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Spaces $C(X)$ with ordered bases ☆

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ABSTRACT

The concept of Σ-base of neighborhoods of the identity of a topological group G is introduced. If the index set $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ is unbounded and directed (and if additionally each subset of Σ which is bounded in $\mathbb{N}^{\mathbb{N}}$ has a bound at Σ) a base $\{U_\alpha : \alpha \in \Sigma\}$ of neighborhoods of the identity of a topological group G with $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ with $\alpha, \beta \in \Sigma$ is called a Σ-base (a Σ_2-base). The case when $\Sigma = \mathbb{N}^{\mathbb{N}}$ has been noticed for topological vector spaces (under the name of $\mathfrak{G}$-base) at [2]. If X is a separable and metrizable space which is not Polish, the space $C_c(X)$ has a Σ-base but does not admit any $\mathfrak{G}$-base. A topological group which is Fréchet–Urysohn is metrizable iff it has a Σ_2-base of the identity. Under an appropriate ZFC model the space $C_c(\omega_1)$ has a Σ_2-base which is not a $\mathfrak{G}$-base. We also prove that (i) every compact set in a topological group with a Σ_2-base of neighborhoods of the identity is metrizable, (ii) a $C_p(X)$ space has a Σ_2-base iff X is countable, and (iii) if a space $C_c(X)$ has a Σ_2-base then X is a C-Suslin space, hence $C_c(X)$ is angelic.

1. Preliminaries

In what follows $\mathbb{N}$ will design the set of positive integers equipped with the discrete topology. The product space $\mathbb{N}^{\mathbb{N}}$ is supposed to be provided with the pointwise partial order, i.e., such that $\alpha \leq \beta$ whenever $\alpha(i) \leq \beta(i)$ for every $i \in \mathbb{N}$. In the sequel Σ will always design a topological subspace of $\mathbb{N}^{\mathbb{N}}$. We shall say that a subset Δ of $\mathbb{N}^{\mathbb{N}}$ is (pointwise) *bounded* if $\sup\{\alpha(k) : \alpha \in \Delta\} < \infty$ for every $k \in \mathbb{N}$, otherwise will be called *unbounded*. A covering $\{A_\alpha : \alpha \in I\}$ of a topological space X is said to *swallow* the compact sets if for each compact set Q in X there is $\gamma \in I$ such that $Q \subseteq A_\gamma$. If the covering $\{A_\alpha : \alpha \in I\}$ consists of compact sets, we shall speak of a *compact covering*.

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Unless otherwise stated X will be a (Hausdorff) completely regular space and $C(X)$ will denote the linear space of real-valued continuous functions defined on X. We shall write $C_p(X)$ or $C_c(X)$ when endowed $C(X)$ with the pointwise or the compact-open topology, respectively. All topological spaces are supposed to be Hausdorff. Let us recall that a topological space X is called *C-Suslin* if there is a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ and a map $T : \Sigma \to \mathcal{P}(X)$ such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$ and if $\{\alpha_n\} \subseteq \Sigma$ converges in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point in X (see [15]). A topological space X is called *web-compact* if there is a map T from a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ into $\mathcal{P}(X)$ such that $\overline{\bigcup\{T(\alpha) : \alpha \in \Sigma\}} = X$ and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point in X (see [13, Definition]). Clearly, every C-Suslin space is web-compact. A topological space X is *angelic* if relatively countably compact sets in X are relatively compact and for every relatively compact subset A of X each point of $\overline{A}$ is the limit of a sequence of A (see [9]). A topological space is *strictly angelic* if X is angelic and every compact subset of X is separable.

A completely regular space X is said to be *M-dominated* by a completely regular space M if there is a compact covering $\mathcal{B}$ of X of the form $\mathcal{B} = \{B_K : K \in \mathcal{K}(M)\}$, where $\mathcal{K}(M)$ stands for the family of all compact sets of M, satisfying that $B_K \subseteq B_Q$ whenever $K \subseteq Q$. If in addition $\mathcal{B}$ swallows the compact sets of X, then X is said to be *strongly M-dominated*, see [3].

In this paper we introduce the notion of a Σ-base of neighborhoods of the identity of a topological group (Definition 3 below), which is a family 'smaller' than a $\mathfrak{G}$-base (see [2]). We show that if X is a separable metrizable space which is not a Polish space, then $C_c(X)$ admits a Σ-base of neighborhoods of the origin but not a $\mathfrak{G}$-base (Theorem 7). We also consider a special type of Σ-bases, named Σ_2-bases, that share some important properties with $\mathfrak{G}$-bases (Definition 11). We show that, under appropriate set-theoretical conditions, there exists a Σ_2-base which is not a $\mathfrak{G}$-base (Example 20). We also prove other results stated in Abstract.

2. Σ-bases and distinguishing examples

A topological group G is said to have a $\mathfrak{G}$-base if there is a base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighborhoods of the identity e in G such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$. Clearly, every metrizable topological group has a $\mathfrak{G}$-base. Conversely, every Fréchet–Urysohn topological group with a $\mathfrak{G}$-base is metrizable, [11, Theorem 1.2].

A space $C_c(X)$ has a $\mathfrak{G}$-base of (absolutely convex) neighborhoods of the origin if and only if X has a covering $\mathcal{A} = \left\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ made up of compact sets such that (i) $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$, and (ii) $\mathcal{A}$ swallows the compact sets (see [6, Theorem 2]). Combining this fact with Christensen's theorem (see [5, Theorem 3.3] or [8, Theorem 6.4]) one gets the following result which will be used, not always with explicit mention, along the paper.

Proposition 1. *For a metrizable space X the following are equivalent*

(1) *X is a Polish space.*
(2) *$C_c(X)$ has a $\mathfrak{G}$-base of neighborhoods of the origin.*

Since $X = \mathbb{R}^{\mathbb{N}}$ is Polish but not hemicompact, the previous proposition ensures that $C_c(X)$ is a non-metrizable locally convex space with a $\mathfrak{G}$-base.

Following [14], a (Hausdorff) topological group G has the *strong Pytkeev property* if there exists a sequence $\mathcal{D}$ of subsets of G satisfying the property: for each neighborhood U of the unit e and each $A \subseteq G$ with $e \in \overline{A} \setminus A$, there is $D \in \mathcal{D}$ such that $D \subseteq U$ and $D \cap A$ is infinite. In [10, Theorem 5] we showed that any topological group G with the strong Pytkeev property admits a *quasi*-$\mathfrak{G}$-base $\{U_\alpha : \alpha \in \Sigma\}$ of the identity, i.e., an ordered base of neighborhoods of e over some $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$.

Very recently T. Banakh [1, Corollary 2.6], being inspired by a question in [10], proved that *for every separable metrizable space X the space $C_c(X)$ has the strong Pytkeev property*; therefore such $C_c(X)$ admits a quasi-$\mathfrak{G}$-base, again by [10, Theorem 2.2]. But it turns out that $C_c(\mathbb{Q})$ has even a quasi-$\mathfrak{G}$-base $\{U_\alpha : \alpha \in \Sigma\}$ for which $\sup\{\alpha(k) : \alpha \in \Sigma\} = \infty$ for each $k \in \mathbb{N}$ despite $C_c(\mathbb{Q})$ does not admit a $\mathfrak{G}$-base (see Remark 2 below). Summarizing, we have the following observation which partially motivates our work to study more carefully such topological groups which admit a Σ-base.

Remark 2. Let X be a separable metric space which is not a Polish space. Then $C_c(X)$ has a quasi-$\mathfrak{G}$-base but $C_c(X)$ does not admit a $\mathfrak{G}$-base.

The concept of quasi-$\mathfrak{G}$-base is rather of a theoretical nature. We propose more practical concept as follows.

Definition 3. A topological group G is said to have a Σ-base if for some unbounded and directed subset Σ of $\mathbb{N}^\mathbb{N}$ the neutral element of G has a base of neighborhoods $\{U_\alpha : \alpha \in \Sigma\}$ such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$ with $\alpha, \beta \in \Sigma$.

The requirement for Σ to be directed is not a serious constraint, since if Γ is any unbounded subset of $\mathbb{N}^\mathbb{N}$ and $\mathcal{F}(\Sigma)$ stands for the family of all finite subsets of Σ then $\Sigma := \{\sup \digamma : \digamma \in \mathcal{F}(\Gamma)\}$, where $\gamma = \sup \digamma \in \mathbb{N}^\mathbb{N}$ is given by $\gamma(i) = \sup\{\alpha(i) : \alpha \in \digamma\}$ for each $i \in \mathbb{N}$, is an unbounded and directed subset of $\mathbb{N}^\mathbb{N}$ of the same cardinality as Γ. A special stronger notion of a Σ-base will be studied in the second part of the paper. The following theorem characterizes those $C_c(X)$ spaces that admit a Σ-base.

Theorem 4. *Let X be a completely regular space. The following are equivalent*

(1) *There is a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ of X, with Σ unbounded and directed, such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \le \beta$ in Σ, that swallows the compact sets.*
(2) *The locally convex space $C_c(X)$ has a Σ-base of absolutely convex neighborhoods of the origin.*

Proof. $1 \Rightarrow 2$. For each compact set $K \subseteq X$ and each $\epsilon > 0$, define

$$[K, \epsilon] := \{f \in C(X) : \sup_{x\in K} |f(x)| \le \epsilon\}.$$

Let $\{A_\alpha : \alpha \in \Sigma\}$ be a compact covering of X indexed by an unbounded directed set Σ in $\mathbb{N}^\mathbb{N}$ such that $A_\alpha \subseteq A_\beta$ if $\alpha \le \beta$. Since Σ is unbounded there is $k \in \mathbb{N}$ such that $\sup\{\alpha(k) : \alpha \in \Sigma\} = \infty$. Then set

$$U_\alpha := [A_\alpha, \alpha(k)^{-1}]$$

for $\alpha \in \Sigma$ and put $\mathfrak{U} = \{U_\alpha : \alpha \in \Sigma\}$. If $f \in C(X) \setminus \{0\}$ and

$$\lambda_\alpha := \sup\{|f(x)| : x \in A_\alpha\},$$

then $\lambda_\alpha^{-1} f(x) \le 1$ for every $x \in A_\alpha$, which means that $f \in \lambda_\alpha \alpha(k) U_\alpha$. This shows that the set U_α is absorbing. On the other hand, given $\alpha, \beta \in \Sigma$, since $(\Sigma, \le)$ is directed there is $\gamma \ge \sup\{\alpha, \beta\}$, so that $A_\alpha \cup A_\beta \subseteq A_\gamma$. Hence, the fact that $\gamma(k) \ge \max\{\alpha(k), \beta(k)\}$ assures that $U_\gamma \subseteq U_\alpha \cap U_\beta$, which shows that $\mathfrak{U}$ is a family of absolutely convex and absorbing sets in $C(X)$ composing a filter base. Since clearly $U_\beta \subseteq U_\alpha$ if $\alpha \le \beta$ and $\{0\} = \bigcap\{U_\alpha : \alpha \in \Sigma\}$, in order to ensure that $\mathfrak{U}$ is a Σ-base of neighborhoods of the origin in $C(X)$ it remains to check that for each $\alpha \in \Sigma$ there is $\gamma \ge \alpha$ such that $U_\gamma \subseteq \frac{1}{2} U_\alpha$. For the latter statement first choose $\beta \in \Sigma$ such that $\beta(k) \ge 2\alpha(k)$. Since $(\Sigma, \le)$ is a directed set, there is $\gamma \in \Sigma$ such that $\gamma \ge \sup\{\alpha, \beta\}$. Hence $\gamma \ge \alpha$ with $\gamma(k) \ge 2\alpha(k)$. If $f \in U_\gamma$, since $A_\alpha \subseteq A_\gamma$ we have

$$\sup_{y\in A_\alpha} |2f(y)| \le 2\sup_{y\in A_\gamma} |f(y)| \le 2\gamma(k)^{-1} \le \alpha(k)^{-1}$$

which means that $2f \in U_\alpha$. Hence $\mathfrak{U}$ is a Σ-base for a locally convex topology τ on $C(X)$ with $\tau_p \le \tau \le \tau_c$. Now assume in addition that $\{A_\alpha : \alpha \in \Sigma\}$ swallows the compact sets of X and let V be a neighborhood of the origin of $C_c(X)$. If Q is a compact set in X with $[Q,\epsilon] \subseteq V$ for some $\epsilon > 0$, choosing $\gamma \in \Sigma$ such that $Q \subseteq A_\gamma$ and $\gamma(k)^{-1} < \epsilon$ then $U_\gamma \subseteq [Q,\epsilon] \subseteq V$. This shows that $\tau = \tau_c$, so $\{U_\alpha : \alpha \in \Sigma\}$ is a Σ-base for $C_c(X)$.

$2 \Rightarrow 1$. First note that if K is a closed subset of $X \setminus \{x\}$ there exists $f \in C(X)$ such that $f(K) = \{0\}$ and $f(x) = 2$. Hence if $\epsilon > 0$, $A \subseteq X$ and $[K,\epsilon] \subseteq [A,1]$, then $A \subseteq K$. Now we proceed exactly as in the second part of the proof of [6, Theorem 2]. Let $\{U_\alpha : \alpha \in \Sigma\}$ be a Σ-base of absolutely convex neighborhoods of the origin in $C_c(X)$. Fix $\alpha \in \Sigma$ and choose a compact set K in X such that $[K,\epsilon] \subseteq U_\alpha$ for some $\epsilon > 0$. Since the closed set

$$A_\alpha := \{x \in X : |f(x)| \leqslant 1,\ \forall f \in U_\alpha\}$$

verifies that $U_\alpha \subseteq [A_\alpha, 1]$, the inclusion $[K,\epsilon] \subseteq [A_\alpha, 1]$ together with the above observation imply that A_α is compact. Finally, if P is any compact subset of X there exists $\beta \in \Sigma$, such that $U_\beta \subseteq [P,1]$, so that $P \subseteq \{x \in X : |f(x)| \leqslant 1, \forall f \in U_\beta\} = A_\beta$. This shows that the family $\mathcal{A} = \{A_\alpha : \alpha \in \Sigma\}$ is a compact covering of X such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \le \beta$ in the unbounded and directed set Σ, that swallows the compact sets. □

Example 5. Let $\omega = \mathbb{N} \cup \{0\}$ be equipped with the discrete topology. If $F = \{n_1, \ldots, n_p\}$ is a finite subset of $\mathbb{N}$, define $\delta_F \in \omega^\omega$ so that $\delta_F(0) = |F|$, $\delta_F(n_i) = 1$ for $1 \le i \le p$ and $\delta_F(j) = 0$ otherwise. Setting $\Sigma := \{\delta_F : F \subseteq \mathbb{N},\ F \text{ finite}\}$ then Σ is a countable subset of ω^ω. If $\alpha, \beta \in \Sigma$, there are finite sets F, G in $\mathbb{N}$ such that $\alpha = \delta_F$ and $\beta = \delta_G$. Setting $H = F \cup G$ then clearly $\alpha, \beta \le \delta_H$, so that Σ is directed. Moreover Σ is unbounded since $\sup\{\alpha(0) : \alpha \in \Sigma\} = \infty$. If $\alpha = \delta_F \in \Sigma$, let us define $A_\alpha = F$. Clearly $\{A_\alpha : \alpha \in \Sigma\}$ is a compact covering of $\mathbb{N}$ such that $A_\alpha \subseteq A_\beta$ if $\alpha \le \beta$ in Σ. Moreover, obviously this covering swallows the compact sets of the discrete space $\mathbb{N}$. According to the preceding theorem, $C_c(\mathbb{N}) = \mathbb{R}^{\mathbb{N}}$ admits a Σ-base of neighborhoods of the origin. Of course, since $\mathbb{R}^{\mathbb{N}}$ is a metrizable locally convex space, it also admits a $\mathfrak{G}$-base. This does not always happen, as the following example shows.

Example 6. There exists a space $C_c(X)$ that admits a Σ-base of neighborhoods of the origin but not a $\mathfrak{G}$-base of neighborhoods of the origin.

Proof. Let us identify the set ω with an enumeration of the set $\mathbb{Q}$ of the rationals. We shall keep the notation ω when consider ω either as a set or equipped with the discrete topology. We shall write Ω when consider ω equipped with the (metrizable) relative topology of $\mathbb{R}$. Let χ_Z be the characteristic function of a set $Z \subseteq \omega$. If $\mathcal{K}(\Omega)$ stands for the family of compact sets of Ω, define $\Sigma \subseteq \omega^\omega$ as follows

$$\Sigma = \{k\chi_Z : Z \in \mathcal{K}(\Omega),\ k \in \mathbb{N}\}$$

so that $\alpha \in \Sigma$ if there are $k \in \mathbb{N}$ and $Z \in \mathcal{K}(\Omega)$ such that $\alpha = \alpha[k,Z] = k\chi_Z$. Thus Σ consists of all ω-valued functions with compact support in Ω which are constant in their support. Now observe that if $\alpha[m,Y], \alpha[n,Z] \in \Sigma$ then $\alpha[m+n, Y \cup Z] \in \Sigma$ and $\alpha[m,Y], \alpha[n,Z] \le \alpha[m+n, Y \cup Z]$, so that Σ is a directed set. On the other hand, if $j \in \omega$ then $\alpha[k,\{j\}](j) = k$, so that $\sup\{\alpha(j) : \alpha \in \Sigma\} = \infty$, which shows that Σ is unbounded. Defining $A_{\alpha[k,Z]} = Z$ for all $k \in \mathbb{N}$ whenever Z is a compact set of Ω, clearly $\{A_\alpha : \alpha \in \Sigma\}$ is a compact covering of Ω. Moreover, if $\alpha[m,Y] \le \alpha[n,Z]$ then $m \le n$ and $Y \subseteq Z$, hence $A_{\alpha[m,Y]} = Y \subseteq Z = A_{\alpha[n,Z]}$. Obviously $\{A_\alpha : \alpha \in \Sigma\}$ swallows the compact sets because contains all compact sets of Ω. So, according to the previous theorem, the space $C_c(\mathbb{Q}) = C_c(\Omega)$ has a Σ-base. On the other hand, since $\mathbb{Q}$ is not a Polish space, according to Proposition 1 the space $C_c(X)$ with $X = \mathbb{Q}$ cannot have a $\mathfrak{G}$-base of neighborhoods of the origin. □

An appropriate modification of the previous example yields the following general result.

Theorem 7. *If X is a separable and metrizable space that is not a Polish space, then $C_c(X)$ admits a Σ-base of neighborhoods of the origin but it does not admit any $\mathfrak{G}$-base.*

Proof. Let us identify now the set ω with an enumeration of a countable dense subspace of X. We keep the notation ω when consider ω either as a set or equipped with the discrete topology and write Ω when consider ω equipped with the relative topology of X. Let χ_Z be the characteristic function of a set $Z \subseteq \omega$. If $\mathcal{F}(\Omega)$ stands for the family of all subsets of Ω with compact closure in X, define $\Sigma \subseteq \omega^\omega$ as

$$\Sigma = \{k\chi_Z : Z \in \mathcal{F}(\Omega),\, k \in \mathbb{N}\}$$

so that $\alpha \in \Sigma$ if there are $k \in \mathbb{N}$ and $Z \in \mathcal{F}(\Omega)$ such that $\alpha = \alpha[k, Z] = k\chi_Z$. Thus Σ consists of all constant ω-valued functions supported in a subset of Ω which is relatively compact in X.

Observe that if $\alpha[m, Y], \alpha[n, Z] \in \Sigma$ then $\alpha[m+n, Y \cup Z] \in \Sigma$ and $\alpha[m, Y], \alpha[n, Z] \leq \alpha[m+n, Y \cup Z]$, so that Σ is a directed set. On the other hand, if $j \in \omega$ we have for instance that $\alpha[k, \{j\}](j) = k$, so that $\sup\{\alpha(j) : \alpha \in \Sigma\} = \infty$, which shows that Σ is unbounded. Defining $A_{\alpha[k,Z]} = \overline{Z}$ for all $k \in \mathbb{N}$ where the closure is in X, then $\{A_\alpha : \alpha \in \Sigma\}$ is a compact covering of X. Indeed, if $x \in X$ there is a sequence S in Ω converging to x. Given that x belongs to the compact set $\overline{S}$, then $A_{\alpha[k,S]} = \overline{S}$ for all $k \in \mathbb{N}$, so that $x \in A_{\alpha[k,S]}$ whatever be $k \in \mathbb{N}$. Moreover, if $\alpha[m, Y] \leq \alpha[n, Z]$ then $m \leq n$ and $Y \subseteq Z$, hence $A_{\alpha[m,Y]} = \overline{Y} \subseteq \overline{Z} = A_{\alpha[n,Z]}$.

We claim that the compact covering $\{A_\alpha : \alpha \in \Sigma\}$ swallows the compact sets of X. So, let $K \subseteq X$ be compact and let $\{a_n : n \in \mathbb{N}\}$ be a countable and dense subspace K_0 of K. If d is an admissible metric on X, for each $n \in \mathbb{N}$ choose $b_{n,m} \in \Omega$ such that

$$d(a_n, b_{n,m}) < 2^{-nm}$$

for every $m \in \mathbb{N}$, so that $\lim_{m\to\infty} b_{n,m} = a_n$ for each $n \in \mathbb{N}$. Although it is not necessary, we shall also assume that each sequence $\{b_{n,m}\}_{m=1}^\infty$ is injective and that $a_n \neq b_{n,m}$ for every $m \in \mathbb{N}$. Now set $B_n := \{b_{n,m} : m \in \mathbb{N}\}$ and

$$B := \bigcup_{n=1}^\infty B_n = \bigcup \left\{b_{n,m} : (n, m) \in \mathbb{N}^2\right\}.$$

Clearly $B \subseteq \omega$ and $\{a_n : n \in \mathbb{N}\} \subseteq \overline{B}$, whence $K \subseteq \overline{B}$. Therefore to prove that $B \in \mathcal{F}(\Omega)$ it suffices to check that $\overline{B}$ it is compact, i.e., we must to justify that each sequence in $\overline{B}$ admits a convergent subsequence.

Let $\{c_p\}_{p=1}^\infty$ be a sequence in $\overline{B}$. Then there exist two sequences $\{n(p)\}_{p=1}^\infty$ and $\{m(p)\}_{p=1}^\infty$ of positive integers such that

$$d(c_p, b_{n(p),m(p)}) < 2^{-p} \tag{2.1}$$

for each $p \in \mathbb{N}$. If $\{n(p)\}_{p=1}^\infty$ contains a constant subsequence $\{n(p_l)\}_{l=1}^\infty$, i.e., such that $n(p_l) = n_0$ for every $l \in \mathbb{N}$, then the sequence $\left\{b_{n(p_l),m(p_l)}\right\}_{l=1}^\infty = \left\{b_{n_0,m(p_l)}\right\}_{l=1}^\infty$ contains a convergent subsequence because the set $\{b_{n_0,m} : m \in \mathbb{N}\} \cup \{a_{n_0}\}$ is compact. Thus, from (2.1) it follows that the sequence $\{c_p\}_{p=1}^\infty$ also contains a convergent subsequence.

If $\{n(p)\}_{p=1}^{\infty}$ does not contain a constant subsequence then there is an strictly increasing sequence $\{p_l\}_{l=1}^{\infty}$ of positive integers such that $n(p_l) < n(p_{l'})$ if $l < l'$. By construction

$$d(a_{n(p_l)}, b_{n(p_l),m(p_l)}) < 2^{-n(p_l)m(p_l)} \text{ and } d(c_{p_l}, b_{n(p_l),m(p_l)}) < 2^{-p_l},$$

whence

$$d(a_{n(p_l)}, c_{p_l}) < 2^{-n(p_l)m(p_l)} + 2^{-p_l} \leqslant 2^{-p_l} + 2^{-p_l} = 2^{1-p_l} \tag{2.2}$$

for each $l \in \mathbb{N}$. Since $\{a_n : n \in \mathbb{N}\}$ is contained in the compact set K, the sequence $\{a_{n(p_l)}\}_{l=1}^{\infty}$ has a convergent subsequence. So inequality (2.2) implies that $\{c_{p_l}\}_{l=1}^{\infty}$ also contains a convergent subsequence.

Finally, as happens in the Example 6 above, if X is not a Polish space Proposition 1 prevents the space $C_c(X)$ to have a $\mathfrak{G}$-base of neighborhoods of the origin. □

3. Boundedly complete sets and Σ_2-bases

In this section we are going to consider a special class of Σ-bases, which we denominate Σ_2-bases, and study some properties of them quite close to those of $\mathfrak{G}$-bases.

Definition 8. A subset Σ of $\mathbb{N}^{\mathbb{N}}$ will be called boundedly complete if each bounded set Δ of Σ has a bound at Σ. In other words, if $\sup\{\alpha(k) : \alpha \in \Delta\} < \infty$ for every $k \in \mathbb{N}$ implies that there is $\gamma \in \Sigma$ such that $\alpha \leq \gamma$ for every $\alpha \in \Delta$.

If Σ is a boundedly complete subset of $\mathbb{N}^{\mathbb{N}}$ then Σ is itself directed. For if $\alpha, \beta \in \Sigma$ then $\{\alpha, \beta\}$ is a bounded subset of Σ, so there is $\gamma \in \Sigma$ such that $\alpha, \beta \leq \gamma$. On the other hand, if $\{U_\alpha : \alpha \in \Sigma\}$ is a base of neighborhoods of a (Hausdorff) locally convex space and Σ is a boundedly complete subset of $\mathbb{N}^{\mathbb{N}}$ then Σ must be unbounded. Otherwise $\sup\{\alpha(k) : \alpha \in \Sigma\} < \infty$ for every $k \in \mathbb{N}$ and hence there exists $\gamma \in \Sigma$ with $\alpha \leq \gamma$ for every $\alpha \in \Sigma$. Consequently $U_\gamma \subseteq \bigcap_{\alpha \in \Sigma} U_\alpha$, a contradiction.

Example 9. *Every cofinal subset Σ of $\mathbb{N}^{\mathbb{N}}$ with respect to the partial order '$\leq$' is boundedly complete.* If $\Delta \subseteq \Sigma$ satisfies that $\sup\{\alpha(k) : \alpha \in \Delta\} < \infty$ for every $k \in \mathbb{N}$, let $\beta(k) := \sup\{\alpha(k) : \alpha \in \Delta\}$. Then $\beta \in \mathbb{N}^{\mathbb{N}}$ and hence there is $\gamma \in \Sigma$ such that $\beta \leq \gamma$.

Proposition 10. *If X is a (completely regular) topological space for which there exists a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ that swallows the compact sets indexed by a boundedly complete subset Σ of $\mathbb{N}^{\mathbb{N}}$ and such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ in Σ, then X is strongly dominated by a second countable space.*

Proof. Consider the mapping $T : \Sigma \to \mathcal{K}(X)$ defined by $T(\alpha) = A_\alpha$. If K is a compact set in Σ, then K is bounded due to $\sup\{\alpha(k) : \alpha \in K\} < \infty$ for every $k \in \mathbb{N}$. Since Σ is supposed to be boundedly complete, there is $\gamma \in \Sigma$ such that $\alpha \leq \gamma$ for every $\alpha \in K$. Consequently $T(K) = \bigcup\{T(\alpha) : \alpha \in K\} \subseteq A_\gamma$. So, setting $B_K := \overline{T(K)}$, then B_K is a closed subset of a compact set of X, hence a compact set. This means that $\mathcal{B} := \{B_K : K \in \mathcal{K}(\Sigma)\}$ is a family of compact sets of X, which clearly covers X. On the other hand, if $K, Q \in \mathcal{K}(\Sigma)$ are such that $K \subseteq Q$ then clearly $T(K) \subseteq T(Q)$, which implies that $B_K \subseteq B_Q$. Finally, if P is a compact set in X there is $\delta \in \Sigma$ with $P \subseteq T(\delta) = B_{\{\delta\}}$. This shows that X is strongly Σ-dominated. Since Σ is a separable metric space, the conclusion follows. □

Definition 11. A Σ-base of neighborhoods of the unit element of a topological group G indexed by a boundedly complete subspace Σ of $\mathbb{N}^{\mathbb{N}}$ will be referred to as a Σ_2-base.

Of course, every $\mathfrak{G}$-base of neighborhoods of the neutral element of a topological group G is a Σ_2-base, with $\Sigma = \mathbb{N}^{\mathbb{N}}$. The proof of the next theorem uses the following

Proposition 12. *([3, Theorem 1]) A compact topological space K is metrizable if and only if the space $(K \times K) \setminus \Delta$ is strongly dominated by a second countable space, where here $\Delta := \{(x, x) : x \in K\}$.*

Theorem 13. *If a topological group G has a Σ_2-base of neighborhoods of the identity, then every compact subset K in G is metrizable. Consequently, G is strictly angelic.*

Proof. Let $\{U_\alpha : \alpha \in \Sigma\}$ be an open Σ_2-base in G. We may assume that all sets U_α are symmetric. We have to show that K is metrizable. To prove this, according to Propositions 10 and 12 it is enough to show that the set $W := (K \times K) \setminus \Delta$ has a compact covering $\{W_\alpha : \alpha \in \Sigma\}$ indexed by a boundedly complete subset $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ that swallows the compact sets. Now, for each $\alpha \in \Sigma$ set

$$W_\alpha := \{(x, y) \in W : xy^{-1} \notin U_\alpha\}.$$

Then W_α is closed in $K \times K$, and hence compact for every $\alpha \in \Sigma$. Let us show that the family $\mathcal{W} := \{W_\alpha : \alpha \in \Sigma\}$ is a compact covering in W as required. Indeed, if $(x, y) \in W$, then $x \neq y$. Hence there exists $\alpha \in \Sigma$ such that $xy^{-1} \notin U_\alpha$. So $(x, y) \in W_\alpha$. Thus $W = \bigcup_{\alpha \in \Sigma} W_\alpha$ and $\mathcal{W}$ is a compact covering such that $W_\alpha \subseteq W_\beta$ whenever $\alpha \leq \beta$ for $\alpha, \beta \in \Sigma$.

We show next that the family $\mathcal{W}$ swallows compact sets in W. Let $Q \subseteq W$ be a compact set. Then the set $T(Q) := \{xy^{-1} : (x, y) \in Q\}$ is compact in G and does not contain the element e. Therefore we can find a neighborhood U_α such that $U_\alpha \cap T(Q) = \emptyset$ for some $\alpha \in \Sigma$, which shows that $Q \subseteq W_\alpha$. Consequently $\mathcal{W}$ swallows the compact sets in W. □

Corollary 14. *If there exists a family $\{A_\alpha : \alpha \in \Sigma\}$ made up of compact sets, indexed by a boundedly complete set Σ such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and satisfying that $\overline{\cup \{A_\alpha : \alpha \in \Sigma\}} = X$, then $C_c(X)$ is strictly angelic.*

Proof. First observe that X is web-compact, so that $C_p(X)$ is angelic. Since the compact-open topology is stronger than the pointwise convergence topology, the angelic lemma [9] guarantees that the space $C_c(X)$ is angelic.

For the second statement set $Y = \cup \{A_\alpha : \alpha \in \Sigma\}$. Now let τ_p and τ_c denote the pointwise and the compact-open topology on $C(Y)$, respectively. Since Σ is unbounded in $\mathbb{N}^{\mathbb{N}}$ there is $k \in \mathbb{N}$ such that $\sup \{\alpha(k) : \alpha \in \Sigma\} = \infty$. Defining $U_\alpha = \{f \in C(Y) : \sup_{y \in A_\alpha} |f(y)| \leq \alpha(k)^{-1}\}$ for $\alpha \in \Sigma$ as in the proof of Theorem 4. Then $\{U_\alpha : \alpha \in \Sigma\}$ is a base of neighborhoods of a Hausdorff locally convex topology τ on $C(Y)$ such that $\tau_p \leq \tau \leq \tau_c$. Moreover, since the index set Σ is a boundedly complete subset of $\mathbb{N}^{\mathbb{N}}$, then $\{U_\alpha : \alpha \in \Sigma\}$ is a Σ_2-base of $(C(Y), \tau)$. So, according to Theorem 13, every τ-compact set in $C(Y)$ is metrizable. Since the restriction map $S : C_c(X) \to (C(Y), \tau)$ defined $S(f) = f|_Y$ is a continuous (linear) injection from $C_c(X)$ into $(C(Y), \tau)$, if K is a compact set in $C_c(X)$ its image $S(K)$ is a compact set in $(C(Y), \tau)$, hence metrizable. Given that S restricts itself to an homeomorphism on K, it follows that K is metrizable in $C_c(X)$ as required. □

Theorem 15. *If $C_c(X)$ has a Σ_2-base of neighborhoods of the origin, then X is a C-Suslin space.*

Proof. Since Σ is unbounded and directed, by Theorem 4 there is in X a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ indexed by Σ such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$. Define the map $T : \Sigma \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ stands for the family of all compact sets of X, by $T(\alpha) = A_\alpha$. If $\{\alpha_n\}$ is a sequence in Σ such that $\alpha_n \to \alpha$ in $\mathbb{N}^{\mathbb{N}}$, since $\Delta = \{\alpha_n : n \in \mathbb{N}\} \subseteq \Sigma$ is a bounded set there is $\gamma \in \Sigma$ with $\alpha_n \leq \gamma$ for every $n \in \mathbb{N}$. Consequently,

$T(\alpha_n) \subseteq A_\gamma$ for every $n \in \mathbb{N}$. Hence, if $x_n \in T(\alpha_n)$ for every $n \in \mathbb{N}$, the sequence $\{x_n\}_{n=1}^\infty$ has a cluster point x in X (contained in A_γ). $\square$

Let $\{U_\alpha : \alpha \in \Sigma\}$ be a Σ_2-base in a topological group G. For every $\alpha = (\alpha_i)_{i\in\mathbb{N}} \in \Sigma$ and each $k \in \mathbb{N}$, set

$$D_k(\alpha) := \bigcap_{\beta\in I_k(\alpha)} U_\beta, \text{ where } I_k(\alpha) = \{\beta \in \Sigma : \beta_i = \alpha_i \text{ for } i = 1, \dots, k\}.$$

Clearly, $\{D_k(\alpha)\}_{k\in\mathbb{N}}$ is an increasing sequence of subsets of G containing the unit. Recall that every Fréchet–Urysohn topological group G satisfies the condition:

(AS) For any family $\{x_{n,k} : (n,k) \in \mathbb{N}\times\mathbb{N}\} \subset G$, with $\lim_n x_{n,k} = x \in G$, $k = 1, 2, \dots$, it is possible to choose strictly increasing sequences of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$, such that $\lim_i x_{n_i,k_i} = x$ (see [4, Lemma 1.3]).

Theorem 16. *If G be a topological group which is Fréchet–Urysohn with a Σ_2-base $\{U_\alpha : \alpha \in \Sigma\}$, then G is metrizable.*

Proof. First observe that for every $\alpha \in \Sigma$ there exists $k \in \mathbb{N}$ such that $D_k(\alpha)$ is a neighborhood of the unit e. Indeed, assume that there exists $\alpha \in \Sigma$ such that $D_k(\alpha)$ is not a neighborhood of the unit e for every $k \in \mathbb{N}$. Hence e belongs to the closure of the set $G \setminus D_k(\alpha)$. Since G is Fréchet–Urysohn, for every $k \in \mathbb{N}$ there is a sequence $\{x_{n,k}\}_{n\in\mathbb{N}}$ in $G \setminus D_k(\alpha)$ converging to e. By (AS) we choose strictly increasing sequences of natural numbers $(n_i)_{i\in\mathbb{N}}$ and $(k_i)_{i\in\mathbb{N}}$, such that $\lim_i x_{n_i,k_i} = e$.

For every $i \in \mathbb{N}$, choose $\alpha_{k_i} \in I_{k_i}(\alpha)$ such that $x_{n_i,k_i} \notin U_{\alpha_{k_i}}$. Since $\sup_{i\in\mathbb{N}} \alpha_{k_i}(j) = \max\{\alpha_{k_1}(j), \dots, \alpha_{k_j}(j)\} < \infty$ for every $j \in \mathbb{N}$, the set $\Delta = \{\alpha_{k_i} : i \in \mathbb{N}\}$ is a subset of Σ bounded in $\mathbb{N}^\mathbb{N}$. By hypothesis there exists $\gamma \in \Sigma$ such that $\sup_{i\in\mathbb{N}} \alpha_{k_i} \le \gamma$. So $x_{n_i,k_i} \notin U_\gamma$ for every $i \in \mathbb{N}$. Thus $x_{n_i,k_i} \nrightarrow e$, a contradiction. Therefore there is $k \in \mathbb{N}$ for which $D_k(\alpha)$ is a neighborhood of e. For every $\alpha \in \Sigma$ choose the minimal $k_\alpha \in \mathbb{N}$ such that $D_{k_\alpha}(\alpha)$ is a neighborhood of e. By the construction of the sets $D_k(\alpha)$, the family $\{\operatorname{int}(D_{k_\alpha}(\alpha))\}_{\alpha\in\Sigma}$ is a countable base of open neighborhoods of e, so G is metrizable. $\square$

Corollary 17. *Let $\{G_t\}_{t\in T}$ be a family of metrizable topological groups. Then the product $G := \prod_{t\in T} G_t$ has a Σ_2-base if and only if T is countable, i.e., when G is metrizable.*

Proof. Let $e := (e_t)$ be the unit vector in G, where e_t is the unit vector in G_t for $t \in T$. Let G_0 be the $\sum$-product in the space G, i.e. $G_0 := \{x = (x_t) \in G : |t \in T : x_t \ne e_t| \le \aleph_0\}$. Then G_0 is a subgroup of G and endowed with the product topology is a Fréchet–Urysohn dense subspace of G, see [12]. Assume that G has a Σ_2-base, then G_0 enjoys also this property. By Theorem 16 we know that G_0 is metrizable, so G is metrizable, too. The converse is clear. $\square$

Since $C_p(X)$ is dense in the product $\mathbb{R}^X$, we apply Corollary 17 to get the following.

Corollary 18. *The space $C_p(X)$ has a Σ_2-base if and only if X is countable.*

4. Existence of proper Σ_2-bases on $C_c([0, \omega_1))$

Let $\mathfrak{d}$ be the *dominating cardinal*, defined as the least cardinality for cofinal subsets of the preordered space $(\mathbb{N}^\mathbb{N}, \le^*)$, where $\alpha \le^* \beta$ stands for the *eventual dominance preorder* defined so that $\alpha(n) \le \beta(n)$ for almost all $n \in \mathbb{N}$, i.e., for all but finitely many values of n. Here $\alpha <^* \beta$ means that there exists $m \in \mathbb{N}$ such that $\alpha(n) < \beta(n)$ for every $n \ge m$. In what follows ω_1 will be the first ordinal of uncountable cardinal, whose cardinality we denote by $\aleph_1$. In ZFC one has $\aleph_1 \leq \mathfrak{d} \leq \mathfrak{c}$. The following result, certainly well known to specialists, of which we provide a detailed proof, will be widely used in the example below.

Lemma 19. *If $\aleph_1 = \mathfrak{d}$ there exists a cofinal ω_1-sequence $\Gamma := \{\beta_\kappa : \kappa < \omega_1\}$ in $(\mathbb{N}^\mathbb{N}, \leq^*)$ such that (i) $\kappa_1 < \kappa_2$ implies that $\beta_{\kappa_1} <^* \beta_{\kappa_2}$, (ii) for each $\alpha \in \mathbb{N}^\mathbb{N}$ the subset*

$$\Delta_\alpha := \{\kappa < \omega_1 : \beta_\kappa \leq^* \alpha\}$$

of $[0, \omega_1)$ is countable, (iii) if $\alpha \leq^* \gamma$ then $\Delta_\alpha \subseteq \Delta_\gamma$, and (iv) every countable subset of $[0, \omega_1)$ is contained in some Δ_γ; in particular, $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} \Delta_\alpha = [0, \omega_1)$.

Proof. Since $\aleph_1 = \mathfrak{d}$ there exists a cofinal set $\mathfrak{D} = \{\delta_\kappa : 0 \leq \kappa < \omega_1\}$ in $(\mathbb{N}^\mathbb{N}, \leq^*)$ with $|\mathfrak{D}| = \aleph_1$. Pick $\beta_0 \in \mathbb{N}^\mathbb{N}$ such that $\delta_0 <^* \beta_0$ and take $\beta_1 \in \mathbb{N}^\mathbb{N}$ such that $\sup\{\delta_1, \beta_0\} <^* \beta_1$. Suppose we have defined $\{\beta_\tau : 0 \leq \tau < \kappa\}$. Since the latter set is countable, we may choose $\beta_\kappa \in \mathbb{N}^\mathbb{N}$ such that $\sup\{\delta_\kappa, \beta_\tau : \tau < \kappa\} <^* \beta_\kappa$. By construction one has that $\beta_\tau <^* \beta_\varepsilon$ whenever $\tau < \varepsilon$ and on the other hand that $\delta_\kappa <^* \beta_\kappa$ for all $\kappa < \omega_1$, which assures that Γ is a cofinal set in $(\mathbb{N}^\mathbb{N}, \leq^*)$. Let us see that each set Δ_α, $\alpha \in \mathbb{N}^\mathbb{N}$, is countable. In fact, given $\alpha \in \mathbb{N}^\mathbb{N}$, the cofinality of $\mathfrak{D}$ allows us to choose $\tau < \omega_1$ such that $\alpha \leq^* \beta_\tau$ and then

$$\Delta_\alpha = \{\kappa < \omega_1 : \beta_\kappa \leq^* \alpha\} \subseteq \{\kappa < \omega_1 : \beta_\kappa <^* \beta_\tau\} \subseteq \{\kappa < \omega_1 : \kappa < \tau\}.$$

Since the latter set is countable, we are done.

If $\alpha \leq^* \gamma$ and $\kappa \in \Delta_\alpha$ then $\beta_\kappa \leq^* \alpha \leq^* \gamma$ and hence $\kappa \in \Delta_\gamma$, so that $\Delta_\alpha \subseteq \Delta_\gamma$. Finally, if M is a countable subset of $[0, \omega_1)$, pick $\varepsilon \in [0, \omega_1)$ such that $\varepsilon > \sup M$. Since $\tau < \varepsilon$ for all $\tau \in M$, then $\beta_\tau <^* \beta_\varepsilon$ for all $\tau \in M$. Hence $M \subseteq \Delta_{\beta_\varepsilon}$. □

Example 20. In any ZFC consistent model for which $\aleph_1 = \mathfrak{d}$ but $\mathfrak{d} < \mathfrak{c}$ there exists a completely regular space X and a compact covering $\{A_\alpha : \alpha \in \Sigma\}$ of X, with $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ and indexed by an unbounded, directed and boundedly complete proper subset Σ of $\mathbb{N}^\mathbb{N}$ that swallows the compact sets of X.

Proof. Assume that $\aleph_1 = \mathfrak{d}$ but $\mathfrak{d} < c$ and let Γ be the cofinal subset of $\mathbb{N}^\mathbb{N}$ of cardinality $\mathfrak{d}$ with respect to the preorder '$\leq^*$' determined in the previous lemma. Then we claim that $\{\Delta_\alpha : \alpha \in \Gamma\}$ is a covering of $X = [0, \omega_1)$ such that $\Delta_\alpha \subseteq \Delta_\beta$ whenever $\alpha \leq \beta$ and Γ is an unbounded subset of $\mathbb{N}^\mathbb{N}$. Indeed, if $\kappa \in [0, \omega_1)$ the previous result yields $\gamma \in \mathbb{N}^\mathbb{N}$ such that $\kappa \in \Delta_\gamma$. Since Γ is cofinal in $\mathbb{N}^\mathbb{N}$ with respect to the eventual dominance preorder, there is $\delta \in \Gamma$ with $\gamma \leq^* \delta$. By Lemma 19 this implies that $\Delta_\gamma \subseteq \Delta_\delta$, so that $\kappa \in \Delta_\delta$. Therefore $\bigcup_{\alpha \in \Gamma} \Delta_\alpha = X$. Now, to see that Γ is an unbounded subset of $\mathbb{N}^\mathbb{N}$, assume by contradiction that $\sup\{\alpha(k) : \alpha \in \Gamma\} < \infty$ for all $k \in \mathbb{N}$. Setting $\beta(k) = \sup\{\alpha(k) : \alpha \in \Gamma\}$ for every $k \in \mathbb{N}$ and choosing $\gamma \in \Gamma$ such that $\beta \leq^* \gamma$ we have that $\bigcup_{\alpha \in \Gamma} \Delta_\alpha \subseteq \Delta_\gamma$, so that $\Delta_\gamma = [0, \omega_1)$. But this is a contradiction, since Δ_γ is countable.

In order to get a directed index set, let us enlarge a little bit the set Γ. For each finite subset $\mathcal{F}$ of Γ choose the supremum $\sup \mathcal{F}$ at $\mathbb{N}^\mathbb{N}$ with respect to the order '$\leq$' and denote by Σ the subset of $\mathbb{N}^\mathbb{N}$ consisting of the suprema of the finite sets of Γ. Of course, according to our hypotheses, $|\Sigma| = |\Gamma| = \mathfrak{d} < \mathfrak{c} = |\mathbb{N}^\mathbb{N}|$, so that Σ is a proper subset of $\mathbb{N}^\mathbb{N}$. Clearly $\{\Delta_\alpha : \alpha \in \Sigma\}$ is still unbounded, $\bigcup_{\alpha \in \Sigma} \Delta_\alpha = X$ and $\Delta_\alpha \subseteq \Delta_\beta$ whenever $\alpha \leq \beta$ in Σ, but now we have the benefit that Σ is directed. In fact, if $\alpha_1, \alpha_2 \in \Sigma$ there are finite sets $\mathcal{F}_1, \mathcal{F}_2 \subseteq \Gamma$ such that $\alpha_1 = \sup \mathcal{F}_1$ and $\alpha_2 = \sup \mathcal{F}_2$, so that if $\gamma := \sup(\mathcal{F}_1 \cup \mathcal{F}_2)$ then clearly $\gamma \in \Sigma$ and $\alpha_1, \alpha_2 \leq \gamma$.

We claim that every compact set $K \subseteq X$ is contained in some Δ_γ with $\gamma \in \Sigma$. In fact, since K is countable, by Lemma 19 there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq \Delta_\alpha$. Since Γ is cofinal with respect to the eventual dominance preorder, there exists $\beta \in \Gamma$ such that $\alpha \leq^* \beta$. But this implies that $\Delta_\alpha \subseteq \Delta_\beta$, which ensures that $K \subseteq \Delta_\beta$ with $\beta \in \Sigma$.

Finally let us see that Σ is boundedly complete. Let P be a bounded subset of Σ and put $\beta(i) = \sup\{\alpha(i) : \alpha \in P\}$ for every $i \in \mathbb{N}$. Now chose $\gamma \in \Gamma$ such that $\beta \leq^* \gamma$ and let $F \subseteq \mathbb{N}$ be a finite set such that $\beta(i) \leq \gamma(i)$ for every $i \in \mathbb{N} \setminus F$. Observe that, since each set $\{\alpha(i) : \alpha \in P\}$ is finite, for each given index $i \in \mathbb{N}$ there exists an element $\alpha_i \in P$ such that $\alpha_i(i) = \beta(i)$. On the other hand, since $\{\gamma, \alpha_j : j \in F\}$ is a finite subset of the directed set Σ there exists some $\delta \in \Sigma$ such that $\gamma \leq \delta$ and $\alpha_j \leq \delta$ for all $j \in F$. This means that $\alpha \leq \delta$ for every $\alpha \in P$, since either $\alpha(i) \leq \beta(i) \leq \gamma(i) \leq \delta(i)$ if $i \in \mathbb{N} \setminus F$ or $\alpha(j) \leq \alpha_j(j) \leq \delta(j)$ if $j \in F$. Consequently, Σ is boundedly complete.

Hence the family $\{A_\alpha : \alpha \in \Sigma\}$ with $A_\alpha := \Delta_\alpha$ satisfies the required conditions. □

Corollary 21. *In any ZFC consistent model for which $\aleph_1 = \mathfrak{d}$ but $\mathfrak{d} < \mathfrak{c}$ there exists a Σ_2-base of absolutely convex neighborhoods of the origin of the space $C_c([0, \omega_1))$ which is not a $\mathfrak{G}$-base.*

Proof. This is a straightforward consequence of Theorem 4 and Example 20. □

Remark 22. According to [7, Theorem 8], the locally convex space $C_c([0, \omega_1))$ admits a $\mathfrak{G}$-base of neighborhoods of the origin if and only if we assume that $\aleph_1 = \mathfrak{d}$. The preceding corollary shows that this space even admits a Σ_2-base, which is not a $\mathfrak{G}$-base if we assume in addition that $\mathfrak{d} < \mathfrak{c}$. This latter condition is consistent with Cichon's diagram and hence can be realized in some model of ZFC.

Problem 23. We do not know whether there exists a topological group with a Σ_2-base that admits no $\mathfrak{G}$-base.

Problem 24. Let X be a separable metric space admitting a compact ordered covering of X indexed by an unbounded and boundedly complete proper subset of $\mathbb{N}^{\mathbb{N}}$ that swallows the compact sets of X. Is then X a Polish space?

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