{1+\varepsilon}{2}}} \right\|_{L^2(\mathbb{R}^2)}$$

$$= C_D \left\| \frac{\mathfrak{L}\big[\chi(x)\, \Delta(x_0)\big](1+i\,\nu_1)\, \mathfrak{L}\big[\chi(y)\, \delta(y-y_0)\big](1+i\,\nu_2)}{(1+\nu_1^2+\nu_2^2)^{\frac{1+\varepsilon}{2}}} \right\|_{L^2(\mathbb{R}^2)}$$

$$\le C_D \left(\int\limits_{\mathbb{R}^2} \frac{\big|\big(e^{-x_0(1+i\,\nu_1)}-1\big)\big|^2 \big|\, e^{-y_0(1+i\,\nu_2)}\big|^2\big)}{(1+\nu_1^2+\nu_2^2)^{1+\varepsilon}}\, d\nu_1\, d\nu_2 \right)^{\frac{1}{2}}$$

and applying Fubini's theorem

$$\le C_D\, e^{-y_0} \left(\int\limits_{\mathbb{R}} \frac{e^{-2\,x_0} - 2\cos x_0\nu_1 + 1}{(1+\nu_1^2)^{\frac{1+\varepsilon}{2}}}\, d\nu_1 \int\limits_{\mathbb{R}} \frac{d\nu_2}{\big(1+\nu_2^2\big)^{\frac{1+\varepsilon}{2}}} \right)^{\frac{1}{2}} := M(x_0,y_0). \tag{2.4}$$

(since these integrals are convergent). From the arbitrariness of $\beta > 0$, we deduce that $\big\|\Delta(x_0)\delta(y-y_0)\big\|_{H^{-(1+\varepsilon)}(D)} \le M(x_0,y_0)$. But the first integral in (2.4) is uniformly convergent with respect to x_0 in $[-1,1]$. Then, $\lim_{(x_0,y_0)\to(0,0)} M(x_0,y_0) = 0$, i.e., $\lim_{(x_0,y_0)\to(0,0)} \big\|\Delta(x_0)\delta(y-y_0)\big\|_{H^{-(1+\varepsilon)}(D)} = 0$. Analogously, it can be proved that $\lim_{(x_0,y_0)\to(0,0)} \big\|\delta(x)\Delta(y_0)\big\|_{H^{-(1+\varepsilon)}(D)} = 0$, and the result follows from (2.3). □

By Lemma 2.1, (2.1) and $\big[$[6], theorem 9 and computations of corollary 15, part b$\big]$, it follows that if $(x_0,y_0)\in D'$, then

$$\lim_{(x_0,y_0)\to(0,0)} \mathfrak{L}\big[\widetilde{G}(x,y,t|x_0,y_0,t_0)\big](s,x,y) = \mathfrak{L}\big[\widetilde{G}(x,y,t|0,0,t_0)\big](s,x,y) \tag{2.5}$$

in the topology of the space $H^{-(1+\varepsilon)}(D)$. So, we need to find $\mathfrak{L}\big[\widetilde{G}(x,y,t\big|x_0,y_0,t_0)\big]$ for $(x_0,y_0)\in D'$. Moreover, in order to find the inverse Laplace transform, it is enough to know the values $\mathfrak{L}\big[\widetilde{G}(x,y,t|0,0,t_0)\big](s,x,y)$ for $s>0$.

2.2. Theoretical Description of $\mathfrak{L}\big[\widetilde{G}(x,y,t\big|x_0,y_0,t_0)\big]$, $(x_0,y_0)\in D'$

To find $\mathfrak{L}\big[\widetilde{G}(x,y,t|x_0,y_0,t_0)\big](x,y,s)$, we switch to polar coordinates ρ,ω in (2.1), (2.2). Defining $\widetilde{\mathcal{L}}(s,x,y) := \mathfrak{L}\big[\widetilde{G}\big](s,x,y)$, by (1.6) it turns out that $\mathcal{L}(s,\rho,\omega)$ must verify the equations

$$\mathcal{X}_s(\mathcal{L}) := \frac{\partial^2 \mathcal{L}}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \mathcal{L}}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 \mathcal{L}}{\partial \omega^2} - (s^2 + 2\,s)\mathcal{L}$$
$$= -e^{-s\,t_0}\frac{\delta(\rho-\rho_0)}{\rho}\delta(\omega-\omega_0). \tag{2.6}$$
$$\frac{\partial \mathcal{L}}{\partial \rho}(s,1,\omega) = 0 \tag{2.7}$$

Given $\gamma \in [0,1[$, there is $\epsilon > 0$ such that $(x_0, y_0) \notin \overline{B_\epsilon(\gamma,0)}$, where $B_\epsilon(\gamma,0)$ is the open ball centered in $(\gamma,0) \in D$ and radius ϵ. It follows from (2.1) that the restriction $\widetilde{\mathcal{L}}_\varepsilon$ to $B_\epsilon(\gamma,0)$ of $\widetilde{\mathcal{L}}$ verifies

$$\frac{\partial^2 \widetilde{\mathcal{L}}_\epsilon}{\partial x^2} + \frac{\partial^2 \widetilde{\mathcal{L}}_\epsilon}{\partial y^2} - (s^2 + 2\,s)\widetilde{\mathcal{L}}_\epsilon = 0 \quad \text{in} \quad B_\epsilon(\gamma,0).$$

Since this equation is elliptic in $B_\epsilon(\gamma,0)$, by the theorem $\big[$[5], chapter 2, theorem 3.2$\big]$ of inner regularity of the solutions of elliptic equations, $\widetilde{\mathcal{L}}_\epsilon$, $\frac{\partial \widetilde{\mathcal{L}}_\epsilon}{\partial x}$ and $\frac{\partial \widetilde{\mathcal{L}}_\epsilon}{\partial y}$ must be continuous in $B_\epsilon(\gamma,0)$ and consequently in $\big\{(x,0) \in D \big| x \in [0,1[\big\}$. This means that for every $0 < \rho < 1$ the limits $\alpha_0(\rho) := \lim_{\omega\to 0} \mathcal{L}(s,\rho,\omega) = \lim_{\omega\to 2\pi} \mathcal{L}(s,\rho,\omega)$ and $\beta_0(\rho) := \lim_{\omega\to 0} \frac{\partial \mathcal{L}}{\partial \omega}(s,\rho,\omega) = \lim_{\omega\to 2\pi} \frac{\partial \mathcal{L}}{\partial \omega}(s,\rho,\omega)$ exist. It follows that if we consider the 4π-periodic respect to ω even function $\widehat{\mathcal{L}}(s,\rho,\omega)$ defined on $]0,1[\times\mathbb{R}$ which coincides with $\mathcal{L}(s,\rho,\omega)$ in $]0,2\,\pi[$ and such that $\widehat{\mathcal{L}}(s,\rho,0) = \alpha_0(\rho)$, we obtain a continuous function such that $\frac{\partial \widehat{\mathcal{L}}}{\partial \omega}$ is continuous at the points $(\rho,0)$, $0 < \rho < 1$ because $\frac{\partial f}{\partial \omega}(\rho,\omega) = -\frac{\partial \widetilde{f}}{\partial x}(x,0)\rho\sin\omega + \frac{\partial \widetilde{f}}{\partial y}(x,0)\;\rho\cos\omega$ for a general function $f(\rho,\omega)$.

Denote by $\widehat{\mathcal{L}}_-$ the restriction to $\Omega_- :=]0,1[\times]-2\pi,0[$ of $\widehat{\mathcal{L}}$. As the partial derivative with respect to ω appearing in $\mathcal{X}_s$ is of even order and $\delta(-w-\omega_0) = \delta(\omega+\omega_0)$, by (2.7) one has

$$\mathcal{X}_s\big(\widehat{\mathcal{L}}_-(s,\rho,\omega)\big) = \mathcal{X}_s\big(\mathcal{L}(s,\rho,-\omega)\big)$$
$$= \mathfrak{g}(s,\rho,-\omega_0) := -e^{-st_0}\frac{\delta(\rho-\rho_0)}{\rho}\delta(\omega+\omega_0) \;\text{ in }\; \Omega_-. \tag{2.8}$$

Given $\varphi(\rho,\omega) \in \mathcal{D}(]0,1[\times]-2\pi,2\pi[)$, its restriction φ_- to Ω_- lies in $\mathcal{C}^\infty(\overline{\Omega_-})$ and its restriction φ_+ to Ω lies in $\mathcal{C}^\infty(\overline{\Omega})$, and hence, the actions $\langle \mathfrak{g}(s,\rho,\omega_0), \varphi_+(\rho,\omega)\rangle_\Omega$ and $\langle \mathfrak{g}(s,\rho,-\omega_0), \varphi_-(\rho,\omega)\rangle_{\Omega_-}$ are well defined. From the definition of derivation of distributions and (2.6) and (2.8), we obtain

$$\big\langle \mathcal{X}_s(\widehat{\mathcal{L}}), \varphi\big\rangle = \int\limits_{]0,1[\times]-2\pi,2\pi[} \varphi\;\mathcal{X}_s(\widehat{\mathcal{L}})\;d\rho\;d\omega = \int\limits_{]0,1[\times]-2\pi,2\pi[} \mathcal{X}'_s(\varphi)\;\widehat{\mathcal{L}}\;d\rho\;d\omega$$
$$= \int\limits_{\Omega} \mathcal{X}'_s(\varphi)\;\widehat{\mathcal{L}}\;d\rho\;d\omega + \int\limits_{\Omega_-} \mathcal{X}'_s(\varphi)\;\widehat{\mathcal{L}}\;d\rho\;d\omega$$
$$= \big\langle \mathcal{L}, \mathcal{X}'_s(\varphi_+)\big\rangle_\Omega + \big\langle \widehat{\mathcal{L}}_-, \mathcal{X}'_s(\varphi_-)\big\rangle_{\Omega_-}$$

and, since for every fixed ω, the functions $\varphi_+(\rho,\omega)$ and $\varphi_-(\rho,\omega)$ have compact support with respect to ρ, after two integrations by parts with respect to ω

$$\begin{aligned}
&= \left\langle \mathcal{X}_s(\mathcal{L}), \varphi_+ \right\rangle_\Omega + \int_0^1 \Big(-\mathcal{L}(s,\rho,0)\frac{\partial \varphi}{\partial \omega}(\rho,0) + \frac{\partial \mathcal{L}}{\partial \omega}(s,\rho,0)\varphi(\rho,0) \Big) \frac{d\rho}{\rho^2} \\
&\quad + \left\langle \mathcal{X}_s(\widehat{\mathcal{L}}_-), \varphi_- \right\rangle_{\Omega_-} + \int_0^1 \Big(\mathcal{L}(s,\rho,0)\frac{\partial \varphi}{\partial \omega}(\rho,0) - \frac{\partial \mathcal{L}}{\partial \omega}(s,\rho,0)\varphi(\rho,0) \Big) \frac{d\rho}{\rho^2} \\
&= \left\langle \mathcal{X}_s(\mathcal{L}), \varphi_+ \right\rangle_\Omega + \left\langle \mathcal{X}_s(\widehat{\mathcal{L}}_-), \varphi_- \right\rangle_{\Omega_-} \\
&= -\frac{e^{-st_0}}{\rho_0}\big(\varphi_+(\rho_0,\omega_0) + \varphi_-(\rho_0,-\omega_0)\big) \\
&= -\frac{e^{-st_0}}{\rho_0}\big(\varphi(\rho_0,\omega_0) + \varphi(\rho_0,-\omega_0)\big) = \langle \widehat{\mathfrak{g}}, \varphi \rangle
\end{aligned}$$

where we have defined $\widehat{\mathfrak{g}} = -e^{-st_0}\frac{\delta(\rho-\rho_0}{\rho}\big(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\big)$. As a consequence,

$$\mathcal{X}_s(\widehat{\mathcal{L}}) = \widehat{\mathfrak{g}} = -e^{-st_0}\frac{\delta(\rho-\rho_0)}{\rho}\big(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\big) \quad \text{in }]0,1[\times]-2\pi,2\pi[. \tag{2.9}$$

By $\big[$[8], chapter 7, theorem 1$\big]$, for every $0<\rho<1$ the even distribution $\widehat{\mathcal{L}}$ has a Fourier cosine expansion in $]-2\pi,2\pi[$

$$\widehat{\mathcal{L}}(s,\rho,\omega) = \sum_{n=0}^{\infty} \mathcal{L}_n(s,\rho)\cos\frac{n}{2}\,\omega \quad \text{in }]-2\pi,2\pi[, \tag{2.10}$$

which will be convergent in the sense of distributions and where, defining $b_n = \frac{2}{\pi}$ if $n \in \mathbb{N}$ and $b_0 = \frac{1}{\pi}$, for every $\rho \in]0,1[$ and s in the domain of $\mathcal{L}[G](s,\rho,\omega)$ we have

$$\forall n \in \mathbb{N}\cup\{0\} \quad \mathcal{L}_n(s,\rho) = \frac{b_n}{2}\int_0^{2\pi} \widehat{\mathcal{L}}(s,\rho,\omega)\cos\frac{n}{2}\,\omega\, d\omega. \tag{2.11}$$

As a consequence, $\mathcal{X}_s(\widehat{\mathcal{L}})$ has a Fourier series expansion

$$\begin{aligned}
&\mathcal{X}_s(\widehat{\mathcal{L}}) \\
&\quad = \sum_{n=0}^{\infty}\Big(\frac{\partial^2 \mathcal{L}_n}{\partial \rho^2}(s,\rho) + \frac{1}{\rho}\frac{\partial \mathcal{L}_n}{\partial \rho}(s,\rho) - \frac{n^2}{4\rho^2}\mathcal{L}_n(s,\rho) - (s^2+2s)\mathcal{L}_n(s,\rho)\Big)\cos\frac{n}{2}\,\omega
\end{aligned} \tag{2.12}$$

which *converges in the sense of distributions* $\Big($[[8], chapter 7, theorem 1]$\Big)$.

Clearly, for every $0<\rho<1$, the series

$$S(\rho,\omega) := -e^{-s\,t_0}\frac{\delta(\rho-\rho_0)}{\rho}\sum_{n\in\mathbb{Z}}\big(\delta(\omega-\omega_0-4n\pi)+\delta(\omega+\omega_0-4n\pi)\big) \tag{2.13}$$

is convergent in $\mathcal{D}(\mathbb{R})$ and defines an even 4π-periodic distribution in $\mathbb{R}$ which has a Fourier expansion

$$S(\rho,\omega) = -e^{-st_0}\frac{\delta(\rho-\rho_0)}{\rho}\sum_{n=0}^{\infty} b_n \cos\frac{n}{2}\,\omega_0\ \cos\frac{n}{2}\,\omega \tag{2.14}$$

convergent in $]-2\pi, 2\pi[$ *in the sense of distributions* $\big($[[8], chapter 7, theorem 1]$\big)$. By (2.9) the expansions (2.12) and (2.14) must be coincident, and hence $\big($[[8], chapter VII, theorem 1]$\big)$, we have

$$\begin{aligned}\mathcal{V}_s(\mathcal{L}_n) &:= \frac{\partial^2\mathcal{L}_n}{\partial\rho^2}(s,\rho) + \frac{1}{\rho}\frac{\partial\mathcal{L}_n}{\partial\rho}(s,\rho) - \left(\frac{n^2}{4\rho^2} + (s^2+2s)\right)\mathcal{L}_n(s,\rho)\\ &= -e^{-st_0}\frac{\delta(\rho-\rho_0)}{\rho}\,b_n\cos\frac{n}{2}\,\omega_0 \end{aligned}\tag{2.15}$$

By restriction to $]0, 2\pi[$ of (2.10), we obtain

$$\mathcal{L}(s,\rho,\omega) = \sum_{n=0}^{\infty}\mathcal{L}_n(s,\rho)\cos\frac{n}{2}\,\omega \quad \text{in } \Omega \tag{2.16}$$

and we only need to find $\mathcal{L}_n(s,\rho)$ from Eq. (2.15).

However, to compute later on the limit (2.5) we shall need more information. By restriction to $]0, 2\pi[$ of (2.13) and (2.14) we obtain

$$\begin{aligned}&\forall\ 0<\rho<1 \qquad -e^{-s\,t_0}\frac{\delta(\rho-\rho_0)}{\rho}\delta(\omega-\omega_0)\\ &= \sum_{n=0}^{\infty}\mathfrak{g}_n(s,\rho,\omega) := \sum_{n=0}^{\infty} -e^{-st_0}\frac{\delta(\rho-\rho_0)}{\rho}\,b_n\cos\frac{n}{2}\,\omega_0\ \cos\frac{n}{2}\,\omega \quad \text{in }]0,2\pi[\end{aligned}\tag{2.17}$$

in the sense of distributions. Let us see that the expansion (2.17) is convergent in $H^{-(1+\varepsilon)}(\Omega)$ with respect to both variables ρ, ω indeed. We need a previous Lemma:

Lemma 2.2. *There is $K(\eta) > 0$ such that*

$$\forall\ \alpha>0 \qquad \mathfrak{i}(\alpha) := \int_{-\infty}^{\infty}\frac{(1+\nu^2)^{\eta}}{\left|(1+i\,\nu)^2+\alpha^2\right|^2}\,d\nu \le \frac{K(\eta)}{\alpha^{2-2\eta}}.$$

Proof. Performing the variable change $x := \sqrt{1+\nu^2}$, after elementary operations, we obtain

$$\mathfrak{i}(\alpha) = 2\int_{1}^{\infty}\frac{x^{2\,\eta+1}}{Q_\alpha(x)\sqrt{x^2-1}}\,dx$$

where $Q_\alpha(x) := (x^4 - 2\ \alpha^2\ x^2 + \alpha^4 + 4\alpha^2)$. To compute this integral, we integrate the complex function

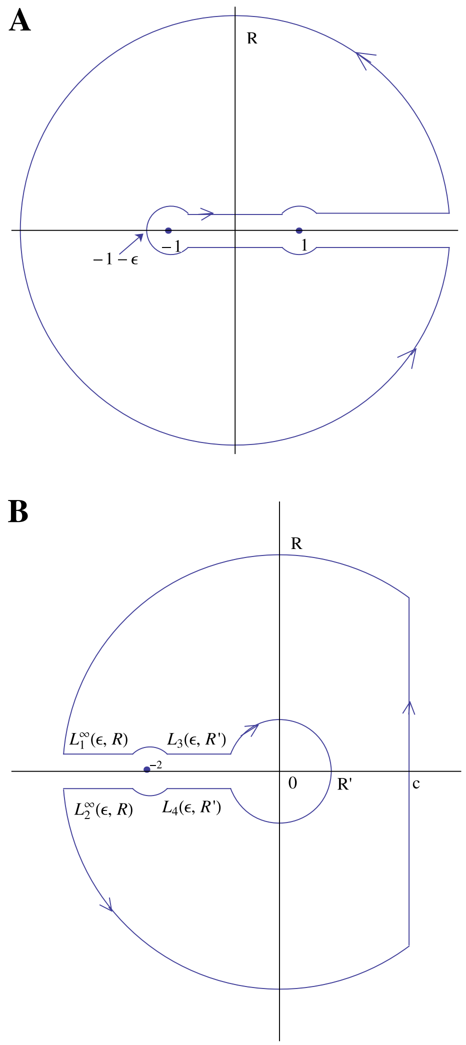

Figure 1. Bromwich's contours in Lemma 2.2 and in Sect. 2.5

$$w(z) := \frac{z^{2\eta+1}}{Q_\alpha(z)\sqrt{z^2-1}}$$

along the circuit displayed in Fig. 1 A, choosing $Arg(z+1) \in]0, 2\pi[$, $Arg(z) \in]0, 2\,\pi[$ and $Arg(z-1) \in]0, 2\pi[$. Letting $A_\alpha := \alpha^4 + 4\,\alpha^2$ and $\varphi_\alpha := \arcsin \frac{2\,\alpha}{\sqrt{A_\alpha}}$, the four simple poles $z_{j,\alpha}$, $1 \le j \le 4$ of $w(z)$ are $z_{1,\alpha} = \sqrt[4]{A_\alpha}\; e^{\frac{\varphi_\alpha}{2} i}$, $z_{2,\alpha} = -z_{1,\alpha}$, $z_{3,\alpha} = \overline{z_{1,\alpha}}$ and $z_{4,\alpha} = -\overline{z_{1,\alpha}}$. This implies $\imath_\alpha > 0$. If $\psi_{1,\alpha} := Arg(z_{1,\alpha}+1)$ and $\psi_{2,\alpha} := Arg(z_{1,\alpha}-1)$, as $4(z_{j,\alpha}^2 - \alpha^2) = 8\,\alpha\, i$ or $4(z_{j,\alpha}^2 - \alpha^2) = -8\,\alpha\, i$, whatever be $j = 1, 2, 3, 4$, after application of L'Hôpital's rule it turns out that the corresponding residues $w_{j\alpha} = \mathrm{Res}_{z=z_{j,\alpha}} w(z)$, $1 \le j \le 4$, are

$$w_{1\alpha} = \frac{A_\alpha^{\frac{\eta}{2}} e^{\left(\eta\varphi_\alpha - \frac{\psi_{1,\alpha}+\psi_{2,\alpha}}{2}\right)\, i}}{8\,\alpha\, i\sqrt{\left|\sqrt{A_\alpha} e^{\varphi_\alpha\, i} - 1\right|}}, \quad w_{2\alpha} = -w_{1\alpha} e^{2\pi\eta i}$$
$$w_{3\alpha} = -\overline{w_{1\alpha}} e^{2\pi\eta i}, \quad w_{4\alpha} = \overline{w_{1,\alpha}} e^{4\pi\eta i}.$$

After standard considerations, the limits for $\epsilon \to 0$ and $R \to \infty$ of the integrals along all the circular arcs are 0, and by the residue theorem, we obtain

$$\frac{1 + 2e^{2\pi\eta i} + e^{4\pi\eta i}}{i} \int_0^1 \frac{x^{2\eta+1}\, dx}{Q_\alpha(x)\sqrt{1-x^2}} + \left(1 - e^{4\pi\eta i}\right) \frac{\mathfrak{i}(\alpha)}{2} = 2\pi\, i \sum_{j=1}^{4} w_{j\alpha}$$

and after some habitual computations, we obtain

$$0 < \mathfrak{i}(\alpha) = \frac{\pi A_\alpha^{\frac{\eta}{2}} \cos\left(\eta\varphi_\alpha - \frac{\psi_{1,\alpha}+\psi_{2,\alpha}}{2} - \pi\eta\right)}{4\alpha\sqrt{\alpha^2+1}\cos\pi\eta} - 2\frac{\cos\pi\eta}{\sin\pi\eta} \int_0^1 \frac{x^{2\eta+1}\, dx}{Q_\alpha(x)\sqrt{1-x^2}}$$
$$\leq \frac{\pi A_\alpha^{\frac{\eta}{2}} \cos\left(\eta\varphi_\alpha - \frac{\psi_{1,\alpha}+\psi_{2,\alpha}}{2} - \pi\eta\right)}{4\alpha\sqrt{\alpha^2+1}\cos\pi\eta} \leq \frac{K(\eta)}{\alpha^{2-2\eta}}.$$

□

Lemma 2.3. *One has*

$$\left\|\mathfrak{g}_n(s,\rho,\omega)\right\|_{H^{-(1+\varepsilon)}(\Omega)} \leq \frac{\left|e^{-st_0}\right|}{\rho_0} \frac{M(\varepsilon,s)}{n^{1+\varepsilon}}. \tag{2.18}$$

and $\sum_{n=0}^{\infty} \mathfrak{g}_n(s,\rho,\omega) = -e^{-st_0}\, \frac{\delta(\rho-\rho_0)}{\rho}\, \delta(\omega-\omega_0)$ *in* $H^{-(1+\varepsilon)}(\Omega)$.

Proof. Let $\chi(\rho)$ and $\chi(\omega)$ be the characteristic functions of the intervals $]0,1[$ and $]0,2\pi[$, respectively. Given $\beta > 0$, there is $\varphi(\rho,\omega) \in \mathcal{D}(\Omega) \cap B_{H_0^{1+\varepsilon}(\Omega)}$ such that

$$\left\|\mathfrak{g}_n(s,\rho,\omega)\right\|_{H^{-(1+\varepsilon)}(\Omega)} - \beta \leq \left|\int_\Omega -b_n\, \frac{\delta(\rho-\rho_0)}{\rho}\, e^{-st_0} \cos\frac{n}{2}\omega\, \varphi(\rho,\omega)\, d\rho\, d\omega\right|$$
$$\leq \left|b_n\right|\left|e^{-st_0}\right| \left|\int_{-\pi}^{\pi} \frac{1}{\rho_0}\, \cos\frac{n}{2}\omega\, \varphi(\rho_0,\omega)\, d\omega\right|$$

and writing $C_0(s) := \left|b_n\right|\left|e^{-st_0}\right|$, by (1.4)

$$= C_0(s) \left|\int_{\mathbb{R}^2} \chi(\rho)\, e^{\rho+\omega}\, \frac{\delta(\rho-\rho_0)}{\rho}\, e^{-\rho}\, \chi(\omega)\, e^{-\omega}\, \cos\frac{n}{2}\omega\, \varphi(\rho,\omega)\, d\rho\, d\omega\right|$$
$$\leq C_0(s) \left\|\chi(\rho)\, \frac{\delta(\rho-\rho_0)}{\rho}\, e^{-\rho-\omega}\, \chi(\omega)\, \cos\frac{n}{2}\omega\right\|_{H^{-(1+\varepsilon)}(\mathbb{R}^2)}$$
$$\times \left\|\, e^{\rho+\omega}\, \varphi(\rho,\omega)\right\|_{H_0^{1+\varepsilon}(\mathbb{R}^2)}$$

$$\leq C_\Omega\ C_0(s) \Big\| \varphi(\rho,\omega) \Big\|_{H_0^{1+\varepsilon}(\mathbb{R}^2)} \times \Big\| \chi(\rho)\ \frac{\delta(\rho-\rho_0)}{\rho}\ e^{-\rho-\omega}\ \chi(\omega)\ \cos\frac{n}{2}\omega \Big\|_{H^{-(1+\varepsilon)}(\mathbb{R}^2)}$$

and by $\big[$[2], theorem 2.6.2$\big]$, writing $C_1(s) = C_\Omega\ C_0(s)$

$$\leq C_1(s) \left(\int\limits_{\mathbb{R}^2} \frac{\Big| \mathcal{F}_{\rho,\omega}\big[\chi(\rho)\ \frac{\delta(\rho-\rho_0)}{\rho}\ e^{-\rho-\omega}\ \chi(\omega)\ \cos\frac{n}{2}\omega\big](\nu_1,\nu_2)\Big|^2}{(1+\nu_1^2+\nu_2^2)^{1+\varepsilon}}\ d\nu_1 d\nu_2 \right)^{\frac{1}{2}}$$

$$= C_1(s) \left(\int\limits_{\mathbb{R}^2} \frac{\Big| \mathcal{F}_{\rho}\big[\chi(\rho)\ \frac{\delta(\rho-\rho_0)}{\rho} e^{-\rho}\big](\nu_1)\ \mathcal{F}_{\omega}\big[\chi(\omega) e^{-\omega}\cos\frac{n}{2}\omega\big](\nu_2)\Big|^2}{(1+\nu_1^2+\nu_2^2)^{1+\varepsilon}} d\nu_1 d\nu_2 \right)^{\frac{1}{2}}$$

$$= C_1(s) \left(\int\limits_{\mathbb{R}^2} \frac{\Big| \mathfrak{L}\big[\chi(\rho) \frac{\delta(\rho-\rho_0)}{\rho}\big](1+i\nu_1)\ \mathfrak{L}\big[\chi(\omega)\cos\frac{n}{2}\omega\big](1+i\nu_2)\Big|^2}{(1+\nu_1^2+\nu_2^2)^{1+\varepsilon}} d\nu_1 d\nu_2 \right)^{\frac{1}{2}}$$

$$\leq C_1(s)\ \frac{(1+e^{\pi})}{\rho_0} \left(\int\limits_{\mathbb{R}^2} \frac{\big|\chi(\rho_0)\ e^{-\rho_0(1+i\ \nu_1)}\big|^2 (1+\nu_2^2)}{(1+\nu_1^2+\nu_2^2)^{1+\varepsilon}\big|(1+i\ \nu_2)^2+\frac{n^2}{4}\big|^2}\ d\nu_1\ d\nu_2 \right)^{\frac{1}{2}}$$

and applying Fubini's theorem and Lemma 2.2 and putting $C_2(s) := C_1(s)\ (1+e^{\pi})$

$$\leq \frac{C_2(s)}{\rho_0}\ e^{-\rho_0} \left(\int\limits_{\mathbb{R}} \frac{d\nu_1}{(1+\nu_1^2)^{\frac{1+\varepsilon}{2}}}\ \int\limits_{\mathbb{R}} \frac{(1+\nu_2^2)^{1-\frac{1+\varepsilon}{2}}}{\big|(1+i\ \nu_2)^2+\frac{n^2}{4}\big|^2)}\ d\nu_2 \right)^{\frac{1}{2}}$$

$$\leq \frac{\big|e^{-st_0}\big|}{\rho_0}\ \frac{M(\varepsilon,s)}{n^{1+\varepsilon}}$$

(since the first integral from the left is clearly convergent). (2.18) follows from the arbitrariness of $\beta > 0$. On the other hand, the estimation (2.18) implies that $\sum_{n=0}^{\infty} \mathfrak{g}_n(s,\rho,\omega)$ is convergent in $H^{-(1+\varepsilon)}(\Omega)$. Then, for every $\varphi(\rho,\omega) \in \mathcal{D}(\Omega)$, by (2.17)

$$\Big\langle \sum_{n=0}^{\infty} \mathfrak{g}_n(s,\rho,\omega), \varphi(\rho,\omega) \Big\rangle = \sum_{n=0}^{\infty} \big\langle \mathfrak{g}_n(s,\rho,\omega), \varphi(\rho,\omega) \big\rangle$$

$$= -e^{-st_0} \sum_{n=0}^{\infty} \Big\langle b_n\ \cos\frac{n}{2}\omega_0 \cos\frac{n}{2}\omega, \frac{\varphi(\rho_0,\omega)}{\rho_0} \Big\rangle = -e^{-st_0} \Big\langle \delta(\omega-\omega_0), \frac{\varphi(\rho_0,\omega)}{\rho_0} \Big\rangle$$

$$= -e^{-st_0} \frac{\varphi(\rho_0,\omega_0)}{\rho_0} = \Big\langle -e^{-st_0} \frac{\delta(\rho-\rho_0)}{\rho}\ \delta(\omega-\omega_0), \varphi(\rho,\omega) \Big\rangle$$

which gives us the desired result. □

Now we are going to prove that changing from polar to cartesian coordinates the corresponding expansion is convergent in $(H^{1+\varepsilon}(D))'$. First, we note that

Lemma 2.4. *The distribution* $\widetilde{\mathfrak{g}_n}(s,x,y) \in \mathcal{D}'(D)$ *given by* $\langle \widetilde{\mathfrak{g}_n}(s,x,y), \widetilde{\varphi}(x,y)\rangle = \langle \mathfrak{g}_n(s,\rho,\omega), \rho\varphi(\rho,\omega)\rangle$ *for every* $\widetilde{\varphi}(x,y) \in \mathcal{D}(D)$ *is well defined and lies in* $\Xi^{-(1+\varepsilon)}(D) \subset (H^{1+\varepsilon}(D))'$.

Proof. Let $\alpha_0 > 0$ such that $0 < \rho_0 - \alpha_0 < \rho_0 + \alpha_0 < 1$ and choose $\xi_0(\rho) \in \mathcal{D}(]0,1[)$ such that $\xi_0(\rho_0) = 1$ and $Supp(\xi_0) \subset]\rho_0 - \frac{\alpha_0}{2}, \rho_0 + \frac{\alpha_0}{2}[$. In cartesian coordinates, one has $Supp(\widetilde{\xi_0}(x,y)) \subset C_{\alpha_0} := \{(x,y) \in D \mid \rho_0 - \alpha_0 < \sqrt{x^2+y^2} < \rho_0 + \alpha_0\}$. Given $\widetilde{\varphi}(x,y) \in \mathcal{D}(D)$, one has $\chi_{C_{\alpha_0}}\widetilde{\varphi} \in \mathcal{C}^\infty(\overline{C_{\alpha_0}}) \subset \Xi^{1+\varepsilon}(C_{\alpha_0})$. From the definition of the complex interpolation method and the computation of $\|\chi_{C_{\alpha_0}}\widetilde{\varphi}\|_{\Xi^{1+\varepsilon}(C_{\alpha_0})}$, after elementary estimations we find $\gamma(\rho_0) > 0$ such that $\|\chi_{C_{\alpha_0}}\widetilde{\varphi}\|_{H^{1+\varepsilon}(C_{\alpha_0})} \le \gamma(\rho_0)\|\chi_{C_{\alpha_0}}\widetilde{\varphi}\|_{\Xi^{1+\varepsilon}(C_{\alpha_0})}$. Using Sobolev's embedding theorem, one has the continuous inclusion $I : H^{1+\varepsilon}(C_{\alpha_0}) \subset \mathcal{C}(\overline{C_{\alpha_0}})$, and hence, there are $M_j(s,\rho_0) > 0, j = 1,2,3$ such that

$$\begin{aligned}
\big|\langle \widetilde{\mathfrak{g}_n}, \widetilde{\varphi}\rangle\big| &= \big|\langle \mathfrak{g}_n, \rho\,\varphi\rangle\big| \\
&= \left| -b_n e^{-st_0}\,\rho_0\,\xi_0(\rho_0)\cos\frac{n}{2}\omega_0 \int_0^{2\pi} \frac{\varphi(\rho_0,\omega)}{\rho_0}\cos\frac{n}{2}\omega\,d\omega \right| \\
&\le b_n\big|e^{-st_0}\big|\,2\pi\big\|\xi_0(\rho)\,\varphi(\rho,\omega)\big\|_{\mathcal{C}(\overline{\Omega})} = M_1(s,\rho_0)\big\|\widetilde{\xi_0}(x,y)\,\widetilde{\varphi}(x,y)\big\|_{\mathcal{C}(\overline{D})} \\
&= M_1(s,\rho_0)\big\|\widetilde{\xi_0}(x,y)\,\widetilde{\varphi}(x,y)\big\|_{\mathcal{C}(\overline{C_{\alpha_0}})} \\
&\le M_2(s,\rho_0)\,\|I\|\,\big\|\chi_{C_{\alpha_0}}(x,y)\,\widetilde{\varphi}(x,y)\big\|_{H^{1+\varepsilon}(C_{\alpha_0})} \\
&\le M_3(s,\rho)\gamma(\rho_0)\big\|\chi_{C_{\alpha_0}}(x,y)\,\widetilde{\varphi}(x,y)\big\|_{\Xi^{1+\varepsilon}(C_{\alpha_0})} \\
&\le M_3(s,\rho)\gamma(\rho_0)\big\|\widetilde{\varphi}(x,y)\big\|_{\Xi^{1+\varepsilon}(D)}
\end{aligned}$$

and the Lemma follows by density of $\mathcal{D}(D)$ in $\Xi^{1+\varepsilon}(D)$. □

Proposition 2.5. *a) We have*

$$\big\|\widetilde{Q}(\widetilde{\mathfrak{g}_n})(s,x,y)\big\|_{H^{-(1+\varepsilon)}(D)} \le \frac{|e^{-st_0}|}{\rho_0}\,\frac{N(\varepsilon,\rho_0,s)}{n^{1+\varepsilon}}.$$

b) $\sum_{n=0}^{\infty}\widetilde{Q}\big(\widetilde{\mathfrak{g}_n}(s,x,y)\big) = -e^{-s\,t_0}\widetilde{Q}(\delta(x-x_0)\delta(y-y_0))$ *in* $H^{-(1+\varepsilon)}(D)$.

c) $\sum_{j=1}^{\infty}\widetilde{\mathfrak{g}_j}(s,x,y) = -e^{-s\,t_0}\delta(x-x_0)\delta(y-y_0)$ *in* $\big(H^{1+\varepsilon}(D)\big)'$.

Proof. a) There is a constant $P(\rho_0) > 0$ such that by (1.5), for every $\widetilde{\varphi}(x,y) \in \mathcal{D}(D) \cap B_{H_0^{1+\varepsilon}(D)}$ one has

$$\begin{aligned}
\big|\langle \widetilde{Q}(\widetilde{\mathfrak{g}_n}(s,x,y)), \widetilde{\varphi}(x,y)\rangle\big| &= \big|\langle \mathfrak{g}_n(s,\rho,\omega), \rho\,\varphi(\rho,\omega)\rangle\big| \\
&\le \big\|\mathfrak{g}_n(s,\rho,\omega)\big\|_{H^{-(1+\varepsilon)}(\Omega)}\big\|\rho\varphi(\rho,\omega)\big\|_{H_0^{1+\varepsilon}(\Omega)} \\
&\le P(\rho_0)\big\|\mathfrak{g}_n(s,\rho,\omega)\big\|_{H^{-(1+\varepsilon)}(\Omega)}\big\|\varphi(\rho,\omega)\big\|_{H_0^{1+\varepsilon}(\Omega)} \\
&\le P(\rho_0)\big\|\mathfrak{g}_n(s,\rho,\omega)\big\|_{H^{-(1+\varepsilon)}(\Omega)}
\end{aligned}$$

and the result follows from Lemma 2.3.

b) By a) the series $\sum_{n=0}^{\infty} \widetilde{Q}(\widetilde{\mathfrak{g}_n}(s,x,y))$ is convergent in $H^{-(1+\varepsilon)}(D)$ to some $\widetilde{z}(x,y,s) \in H^{-(1+\varepsilon)}(D)$. Given $\widetilde{\varphi}(x,y) \in \mathcal{D}(D)$, it follows from a) and Lemma 2.3 that $\sum_{n=0}^{\infty}\langle \mathfrak{g}_n(s,\rho,\omega), \varphi(\rho,\omega)\rangle$ is convergent, and by (1.5) and Lemma 2.3, we have

$$\begin{aligned}
\big\langle \widetilde{z}(x,y,s), \widetilde{\varphi}(x,y)\big\rangle &= \lim_{k\to\infty} \Big\langle \sum_{n=0}^{k} \widetilde{\mathfrak{g}_n}(s,x,y), \widetilde{\varphi}(x,y)\Big\rangle \\
&= \lim_{k\to\infty} \Big\langle \sum_{n=0}^{k} \mathfrak{g}_n(s,\rho,\omega), \rho\varphi(\rho,\omega)\Big\rangle \\
&= \Big\langle \sum_{n=0}^{\infty} \mathfrak{g}_n(s,\rho,\omega), \rho\varphi(\rho,\omega)\Big\rangle \\
&= -e^{-st_0}\varphi(\rho_0,\omega_0) = -e^{-st_0}\widetilde{\varphi}(x_0,y_0) \\
&= \big\langle -e^{-st_0}\widetilde{Q}(\delta(x-x_0)\delta(y-y_0)), \widetilde{\varphi}(x,y)\big\rangle
\end{aligned}$$

and so $\widetilde{z}(x,y,s) = -e^{-st_0}\widetilde{Q}(\delta(x-x_0)\delta(y-y_0))$ in $H^{-(1+\varepsilon)}(D)$.

c) For every $m \in \mathbb{N} \cup \{0\}$, as $\widetilde{\mathfrak{g}}_m(s,x,y) \in \Xi^{-(1+\varepsilon)}(D) \subset \big(H^{1+\varepsilon}(D)\big)'$ (Lemma 2.4), by $\big[$[6], theorem 13 and corollary 14$\big]$ there is $\widetilde{V}_m \in H^{-(\varepsilon+2\xi)}(D)$ verifying $\widetilde{\mathcal{X}}_s(\widetilde{V}_m) = \widetilde{\mathfrak{g}_m}$. On the other hand, let $\widetilde{S}_m(s,x,y) := \sum_{j=1}^{m} \widetilde{\mathfrak{g}}_j(s,x,y)$. Assume that the sentence c) is false. Then, there is a subsequence $\{\widetilde{S}_{m_k}\}_{m=1}^{\infty}$ and $\alpha > 0$ such that $\big\|\widetilde{S}_{k_m} + e^{-st_0}\delta(x-x_0)\delta(y-y_0)\big\| \geq \alpha$ for every $m \in \mathbb{N}$. By part b), the sequence $\{\widetilde{Q}(\widetilde{S}_{k_m})\}_{m=1}^{\infty}$ is convergent in $H^{-(1+\varepsilon)}(D)$, and using the argumentation of $\big[$[3], §14, 4. (3)$\big]$, we find a subsequence (again denoted by $\{\widetilde{S}_{k_m}\}_{m=1}^{\infty}$) and elements $\{\Phi_m\}_{m=1}^{\infty} \subset \big(H^{1+\varepsilon}(D)\big)'$ such that $\widetilde{Q}(\Phi_m) = \widetilde{Q}(\widetilde{S}_{k_m})$, there exists $\Phi := \lim_{m\to\infty} \Phi_m$ in the topology of $\big(H^{1+\varepsilon}(D)\big)'$, and moreover,

$$\widetilde{Q}(\Phi) = \lim_{m\to\infty} \widetilde{Q}(\widetilde{S}_{k_m}) = -e^{-st_0}\widetilde{Q}(\delta(x-x_0)\delta(y-y_0)). \tag{2.19}$$

By $\big[$[6], theorem 9 and corollary 14$\big]$, we have $\widetilde{\mathcal{X}}_s^{-1}(\Phi) = \lim_{m\to\infty} \widetilde{\mathcal{X}}_s^{-1}(\Phi_m)$ in $H^{-(\varepsilon+2\xi)}(D)$. But, given $\widetilde{\varphi}(x,y) \in \mathcal{D}(D)$, by $\big[$[6], proposition 8, part a) and corollary 14$\big]$, one has $(\widetilde{\mathcal{X}}_s^{-1})'(\widetilde{\varphi}) \in H_0^{1+\varepsilon+2\xi}(D) \subset H_0^{1+\varepsilon}(D)$ and then there is a sequence $\{\widetilde{\nu}_j\}_{j=1}^{\infty} \subset \mathcal{D}(D)$ such that $(\widetilde{\mathcal{X}}_s^{-1})'(\widetilde{\varphi}) = \lim_{j\to\infty} \widetilde{\nu}_j$ in $H_0^{1+\varepsilon+2\xi}(D) \subset H^{1+\varepsilon}(D)$. As a consequence,

$$\begin{aligned}
\big\langle \widetilde{S}_{k_m}, (\widetilde{\mathcal{X}}_s^{-1})'(\widetilde{\varphi})\big\rangle &= \lim_{j\to\infty}\big\langle \widetilde{S}_{k_m}, \widetilde{\nu}_j\big\rangle = \lim_{j\to\infty}\big\langle \widetilde{Q}(\widetilde{S}_{k_m}), \widetilde{\nu}_j\big\rangle = \lim_{j\to\infty}\big\langle \widetilde{Q}(\Phi_m), \widetilde{\nu}_j\big\rangle \\
&= \lim_{j\to\infty}\big\langle \Phi_m, \widetilde{\nu}_j\big\rangle = \big\langle \Phi_m, (\widetilde{\mathcal{X}}_s^{-1})'(\widetilde{\varphi})\big\rangle
\end{aligned}$$

and then

$$\big\langle \widetilde{\mathcal{X}}_s^{-1}(\widetilde{S}_{k_m}), \widetilde{\varphi}\big\rangle = \big\langle \widetilde{S_{k_m}}, (\widetilde{\mathcal{X}}_s^{-1})'(\widetilde{\varphi})\big\rangle = \big\langle \Phi_m, (\widetilde{\mathcal{X}}_s^{-1})'(\widetilde{\varphi})\big\rangle = \big\langle \widetilde{\mathcal{X}}_s^{-1}(\Phi_m), \widetilde{\varphi}\big\rangle.$$

It follows $\widetilde{\mathcal{X}}_s^{-1}(\Phi_m) = \widetilde{\mathcal{X}}_s^{-1}(\widetilde{S}_{k_m}) = \sum_{j=1}^{k_m} \widetilde{V}_j \in H^{-(\varepsilon+2\xi)}(D)$, and by the action of $\widetilde{\mathcal{X}}_s$, we obtain $\Phi_m = \widetilde{S}_{k_m}$ and so $\Phi = \lim_{m\to\infty}\Phi_m = \lim_{m\to\infty}\widetilde{S}_{k_m}$. But by (2.19), the same argumentation as above working with Φ and

$-e^{-st_0}\delta(x-x_0)\delta(y-y_0)$ instead of Φ_m and $\widetilde{S}_{k_m}$ shows that $\Phi = -e^{-st_0}\delta(x-x_0)\delta(y-y_0)$, a contradiction with the definition of $\{\widetilde{S}_{k_m}\}_{m=1}^{\infty}$. □

An immediate consequence of c) and $\big[$[6], theorem 9$\big]$ is that

$$\widetilde{\mathcal{L}}(s,x,y) = \mathfrak{L}\big[\widetilde{G}(x,y,t|x_0,y_0,t_0)\big](s,x,y) = \sum_{n=1}^{\infty} \widetilde{V}_n(s,x,y) \text{ in } H^{-(\varepsilon+2\xi)}(D). \tag{2.20}$$

Moreover, by (2.15) we obtain for every $n \in \mathbb{N} \cup \{0\}$

$$\mathcal{X}_s(V_n(s,\rho,\omega)) = \mathfrak{g}_n(s,\rho,\omega) = \mathcal{X}_s(\mathcal{L}_n(s,\rho)\cos\frac{n}{2}\omega) = \mathcal{V}_s\big(\mathcal{L}_n(s,\rho,\omega)\big)\cos\frac{n}{2}\omega$$

and hence, by Lemma 2.4 and $\big[$[6], theorem 13 and corollary 14$\big]$

$$\mathcal{L}_n(\rho,\omega)\cos\frac{n}{2}\omega = \mathcal{X}_s^{-1}(\mathfrak{g}_n(s,\rho,\omega)) = V_n(s,\rho,\omega). \tag{2.21}$$

2.3. Computation of $\mathcal{L}_n(s,\rho)$

(2.15) is a non homogeneous modified Bessel equation of order $\frac{n}{2}$, $n \in \mathbb{N}\cup\{0\}$. The general solution of the associated homogeneous equation is

$$C_1 I_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho) + C_2 K_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho).$$

According to $\big[$[4], chapter 14, section 4.6, example 4$\big]$, the solution of (2.15) must be of the form

$$\mathcal{L}_n(s,\rho) = \begin{cases} M_n(s,\rho_0)\, I_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho) + N_n(s,\rho_0)\, K_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho) & \text{if} \quad 0<\rho<\rho_0 \\ R_n(s,\rho_0)\, I_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho) + S_n(s,\rho_0)\, K_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho) & \text{if} \quad \rho_0<\rho<1 \end{cases} \tag{2.22}$$

where the functions $M_n(s,\rho_0), N_n(s,\rho_0), R_n(s,\rho_0)$ and $S_n(s,\rho_0)$ must verify the conditions

a) continuity of $\mathcal{L}_n(s,\rho)$ in $\rho=\rho_0$:

$$\begin{aligned} &M_n(s,\rho_0)\, I_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho_0) + N_n(s,\rho_0)\, K_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho_0) \\ &\quad = R_n(s,\rho_0) I_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho_0) + S_n(s,\rho_0) K_{\frac{n}{2}}(\sqrt{s^2+2s}\,\rho_0) \end{aligned} \tag{2.23}$$

b) boundary condition $\frac{\partial \mathcal{L}_n}{\partial \rho}(s,1)=0$:

$$R_n(s,\rho_0) I'_{\frac{n}{2}}(\sqrt{s^2+2\,s}) + S_n(s,\rho_0) K'_{\frac{n}{2}}(\sqrt{s^2+2\,s}) = 0 \tag{2.24}$$

c) jump of the derivative of $\mathcal{L}_n(s,\rho)$ in $\rho=\rho_0$:

$$\begin{aligned} &R_n(s,\rho_0) I'_{\frac{n}{2}}(\sqrt{s^2+2\,s}\,\rho_0) + S_n(s,\rho_0) K'_{\frac{n}{2}}(\sqrt{s^2+2\,s}\,\rho_0) \\ &\quad - M_n(s,\rho_0) I'_{\frac{n}{2}}(\sqrt{s^2+2\,s}\,\rho_0) - N_n(s,\rho_0) K'_{\frac{n}{2}}(\sqrt{s^2+2\,s}\,\rho_0) \\ &= \frac{C_n(\omega_0)}{\rho_0\sqrt{s^2+2\,s}}\, e^{-s\,t_0} \end{aligned} \tag{2.25}$$

where $C_n(\omega_0) = -\frac{1}{\pi}$ if $n=0$ and $C_n(\omega_0) = -\frac{2\cos\frac{n}{2}\omega_0}{\pi}$ if $n \in \mathbb{N}$.

d) Since there are four unknown functions, we need another condition. Choosing $\kappa > 0$ such that $x_0^2 + y_0^2 > \kappa^2$, by (2.1) the restriction $\mathcal{L}_\kappa\big[\widetilde{G}\big]$ to $B_\kappa(0,0)$ of $\mathcal{L}\big[\widetilde{G}\big]$ verifies the equation

$$\frac{\partial^2 \mathcal{L}_\kappa[\widetilde{G}]}{\partial x^2} + \frac{\partial^2 \mathcal{L}_\kappa[\widetilde{G}]}{\partial y^2} - (s^2 + 2\,s)\mathcal{L}_\kappa[\widetilde{G}] = 0 \quad \text{in} \quad B_\kappa(0,0).$$

Since this equation is elliptic in $B_\kappa(0,0)$, by $\big[$[5], chapter 2, theorem 3.2$\big]$, $\mathcal{L}_\kappa[\widetilde{G}]$ must be continuous in $B_\kappa(0,0)$. Then, $\mathcal{L}(s,\rho,\omega)$ can be continuously extended to $[0,\kappa[\times[0,2\pi]$ and must be bounded in $\big[0,\frac{\kappa}{2}\big]\times[0,2\pi]$. By (2.11), for every $n \in \mathbb{N}\cup\{0\}$, we deduce $\sup_{\rho\in]0,\frac{\kappa}{2}]}|\mathcal{L}_n(s,\rho)| < \infty$, and by (2.22), (1.8) and (1.10), we obtain necessarily $N_n = 0$ for every $n \in \mathbb{N}\cup\{0\}$.

Using $\big[$[1], 9.6.15$\big]$ and solving the system (2.23), (2.24), (2.25), we obtain

$$\begin{aligned} S_n(s,\rho_0) &= -\,C_n(\omega_0)\; e^{-s\;t_0}\; I_{\frac{n}{2}}(\rho_0\sqrt{s^2+2s}),\\ M_n(s,\rho_0) &= -\,C_n(\omega_0)\; e^{-s\;t_0}\\ &\quad\times\left(K_{\frac{n}{2}}(\rho_0\sqrt{s^2+2s}) - I_{\frac{n}{2}}(\rho_0\sqrt{s^2+2s})\frac{K'_{\frac{n}{2}}(\sqrt{s^2+2s})}{I'_{\frac{n}{2}}(\sqrt{s^2+2s})}\right),\\ R_n(s,\rho_0) &= C_n(\omega_0)\; e^{-s\;t_0} I_{\frac{n}{2}}\left(\rho_0\sqrt{s^2+2s}\right)\frac{K'_{\frac{n}{2}}(\sqrt{s^2+2s})}{I'_{\frac{n}{2}}(\sqrt{s^2+2s})}. \end{aligned}$$

Hence, if for every $n \in \mathbb{N}\cup\{0\}$, we define

$$\begin{aligned} F_n(z,u,v) \;&=\; I_{\frac{n}{2}}(uz)\left(-K_{\frac{n}{2}}(vz) + I_n(vz)\frac{K'_{\frac{n}{2}}(z)}{I'_{\frac{n}{2}}(z)}\right),\\ f_n(s,u,v) &:= F_n(\sqrt{s^2+2s},u,v) \end{aligned} \tag{2.26}$$

we obtain

$$\mathcal{L}_n(s,\rho) := \begin{cases} L_{1n}(s,\rho,\rho_0) := e^{-st_0}C_n(\omega_0)f_n(s,\rho,\rho_0) & \text{if} \quad 0 \le \rho \le \rho_0\\ L_{2n}(s,\rho,\rho_0) := e^{-st_0}C_n(\omega_0)f_n(s,\rho_0,\rho) & \text{if} \quad \rho \le \rho_0 \le 1. \end{cases} \tag{2.27}$$

2.4. Computation of $\mathfrak{L}\big[\widetilde{G}(x,y,t|0,0,t_0)\big](s,x,y)$

Now we are going to compute the limit (2.5).

Lemma 2.6. *If $s > 0$ and $n \in \mathbb{N}$, there is $C(s) > 0$ such that* $\big|L_{1n}(s,\rho,\rho_0)\big| \le \frac{C(s)}{n}\left(\frac{\rho^n}{\rho_0^n} + \rho^n\rho_0^n\right)$.

Proof. Write $\psi(s) := \sqrt{s^2+2\,s}$ in order to simplify. Given $s > 0$, let

$$a_n(s,v) := \sqrt{1+\frac{4v^2\psi(s)^2}{n^2}},\quad b_n(s,v) := \log\frac{2v\psi(s)}{n\left(1+\sqrt{1+\frac{4v^2\psi(s)^2}{n^2}}\right)}$$

and $\tau_n(s,v) := a_n(s,v) + b_n(s,v)$. If $0 < u < v \leq 1$, we have

$$a_n(s,v) - a_n(s,u) = \frac{4(v^2-u^2)\psi(s)^2}{n^2\big(a_n(s,u)+a_n(s,v)\big)}$$

and

$$\begin{aligned}\left(\frac{1+a_n(s,v)}{1+a_n(s,u)}\right)^{\frac{n}{2}} &= \left(1+\frac{a_n(s,v)-a_n(s,u)}{1+a_n(s,u)}\right)^{\frac{n}{2}}\\ &\leq \left(1+2\frac{(v-u)\psi(s)}{n}\right)^{\frac{n}{2}} \leq \left(1+\frac{2\psi(s)}{n}\right)^{\frac{n}{2}}.\end{aligned}$$

Since $\lim_{n\to\infty}\left(1+\frac{2\psi(s)}{n}\right)^{\frac{n}{2}} = e^{\psi(s)}$, there are $C_1(s) > 0, C_2(s) > 0$ independent of u, v and n such that

$$\begin{aligned}e^{\frac{n}{2}\left(\tau_n(s,u)-\tau_n(s,v)\right)} &= e^{\frac{n}{2}\left(a_n(s,u)-a_n(s,v)\right)}\left(\frac{u(1+a_n(s,v))}{v(1+a_n(s,u))}\right)^{\frac{n}{2}}\\ &\leq \left(\frac{u}{v}\right)^{\frac{n}{2}} C_1(s)\big(e^{a_n(s,u)-a_n(s,v)}\big)^{\frac{n}{2}} \leq C_2(s)\left(\frac{u}{v}\right)^{\frac{n}{2}}.\end{aligned}$$

By the uniform asymptotic expansions [[1], 9.7.7, 9.7,8, 9.7.9, 9.7.10, 9.7.11] and the general properties of such expansions [[7], chapter 1, section 8.1] there are $M_1(s) > 0, M_2(s) > 0$ independent of n such that

$$\begin{aligned}\left|I_{\frac{n}{2}}(\rho\psi(s))I_{\frac{n}{2}}(\rho_0\psi(s))\frac{K'_{\frac{n}{2}}(\psi(s))}{I'_{\frac{n}{2}}(\psi(s))}\right| &\leq M(s)\left|\frac{e^{\frac{n}{2}(\tau_n(s,\rho)+\tau_n(s,\rho_0))}\ e^{-n\tau_n(s,1)}}{n\sqrt{a_n(s,\rho)\ a_n(s,\rho_0)}}\right|\\ &\leq \frac{M_1(s)}{n}\ e^{\frac{n}{2}(\tau_n(s,\rho)-\tau_n(s,1))}\ e^{\frac{n}{2}(\tau_n(s,\rho_0)-\tau_n(s,1))} \leq \frac{M_2(s)}{n}(\rho\rho_0)^{\frac{n}{2}}.\end{aligned}$$

Analogously,

$$\left|I_{\frac{n}{2}}(\rho\psi(s))K_{\frac{n}{2}}(\rho_0\psi(s))\right| \leq \frac{M_3(s)}{n}\frac{e^{\frac{n}{2}(\tau_n(s,\rho)-\tau_n(s,\rho_0))}}{\sqrt{a_n(s,\rho)\ a_n(s,\rho_0)}} \leq \frac{M_4(s)}{n}\ \left(\frac{\rho}{\rho_0}\right)^{\frac{n}{2}}$$

and now the result is immediate. □

Proposition 2.7. *$\lim_{(x_0,y_0)\to(0,0)}\sum_{n=1}^{\infty}\widetilde{V_n}(s,x,y) = 0$ in $H^{-(\varepsilon+2\xi)}(D)$ for every $s > 0$ and $(x_0,y_0) \in D'$.*

Proof. Given $0 < \rho_0 < 1$, by (2.20) the series $\sum_{n=1}^{\infty}\widetilde{V_n}(s,x,y)$ is convergent in $H^{-(\varepsilon+2\,\xi)}(D)$. Remark that as $\lim_{x\to0^+}I_n(x)K_n(x) = 0$, by (2.27), (2.26) and the formula of change in variables in a double integral we see that $\sum_{n=1}^{k}\widetilde{V_n}(s,x,y) \in L^2(D)$ for every $k \in \mathbb{N}$. Then, there is $\widetilde{\varphi}_k(x,y) \in B_{H_0^{\varepsilon+2\,\xi}(D)}$ such that

$$\Big\|\sum_{n=1}^{\infty}\widetilde{V_n}(s,x,y)\Big\|_{H^{-\varepsilon-2\,\xi}(D)} = \lim_{k\to\infty}\Big|\int_D\sum_{n=1}^{k}\widetilde{V_n}(s,x,y)\widetilde{\varphi}_k(x,y)\,dx\,dy\Big|$$

and by Hölder's inequality and switching to polar coordinates, by (2.21)

$$\leq \lim_{k\to\infty} \Big\| \sum_{n=1}^{k} \widetilde{V_n}(s,x,y) \Big\|_{L^2(D)} \|\widetilde{\varphi}_k\|_{L^2(D)}$$

$$\leq \lim_{k\to\infty} \Big\| \sqrt{\rho} \sum_{n=1}^{k} \mathcal{L}_n(s,\rho)\ \cos\frac{n}{2}\omega \Big\|_{L^2(\Omega)}$$

and by Fubini's theorem and the orthogonality of the trigonometrical system

$$= \lim_{k\to\infty} \left(\sum_{n=1}^{k} \int_0^1 \rho \mathcal{L}_n(s,\rho)^2 d\rho \int_0^{2\pi} \cos^2\frac{n}{2}\ \omega\ d\omega \right)^{\frac{1}{2}}$$

$$= \lim_{k\to\infty} \left(\pi \sum_{n=1}^{k} \int_0^1 \rho \mathcal{L}_n(s,\rho)^2 d\rho \right)^{\frac{1}{2}}$$

and by Lemma 2.6, since $\lim_{\rho_0\to 0} \rho_0 \log \rho_0 = 0$, for some $C_1(s) > 0, C_2(s) > 0$, using Hölder's inequality

$$= \lim_{k\to\infty} \sqrt{\pi} \left(\sum_{n=1}^{k} \left(\int_0^{\rho_0} \rho \big| L_{1n}(s,\rho,\rho_0) \big|^2\ d\rho + \int_{\rho_0}^{1} \rho \big| L_{1n}(s,\rho_0,\rho) \big|^2\ d\rho \right) \right)^{\frac{1}{2}}$$

$$\leq \lim_{k\to\infty} C_1(s) \left(\sum_{n=1}^{k} \left(\int_0^{\rho_0} \frac{\rho^{2n+1}}{n^2} \left(\frac{1}{\rho_0^n} + \rho_0^n \right)^2\ d\rho \right.\right.$$

$$\left.\left. + \int_{\rho_0}^{1} \frac{\rho\rho_0^{2n}}{n^2} \left(\frac{1}{\rho^n} + \rho^n \right)^2\ d\rho \right) \right)^{\frac{1}{2}}$$

$$\leq \lim_{k\to\infty} C_1(s) \left(\sum_{n=1}^{k} \frac{\rho_0^2 + \rho_0^{4n+2}}{n^2(n+1)} + 2\rho_0^2 \left(\log\frac{1}{\rho_0} + \frac{1}{4} - \frac{\rho_0^4}{4} \right) \right.$$

$$\left. +2 \sum_{n=2}^{k} \frac{\rho_0^2}{n^2} \left(\frac{1-\rho_0^{2n-2}}{2n-2} + \frac{\rho_0^{2n-2}}{2n+2} \right) \right)^{\frac{1}{2}} \leq C_2(s) \sqrt{\rho_0} \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{\frac{1}{2}}.$$

Hence, $\lim_{(x_0,y_0)\to(0,0)} \Big\| \sum_{n=1}^{\infty} \widetilde{V_n}(s,x,y) \Big\|_{H^{-(\varepsilon+2\ \xi)}(D)} = 0.$ □

Proposition 2.8. *For every $s > 0$, if $(x_0, y_0) \in D'$, one has*

$$\lim_{(x_0,y_0)\to(0,0)} \widetilde{V_0}(s,x,y) = -\frac{e^{-st_0}}{\pi} \widetilde{h_0}(s,x,y) :$$

$$= -\frac{e^{-st_0}}{\pi} \Big(-K_0\big(\sqrt{x^2+y^2}\psi(s)\big)$$

$$-I_0\big(\sqrt{x^2+y^2}\psi(s)\big) \frac{K_1(\psi(s))}{I_1(\psi(s))} \Big)$$

in $H^{-(\varepsilon+2\ \xi)}(D)$.

Proof. Let $0 < \rho_0 < 1$. There is $\widetilde{\varphi}(x,y) \in B_{H_0^{\varepsilon+2\,\xi}(D)}$ such that

$$\begin{aligned}&\Big\|\widetilde{V_0}(s,x,y) + \frac{e^{-st_0}}{\pi}\widetilde{h_0}(s,x,y)\Big\|_{H^{-(\varepsilon+2\,\xi)}(D)}\\ &\quad = \Big|\int\limits_D \Big(\widetilde{V_0}(s,x,y) + \frac{e^{-st_0}}{\pi}\widetilde{h_0}(s,x,y)\Big)\widetilde{\varphi}(x,y)\,dx\,dy\Big|\end{aligned}$$

and by Hölder's inequality and switching to polar coordinates, by (2.21)

$$\begin{aligned}&\le \Big\|\widetilde{V_0}(s,x,y) + \frac{e^{-st_0}}{\pi}\widetilde{h_0}(s,x,y)\Big\|_{L^2(D)}\big\|\widetilde{\varphi}\big\|_{L^2(D)}\\ &\le \Big\|\sqrt{\rho}\Big(\mathcal{L}_0(s,\rho) + \frac{e^{-st_0}}{\pi}h_0(s,\rho)\Big)\Big\|_{L^2(\Omega)}\\ &= \frac{\sqrt{2\pi}}{\pi\ e^{st_0}}\left(\int\limits_0^{\rho_0}\rho\Big|f_0(s,\rho,\rho_0) - h_0(s,\rho)\Big|^2\,d\rho\right.\\ &\qquad\left. + \int\limits_{\rho_0}^1 \rho\Big|f_0(s,\rho_0,\rho) - h_0(s,\rho)\Big|^2\,d\rho\right)^{\frac{1}{2}}. \qquad (2.28)\end{aligned}$$

Having in mind the equalities $\lim_{\rho\to 0}\rho\log\frac{\rho\psi(s)}{2} = 0$, $\lim_{\rho\to 0}\rho\log^2\frac{\rho\psi(s)}{2} = 0$, by (1.7) we find $A_1(s) > 0$ such that

$$\begin{aligned}&B_1(\rho_0,s) := \int\limits_{\rho_0}^1 \rho\Big|f_0(s,\rho_0,\rho) - h_0(s,\rho)\Big|^2\,d\rho\\ &= \big(I_0(\rho_0\psi(s)) - 1\big)^2\int\limits_{\rho_0}^1\rho\left(K_0\big(\rho\psi(s)\big) + I_0\big(\rho\psi(s)\big)\frac{K_1(\psi(s))}{I_1(\psi(s))}\right)^2\,d\rho\\ &\le \big|\mathfrak{A}_0(\rho_0\psi(s))\big|\int\limits_{\rho_0}^1 \rho I_0\big(\rho\psi(s)\big)^2\left(-\log\frac{\rho\psi(s)}{2} + \frac{\mathfrak{K}_0(\rho\psi(s))}{I_0\big(\rho\psi(s)\big)} + \frac{K_1(\psi(s))}{I_1(\psi(s))}\right)^2 d\rho\\ &\le A_1(s)\big|\mathfrak{A}_0(\rho_0\psi(s))\big|\end{aligned}$$

and by (1.7) $\lim_{\rho_0\to 0} B_1(\rho_0,s) = 0$.

Analogously, using Hölder's inequality, (1.7) and (1.8), we find a constant $A_2(s) > 0$ such that

$$\begin{aligned}&B_2(\rho_0,s):\\ &\quad = \int\limits_0^{\rho_0}\rho\big|f_0(s,\rho,\rho_0) - h_0(s,\rho)\big|^2\,d\rho \le 2\int\limits_0^{\rho_0}\rho\Big(\big|f_0(s,\rho,\rho_0)\big|^2 + \big|h_0(s,\rho)\big|^2\Big)\,d\rho\end{aligned}$$

$$\leq 4\int_0^{\rho_0} \rho\left(\left|I_0(\rho\psi(s))K_0(\rho_0\psi(s))\right|^2 + \left|I_0(\rho\psi(s))I_0(\rho_0\psi(s))\frac{K_1(\psi(s))}{I_1(\psi(s))}\right|^2\right)\,d\rho$$
$$+4\int_0^{\rho_0} \rho\left(\left|K_0(\rho\psi(s))\right|^2 + \left|I_0(\rho\psi(s))\frac{K_1(\psi(s))}{I_1(\psi(s))}\right|^2\right)\,d\rho \leq \rho_0 A_2(s).$$

Then, $\lim_{\rho_0\to 0} B_2(\rho_0, s) = 0$, and the result follows from (2.28). □

2.5. Computation of $\widetilde{G}(x, y, t|0, 0, t_0)$

Clearly, it follows from (2.5), (2.20) and propositions 2.7 and 2.8

$$\mathfrak{L}\big[\widetilde{G}(x, y, t|0, 0, t_0)\big](s, x, y) = \mathfrak{L}\big[G(\rho, \omega, t|0, 0, t_0)\big](s, \rho, \omega) = -\frac{e^{-st_0}}{\pi}\,h_0(s, \rho) \tag{2.29}$$

and by the translation theorem, we only need to find $\mathcal{L}^{-1}\big[h_0(s, \rho)\big](\rho, t)$.

If we consider $s \in \mathbb{C}$, as $\psi(s) = \sqrt{s(s+2)}$ the complex function $h_0(s, \rho)$ has two branch points $s = 0$ and $s = -2$. We define a holomorphic branch of $h_0(s, \rho)$ on $\mathbb{C}\backslash]-\infty, 0]$ taking

$$s+2 = |s+2|e^{i\,Arg(s+2)},\ Arg(s+2) \in]-\pi, \pi[,\ s = |s|e^{i\,Arg\,s},\ Arg\,s \in]-\pi, \pi[.$$

With this selection of arguments, the function $\psi(s)$ turns out to be holomorphic in $\mathbb{C}\backslash[-2, 0]$ indeed and clearly $\psi(\overline{s}) = \overline{\psi(s)}$ for $s \in \mathbb{C}\backslash]-\infty, 0]$. Moreover, by $\big[$[1], 9.1.40 and 9.6.32$\big]$ and $\big[$[1], 9.1.27, 9.1.28, 9.6.26 and 9.6.27$\big]$, we have

$$\forall\, z \in \mathbb{C}\backslash]-\infty, 0] \quad \overline{Z_0(z)} = Z_0(\overline{z}), \tag{2.30}$$

$Z_0(z)$ being any of the functions $J_0(z), Y_0(z), I_0(z), K_0(z)$ or their derivatives.

By $\big[$[1], 9.1.28, 9.6.27 and 9.6.3$\big]$, one has $I_1(z) = I_0'(z) = i\,J_0'(z\,i) = -i\,J_1(z\,i)$. By $\big[$[10], section 15.25 $\big]$, all zeros of $J_1(z)$ are real and form a countable set. Let $\{\alpha_m\}_{m=1}^{\infty}$ be the set of non negative zeros of J_1 greater than 0. By $\big[$[1], 9.1.10$\big]$, the set of non null zeros of J_1 is $\{\pm\alpha_m\}_{m=1}^{\infty}$. Then, a point $s \in \mathbb{C}\backslash]-\infty, 0]$ will be a pole of $h_0(s, \rho)$ if and only if

$$i\sqrt{s^2+2s} = \pm\alpha_m \implies s := s_m := -1 \pm \sqrt{1-\alpha_m^2} = -1 \pm \sqrt{\alpha_m^2 - 1}\,i.$$

We will write $s_m^+ := -1 + \sqrt{\alpha_m^2-1}\,i$ and $s_m^- := -1 - \sqrt{\alpha_m^2-1}\,i = \overline{s_m^+}$.

Lemma 2.9. *Let* $0 < \rho \leq 1$ *and* $c > 0$. *Then,* $\lim_{|s|\to\infty, Re(s)\leq c} h_0(s, \rho) = 0$.

Proof. Assume $Re(s) \leq c$. We apply the asymptotic expansions for $|z| \to \infty, n \in \mathbb{N}\cup\{0\}$ $\big[$[10], §7.23, (1) and (2)$\big]$

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}}\,e^{-z}\sum_{k=0}^{\infty}\frac{a_k(n)}{z^k} \quad \text{if } |Arg\,z| \leq \frac{3\pi}{2} - \epsilon,\ \epsilon > 0. \tag{2.31}$$

and

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}(-1)^k\frac{a_k(n)}{z^k} \pm i\,e^{\pm n\pi i}\,\frac{e^{-z}}{\sqrt{2\pi z}}\sum_{k=0}^{\infty}\frac{a_k(n)}{z^k} \tag{2.32}$$

valid for $-\frac{\pi}{2}+\epsilon \leq \pm Arg\ z \leq \frac{3\pi}{2}-\epsilon,\ \epsilon>0$, respectively, and where $a_0(n)=1$ and

$$\forall\, k \in \mathbb{N} \quad a_k(n) = \frac{(4n^2-1^2)(4n^2-3^2)\cdots(4n^2-(2k-1)^2)}{k!\, 8^k}.$$

Assume $Im(s)>0$ (the argumentation in the case $Im(s)\leq 0$ is similar). If $-1\leq Re(s)\leq c$, we have $\cos(Arg\ \psi(s))\geq 0$ and by (2.32) and $\big[$[7], chapter 1, section 8.1$\big]$

$$\frac{I_0(\rho\psi(s))}{I_1(\psi(s)} \sim \frac{e^{-(1-\rho)\psi(s)}}{\rho}\, \frac{1+O\left(\frac{1}{\rho\psi(s)}\right)+ie^{-2\rho\psi(s)}\left(1+O\left(\frac{1}{\rho\psi(s)}\right)\right)}{1+O\left(\frac{1}{\psi(s)}\right)-ie^{-2\psi(s)}\left(1+O\left(\frac{1}{\psi(s)}\right)\right)}.$$

Then, by (2.31), it follows easily $\lim |s| \to \infty, Re(s)\in[-1,c]=0$.

In the case $Re(s)\leq -1$, we have $\cos(Arg\ \psi(s))<0$ and the proof is much more delicate. We need to use (2.32) and (2.31) in its full strength. By $\big[$[7], chapter 1, section 8.1$\big]$ and (2.31) and (2.32), we have

$$\begin{aligned}
&-K_0(\rho\psi(s))I_1(\psi(s)) - I_0(\rho\psi(s))K_1(\psi(s))\\
&\quad\sim \frac{e^{-(1+\rho)\psi(s)}}{2\sqrt{\rho}\psi(s)}\left(\sum_{k=0}^{\infty}\frac{a_k(0)}{(\rho\psi(s))^k}\right)\left(-e^{2\psi(s)}\left(\sum_{u=0}^{\infty}\frac{(-1)^u a_u(1)}{\psi(s)^u}\right)+i\sum_{u=0}^{\infty}\frac{a_u(1)}{\psi(s)^u}\right)\\
&\quad+\frac{e^{-(1+\rho)\psi(s)}}{2\sqrt{\rho}\psi(s)}\left(\sum_{k=0}^{\infty}\frac{a_k(1)}{\psi(s)^k}\right)\left(-e^{2\rho\psi(s)}\left(\sum_{u=0}^{\infty}\frac{(-1)^u a_u(0)}{(\rho\psi(s))^u}\right)-i\sum_{u=0}^{\infty}\frac{a_u(0)}{(\rho\psi(s))^u}\right)\\
&\quad=-\frac{e^{(1-\rho)\psi(s)}}{2\sqrt{\rho}\psi(s)}\left(\sum_{k=0}^{\infty}\frac{a_k(0)}{(\rho\psi(s))^k}\right)\left(\sum_{u=0}^{\infty}\frac{(-1)^u a_u(1)}{\psi(s)^u}\right)\\
&\quad-\frac{e^{(1-\rho)\psi(s)}}{2\sqrt{\rho}\psi(s)}\left(\sum_{k=0}^{\infty}\frac{a_k(1)}{\psi(s)^k}\right)\left(\sum_{u=0}^{\infty}\frac{(-1)^u a_u(0)}{(\rho\psi(s))^u}\right)
\end{aligned} \tag{2.33}$$

because changing the name of the summation indexes

$$\begin{aligned}
\left(\sum_{k=0}^{\infty}\frac{a_k(0)}{(\rho\psi(s))^k}\right)\left(\sum_{u=0}^{\infty}\frac{a_u(1)}{\psi(s)^u}\right) &= \sum_{k=0,u=0}^{\infty}\frac{a_k(0)a_u(1)}{\rho^k\psi(s)^{k+u}}\\
&=\left(\sum_{k=0}^{\infty}\frac{a_k(1)}{\psi(s)^k}\right)\left(\sum_{u=0}^{\infty}\frac{a_u(0)}{(\rho\psi(s))^u}\right).
\end{aligned}$$

Since ρ is fixed, one has

$$\left(1+O\left(\frac{1}{\psi(s)}\right)\right)\left(1+O\left(\frac{1}{\rho\psi(s)}\right)\right)=1+O\left(\frac{1}{\psi(s)}\right),$$

and once again by (2.33) and (2.32), the definition of asymptotic expansion implies that

$$h_0(s,\rho) = -\sqrt{\frac{\pi}{2\rho\psi(s)}}\,\frac{e^{(2-\rho)\psi(s)}\left(2+O\left(\frac{1}{\psi(s)}\right)\right)}{e^{2\psi(s)}\left(1+O\left(\frac{1}{\psi(s)}\right)\right)-i\left(1+O\left(\frac{1}{\psi(s)}\right)\right)}$$

and now the result follows easily. □

Given $c > 0$, by (2.31) and (2.32), there is $C(c) > 0$ such that

$$\left|h_0(c+y\,i,\rho)\right| \le \frac{C(c)}{|c+y\,i|^{\frac{1}{2}}} \Longrightarrow \left|\frac{h_0(c+y\,i,\rho)}{c+y\,i}\right| \le \frac{C(c)}{|c+y\,i|^{\frac{3}{2}}} \tag{2.34}$$

and by Lemma 2.9, (2.31) and (2.32)

$$\lim_{|s|\to\infty, Re(s)\in[-1,c]} \frac{h_0(s,\rho)}{s} = 0, \quad \lim_{s\to 0}\left|s\frac{h_0(s,\rho)}{s}\right| = \infty \text{ and } \lim_{s\to 0}|s\,h_0(s,\rho)| = 0. \tag{2.35}$$

By (2.34) $\mathfrak{L}^{-1}\left[\frac{h_0(s,\rho)}{s}\right](\rho,t)$ can be obtained by Bromwich's formula. By Lemma 2.9, as $h_0(s,\rho)$ has no pole with real part greater than c, we have $\mathfrak{L}^{-1}\left[\frac{h_0(s,\rho)}{s}\right](\rho,t) = 0$ if $t < 0$. However, if $t \ge 0$, by (2.35) $\mathfrak{L}^{-1}\left[\frac{h_0(s,\rho)}{s}\right](\rho,t)$ must be computed using the contour of Fig. 1 B, with a *fixed* $0 < R' < \frac{1}{2}$.

a): Residues. If $\alpha > 0$, by $\left[[1], 9.6.3 \text{ and } 9.1.10\right]$, one has $I_0(\alpha i) = J_0(\alpha) = I_0(-\alpha i)$, $I_1(\alpha i) = iJ_1(\alpha)$, $I_1(-\alpha i) = -iJ_1(\alpha)$, and by derivation, we obtain from $\left[[1], 9.6.3\right]$ $I_1'(\alpha i) = J_1'(\alpha)$. Moreover, by $\left[[1], 9.6.4 \text{ and } 9.1.4\right]$ we have $K_n(\alpha i) = -\frac{\pi}{2}\,i\,e^{-\frac{n\pi}{2}\,i}\big(J_n(\alpha) - i\,Y_n(\alpha)\big)$ for every $n \in \mathbb{N}\cup\{0\}$. Then, as $\psi(s_m^+) = \alpha_m\,i$ and $J_1(\alpha_m) = 0, m \in \mathbb{N}$, by L'Hôpital's rule we obtain

$$\begin{aligned}
\operatorname{Res}\left(e^{st}\frac{h_0(s,\rho)}{s}\right)_{s=s_{0,m}^+} &= \lim_{s\to s_m^+}(s-s_m^+)e^{st}\frac{h_0(s,\rho)}{s}\\
&= -e^{s_m^+t}\frac{I_0(\rho\alpha_m i)}{s_m^+}\,K_1(\alpha_m i)\lim_{s\to s_m^+}\frac{\psi(s)}{(s+1)I_1'(\psi(s))}\\
&= -\frac{\pi}{2}\,\frac{e^{s_m^+t}}{s_m^+}\,\frac{\alpha_m\,i}{\sqrt{\alpha_m^2-1}}\,\frac{J_0(\rho\alpha_m)}{J_1'(\alpha_m)}Y_1(\alpha_m)
\end{aligned}$$

Using (2.30), we obtain $\operatorname{Res}\left(e^{st}\frac{h_0(s,\rho)}{s}\right)_{s=s_m^-} = \overline{\operatorname{Res}\left(e^{st}\frac{h_0(s,\rho)}{s}\right)_{s=s_m^+}}$. Consequently,

$$\begin{aligned}
\mathfrak{R}_0(\rho,t) &:= \sum_{m=1}^{\infty}\left(\operatorname{Res}\left(e^{st}\frac{h_0(s,\rho)}{s}\right)_{s=s_m^+} + \operatorname{Res}\left(e^{st}\frac{h_0(s,\rho)}{s}\right)_{s=s_m^-}\right)\\
&= -\pi e^{-t}\sum_{m=1}^{\infty}\mathfrak{h}_{1m}(\rho,t):\\
&= -\pi e^{-t}\sum_{m=1}^{\infty}\frac{J_0(\rho\alpha_m)Y_1(\alpha_m)}{\alpha_m\sqrt{\alpha_m^2-1}J_1'(\alpha_m)}\Big(\sqrt{\alpha_m^2-1}\cos\sqrt{\alpha_m^2-1}t\\
&\quad + \sin\sqrt{\alpha_m^2-1}t\Big)
\end{aligned} \tag{2.36}$$

b): Integrals along $L_1^\infty(\epsilon,R)$ and $L_2^\infty(\epsilon,R)$. Remark that given $\alpha > 0$ and $n \in \mathbb{N}\cup\{0\}$, by $\left[[1], 9.6.4 \text{ and } 9.1.4\right]$, one has

$$\lim_{x\to\alpha,\epsilon\to 0^+} K_n(-x+i\,\epsilon) = -\frac{\pi}{2}\,e^{-\frac{n\pi}{2}\,i}\big(J_n(\alpha i) - iY_n(\alpha i)\big),$$

by $\left[[1], 9.6.5\right]$ $Y_n(\alpha\,i) = e^{\frac{(n+1)\pi}{2}\,i}I_n(\alpha) - \frac{2}{\pi}\,e^{-\frac{n\pi}{2}\,i}K_n(\alpha)$ and by $\left[[1], 9.6.3\right]$ $J_n(\alpha\,i) = e^{\frac{n\pi}{2}\,i}I_n(\alpha)$. Then, by (2.30) and $\left[[1], 9.6.10\right]$, defining $\vartheta :=$

$|x(x+2)|$, we obtain

$$\begin{aligned}\mathfrak{I}_1(\rho,t) &:= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\left(\int\limits_{L_1^\infty(\epsilon,R)} e^{st}\frac{h_0(s,\rho)}{s}\,ds + \int\limits_{L_2^\infty(\epsilon,R)} e^{st}\frac{h_0(s,\rho)}{s}\,ds\right)\\ &= \frac{1}{2}\int\limits_{-\infty}^{-2}\frac{e^{xt}}{x}Im\Big(i\,J_0(\rho\sqrt{\vartheta}\,i)+Y_0(\rho\sqrt{\vartheta}\,i)\\ &\quad+\frac{I_0(-\rho\sqrt{\vartheta})}{I_1(-\sqrt{\vartheta})}\big(J_1(\sqrt{\vartheta}\,i)-iY_1(\sqrt{\vartheta}\,i)\big)\Big)\,dx\\ &= \int\limits_{-\infty}^{-2}\frac{e^{xt}}{x}\left(I_0(\rho\sqrt{\vartheta})-\frac{I_0(\rho\sqrt{\vartheta})}{I_1(\sqrt{\vartheta})}I_1(\sqrt{\vartheta})\right)\,dx=0.\end{aligned}$$

c): Integrals along $L_3(\epsilon,R')$ and $L_4(\epsilon,R')$). Analogously to part a), we obtain

$$\begin{aligned}\mathfrak{I}_2(\rho,t) &:= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\left(\int\limits_{L_3(\epsilon,R')} e^{st}\frac{h_0(s,\rho)}{s}\,ds + \int\limits_{L_4(\epsilon,R')} e^{st}\frac{h_0(s,\rho)}{s}\,ds\right)\\ &= \frac{-1}{\pi}\int\limits_{-2}^{-R'} Im\left(\frac{e^{xt}}{x}\left(K_0(\rho\sqrt{\vartheta}\,i)+\frac{I_0(\rho\sqrt{\vartheta}\,i)}{I_1(\sqrt{\vartheta}\,i)}\,K_1(\sqrt{\vartheta}\,i)\right)\right)\,dx\\ &= -\frac{1}{2}\int\limits_{-2}^{-R'}\frac{e^{x\,t}}{x}\left(-J_0(\rho\sqrt{\vartheta})+\frac{J_0(\rho\sqrt{\vartheta})}{J_1(\sqrt{\vartheta})}J_1(\sqrt{\vartheta})\right)\,dx=0.\end{aligned}$$

Finally, by Bromwich's formula, Lemma 2.9, Jordan's Lemma and Cauchy's theorem, we arrive at

$$\mathfrak{L}^{-1}\left[\frac{h_0(s,\rho)}{s}\right](\rho,t)=\mathfrak{R}_0(t)+\frac{1}{2\pi i}\int\limits_{-\pi}^{\pi} e^{R'e^{\theta\,i}t}h_0(R'e^{\theta\,i},\rho)\,i\,d\theta \tag{2.37}$$

It follows that

$$\begin{aligned}\mathfrak{L}^{-1}\big[h_0(s,\rho)\big](\rho,t) &= \frac{\partial}{\partial t}\mathfrak{L}^{-1}\left[\frac{h_0(s,\rho)}{s}\right](\rho,t)\\ &= \frac{\partial\mathfrak{R}_0}{\partial t}(\rho,t)+\frac{1}{2\pi i}\int\limits_{-\pi}^{\pi} R'e^{\theta\,i}e^{R'e^{\theta\,i}t}h_0(R'e^{\theta\,i},\rho)\,i\,d\theta\\ &= \frac{\partial\mathfrak{R}_0}{\partial t}(\rho,t)+\frac{1}{2\pi i}\int\limits_{\Gamma(R')} e^{st}h_0(s,\rho)ds\end{aligned}$$

where $\Gamma(R')$ is the circumference centered in $s=0$ with radius R' oriented in the positive sense. Since this equality holds for every $0<R'<\frac{1}{2}$, by (2.35) we obtain

$$\mathfrak{L}^{-1}\big[h_0(s,\rho)\big](\rho,t) = \frac{\partial \mathfrak{R}_0}{\partial t}(\rho,t) + \lim_{R'\to 0}\frac{1}{2\pi i}\int\limits_{\Gamma(R')} e^{st}h_0(s,\rho)\,ds = \frac{\partial \mathfrak{R}_0}{\partial t}(\rho,t)$$

$$= -\,\mathfrak{R}_0(\rho,t) + \pi e^{-t}\sum_{m=1}^{\infty}\frac{J_0(\rho\alpha_m)Y_1(\alpha_m)}{\alpha_m J_1'(\alpha_m)}$$

$$\times\left(\sqrt{\alpha_m^2-1}\sin\sqrt{\alpha_m^2-1}\,t - \cos\sqrt{\alpha_m^2-1}\,t\right) \quad (2.38)$$

in the sense of distributions. Then, by (2.29) and the second translation theorem for Laplace transforms

$$G(\rho,\omega,t\big|0,0,t_0) = -\frac{H(t-t_0)}{\pi}\,\frac{\partial \mathfrak{R}_0}{\partial t}(\rho,t-t_0)$$

$$= -H(t-t_0)e^{-(t-t_0)}\sum_{j=1}^{3}\sum_{m=1}^{\infty}\mathfrak{h}_{jm}(\rho,\omega,t-t_0) \quad (2.39)$$

where $\mathfrak{h}_{1m}$ has been defined in (2.36) and

$$\mathfrak{h}_{2m}(\rho,\omega,t) := \frac{J_0(\rho\,\alpha_m)Y_1(\alpha_m)}{\alpha_m J_1'(\alpha_m)}\sqrt{\alpha_m^2-1}\sin\sqrt{\alpha_m^2-1}\,t$$

and

$$\mathfrak{h}_{3m}(\rho,\omega,t) := \frac{J_0(\rho\,\alpha_m)Y_1(\alpha_m)}{\alpha_m J_1'(\alpha_m)}\cos\sqrt{\alpha_m^2-1}\,t.$$

3. The Counterexample

To establish our counterexample, we need some information about the pointwise convergence of the latter series:

Proposition 3.1. *a) The series* $\sum_{m=1}^{\infty}\mathfrak{h}_{1m}(\rho,t-t_0)$ *and* $\sum_{m=1}^{\infty}\mathfrak{h}_{3m}(\rho,t-t_0)$ *are pointwise absolutely convergent in* $\mathbb{R}\times\mathbb{R}$.

b) $\sum_{m=1}^{\infty}\mathfrak{h}_{2m}(\rho,t-t_0)$ *is pointwise convergent in every point* $(\rho,t)\in\,]0,\infty[\times\mathbb{R}$ *such that for every* $k\in\mathbb{N}\cup\{0\}$ *the conditions*

$$\rho+t-t_0\neq 2k,\quad \rho-(t-t_0)\neq 2k,\quad -\rho+t-t_0\neq 2k,\quad -\rho-(t-t_0)\neq 2k \quad (3.1)$$

hold. In these points, the equality (2.38) *holds in the ordinary sense.*

Proof. a) From McMahon's asymptotic expression $\big[$[1], §9.5.13 $\big]$ of the zeros $\{\alpha_m\}_{m=1}^{\infty}$, one has

$$\alpha_m = \left(m-\frac{3}{4}\right)\pi + O\left(\frac{1}{m}\right). \quad (3.2)$$

Then, by the asymptotic expansions $\big[[1]$, 9.2.2 and 9.2.11$\big]$, there is $M_1 > 0$ such that for large m

$$\left|1 - \frac{Y_1(\alpha_m)}{J_1'(\alpha_m)}\right| = \frac{\left|O\left(\frac{1}{\alpha_m}\right)\right|}{\left|\cos\left(\alpha_m - \frac{\pi}{4}\right) + O\left(\frac{1}{\alpha_m}\right)\right|} \leq \frac{\left|O\left(\frac{1}{\alpha_m}\right)\right|}{\left|\cos\left(\alpha_m - \frac{\pi}{4}\right)\right| - \left|O\left(\frac{1}{\alpha_m}\right)\right|}$$
$$\leq \frac{\left|O\left(\frac{1}{m}\right)\right|}{\left|(-1)^{m-1}\cos\left(O\left(\frac{1}{m}\right)\right)\right| - \left|O\left(\frac{1}{m}\right)\right|} \leq \frac{M_1}{m}. \tag{3.3}$$

Using $\big[[1]$, 9.5.2 $\big]$, it follows that there are $\mathfrak{M}_1 > 0, \mathfrak{M}_2 > 0$ such that

$$\forall\, m \in \mathbb{N} \quad \mathfrak{M}_1 \leq \left|\frac{Y_1(\alpha_m)}{J_1'(\alpha_m)}\right| \leq \mathfrak{M}_2. \tag{3.4}$$

It follows easily from the asymptotic expansion $\big[[1]$, 9.2.1$\big]$ of $J_0(\alpha_m\rho)$ and (3.4) that the series $\sum_{m=1}^{\infty} \mathfrak{h}_{1m}(\rho, t - t_0)$ is pointwise absolutely convergent in $\mathbb{R} \times \mathbb{R}$ and uniformly convergent in a suitable neighborhood of every $(\rho, t) \in]0, \infty[\times\mathbb{R}$ and $\sum_{m=1}^{\infty} \mathfrak{h}_{3m}(\rho, t - t_0)$ is pointwise absolutely convergent in $\mathbb{R} \times \mathbb{R}$.

b) By the asymptotic expansion $\big[[1]$, 9.2.1$\big]$ of $J_0(\alpha_m\rho)$ and (3.4), it is easy to see that $\sum_{m=1}^{\infty} \mathfrak{h}_{2m}(\rho, t - t_0)$ has the same character that the series

$$\sum_{m=1}^{\infty} \frac{\cos(\rho\alpha_m - \frac{\pi}{4})}{\sqrt{\alpha_m} J_1'(\alpha_m)} Y_1(\alpha_m) \sin\alpha_m\ (t - t_0)$$

and

$$\sum_{m=1}^{\infty} \frac{\cos(\rho m\pi + \sigma)}{\sqrt{\alpha_m} J_1'(\alpha_m)} Y_1(\alpha_m) \sin(m\pi\ (t - t_0) + \varsigma) \tag{3.5}$$

where $\sigma := -(3\rho + 1)\frac{\pi}{4}$ and $\varsigma := \left(-\frac{3\pi}{4}\right)(t - t_0)$. Now, by (3.3) and (3.2), it turns out that (3.5) has the convergence character of the series

$$\sum_{m=1}^{\infty} \frac{\cos(\rho m\pi + \sigma)}{\sqrt{\alpha_m}} \sin(m\pi\ (t - t_0) + \varsigma) = \sum_{j=1}^{2} \sum_{m=1}^{\infty} \frac{1}{2\sqrt{\alpha_m}} \sin(a_j m\pi + b_j) \tag{3.6}$$

where we have written $a_1 := \rho + (t - t_0), b_1 := \sigma + \varsigma, a_2 := -\rho + t - t_0, b_2 := -(\sigma - \varsigma)$. Finally, for every $j = 1, 2$, we have

$$\left|\sum_{m=1}^{h} \sin(a_j m\pi + b_j)\right| = \left|Im\left(e^{ib_j} \sum_{m=1}^{h} e^{a_j m\pi i}\right)\right|$$
$$\leq \left|e^{ib_j} \frac{e^{a_j\pi\, i}(e^{a_j\pi h\, i} - 1)}{e^{a_j\pi\, i} - 1}\right| \leq \frac{1}{\sqrt{2}\sqrt{1 - \cos(a_j\pi)}}. \tag{3.7}$$

By (3.2) and Abel's criterion, the series (3.6) are convergent in the points verifying (3.1). Moreover, it is clear that for every such point (ρ, t) there is a suitable neighborhood W of t such that the partial sums (3.7) are uniformly bounded with respect to $t \in W$. Then, (3.6) are uniformly convergent series in W, and the conclusion follows easily. □

Lemma 3.2. *The series*

$$S_1(t) = \sum_{m=1}^{\infty} \frac{\cos(2\sqrt{\alpha_m^2 - 1}\; t)}{\alpha_m},\; S_2(t) = \sum_{m=1}^{\infty} \frac{\cos(2\alpha_m\; t)}{\alpha_m},$$
$$S_3 = \sum_{m=1}^{\infty} \frac{\cos 2\left(m\pi - \frac{3\pi}{4}\right)\; t}{\alpha_m} \tag{3.8}$$

are convergent for every $t \neq k \in \mathbb{Z}$.

Proof. By (3.2), it is elementary to see that $S_1(t)$ has the same character that the series $S_2(t)$. By (3.4)

$$\sum_{m=1}^{\infty} \frac{\cos(2\alpha_m\; t) - \cos 2\left(m\pi - \frac{3\pi}{4}\right) t}{\alpha_m}$$
$$= -2 \sum_{m=1}^{\infty} \frac{\sin \eta_m\; t}{\alpha_m} \sin\left(\alpha_m\; t + \left(m\pi - \frac{3\pi}{4}\right) t\right)$$

where $\left|\eta_m\right| \leq M/(m\pi - \frac{3}{4}\pi)$ for some $M > 0$. This implies that the last series is absolutely convergent, and so, $S_2(t)$ has the same character that the series $S_3(t)$. But

$$\forall\; h \in \mathbb{N} \quad \left|\sum_{m=1}^{h} \cos 2\left(m\pi - \frac{3\pi}{4}\right) t\right| = \left|Re\left(e^{-\frac{3\pi}{2} t\; i} \sum_{m=1}^{h} e^{2m\pi\; t\; i}\right)\right|$$
$$\leq \left|e^{-\frac{3\pi}{2} t\; i}\; \frac{e^{2\pi\; t\; i}(e^{2\pi\; t\; h\; i} - 1)}{e^{2\pi\; t\; i} - 1}\right| \leq \frac{1}{\sqrt{2}\sqrt{1 - \cos(2\pi\; t)}} \tag{3.9}$$

and by Abel's criterion, the series $S_3(t)$ converges if $t \neq k \in \mathbb{Z}$. □

Proposition 3.3. *For every $t \neq k \in \mathbb{N} \cup \{0\}$, the function $\widetilde{G}(x, y, t\big|0, 0, t_0)$ does not belong to $L^2(D)$ and hence $\widetilde{G}(x, y, t\big|0, 0, t_0) \notin L^2(D \times [0, T])$, whatever be $T > 0$.*

Proof. By (2.39) and proposition 3.1, $\widetilde{G}(x, y, t\big|0, 0, t_0)$ is defined (except in a numerable set) by the sum of three pointwise convergent series. So, almost everywhere for $t > t_0$, one has

$$\left\|\widetilde{G}(x, y, t\big|0, 0, t_0)\right\|_{L^2(D)}$$
$$\geq e^{-(t-t_0)} \left(\left\|\sum_{m=1}^{\infty} \widetilde{\mathfrak{h}}_{2m}(x, y, t - t_0)\right\|_{L^2(D)} - \sum_{j=1, j=3} \left\|\sum_{m=1}^{\infty} \widetilde{\mathfrak{h}}_{jm}(x, y, t - t_0)\right\|_{L^2(D)}\right).$$

By (3.4), the asymptotic expansion $\big[$[1], 9.2.1$\big]$ of $J_0(\alpha_m \rho)$) and the orthogonality of functions $\{\rho J_0(\alpha_m \rho)\}_{m=1}^{\infty}$ in $[0, 1]$ ($\big[$[1], 11.4.5$\big]$), there are $\mathfrak{M}_3 >$

$0, \mathfrak{M}_4 > 0$ such that for every $h \in \mathbb{N}$

$$\begin{aligned}
&\left\| \sqrt{\rho} \sum_{m=1}^{h} \mathfrak{h}_{2m}(\rho, \omega, t - t_0) \right\|_{L^2(\Omega)}^2 \\
&\quad = 2\pi \int_0^1 \rho \sum_{m=1}^{h} \frac{J_0(\alpha_m \rho)^2 Y_0'(\alpha_m)^2}{\alpha_m^2 J_1'(\alpha_m)^2} (\alpha_m^2 - 1) \sin^2 \sqrt{\alpha_m^2 - 1}\,(t - t_0)\, d\rho \\
&\quad \geq \frac{\mathfrak{M}_1^2}{2} \int_0^1 \rho \sum_{m=1}^{h} \frac{2}{\pi \alpha_m \rho} \cos^2 \left(\alpha_m \rho - \frac{\pi}{4} \right) \sin^2 \sqrt{\alpha_m^2 - 1}\,(t - t_0)\, d\rho \\
&\qquad - \mathfrak{M}_3 \int_0^1 \sum_{m=1}^{h} \frac{2}{\pi \alpha_m^2}\, d\rho \\
&\geq \frac{\mathfrak{M}_1^2}{4\pi} \int_0^1 \sum_{m=1}^{h} \frac{1}{\alpha_{0m}} \left(\sin(\alpha_m \rho + \beta_m) - \sin(\alpha_m \rho - \gamma_{0m}) \right)^2 \, d\rho - \mathfrak{M}_4
\end{aligned}$$

(where $\beta_m := \sqrt{\alpha_m^2 - 1}\,(t - t_0) - \frac{\pi}{4}$ and $\gamma_m := \sqrt{\alpha_m^2 - 1}\,(t - t_0) + \frac{\pi}{4}$)

$$\begin{aligned}
&\geq \frac{\mathfrak{M}_1^2}{4\pi} \int_0^1 \sum_{m=1}^{h} \frac{1}{\alpha_m} \left(\sin^2(\alpha_m \rho + \beta_m) + \sin^2(\alpha_m \rho - \gamma_m) \right) \, d\rho \\
&\quad - \frac{\mathfrak{M}_1^2}{2\pi} \left| \int_0^1 \sum_{m=1}^{h} \frac{1}{\alpha_m} \sin(\alpha_m \rho + \beta_m) \sin(\alpha_m \rho - \gamma_m) \, d\rho \right| - \mathfrak{M}_4 \\
&= \frac{\mathfrak{M}_1^2}{4\pi} \sum_{m=1}^{h} \frac{1}{\alpha_m} \left(\frac{1}{2} - \frac{\sin 2(\alpha_m + \beta_m) - \sin 2\beta_m}{4\alpha_m} \right. \\
&\qquad \left. + \frac{1}{2} - \frac{\sin 2(\alpha_m - \gamma_m) + \sin 2\gamma_m}{4\alpha_m} \right) \\
&\quad - \frac{\mathfrak{M}_1^2}{4\pi} \left| \sum_{m=1}^{h} \frac{1}{\alpha_m} \left(\cos(\beta_m + \gamma_m) \right. \right. \\
&\qquad \left. \left. - \frac{\sin(2\alpha_m + \beta_m - \gamma_m) - \sin(\beta_m - \gamma_m)}{2\alpha_m} \right) \right| - \mathfrak{M}_4 \\
&\geq \frac{\mathfrak{M}_1^2}{4\pi} \left(\sum_{m=1}^{h} \frac{1}{\alpha_m} - \sum_{m=1}^{h} \frac{1}{\alpha_m^2} - \left| \sum_{m=1}^{h} \frac{1}{\alpha_m} \cos 2\sqrt{\alpha_m^2 - 1}\,(t - t_0) \right| \right. \\
&\qquad \left. - \sum_{m=1}^{h} \frac{1}{\alpha_m^2} \right) - \mathfrak{M}_4
\end{aligned}$$

which lets to ∞ for $h \to \infty$ by (3.2), the properties of harmonic series and Lemma 3.2. Moreover, $\|\sqrt{\rho} \sum_{m=1}^{h} \mathfrak{h}_{2m}\|_{L^2(\Omega)}^2$ turns out to be increasing with

h. Lebesgue's monotone convergence theorem gives $\left\|\sqrt{\rho}\sum_{m=1}^{\infty}\mathfrak{h}_{2m}\right\|_{L^2(\Omega)}^2 = \lim_{h\to\infty}\left\|\sqrt{\rho}\sum_{m=1}^{h}\mathfrak{h}_{2m}\right\|_{L^2(\Omega)}^2 = \infty$.

In the same way, it can be proved that $\left\|\sqrt{\rho}\sum_{m=1}^{\infty}\mathfrak{h}_{jm}\right\|_{L^2(\Omega)}^2 < \infty, j = 1, j = 3$ and the result follows. □

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