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# The Ascoli property for function spaces and the weak topology of Banach and Fréchet spaces

by

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**Abstract.** Following Banakh and Gabriyelyan (2016) we say that a Tychonoff space $X$ is an Ascoli space if every compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous; this notion is closely related to the classical Ascoli theorem. Every $k_{\mathbb{R}}$-space, hence any $k$-space, is Ascoli.

Let $X$ be a metrizable space. We prove that the space $C_k(X)$ is Ascoli iff $C_k(X)$ is a $k_{\mathbb{R}}$-space iff $X$ is locally compact. Moreover, $C_k(X)$ endowed with the weak topology is Ascoli iff $X$ is countable and discrete.

Using some basic concepts from probability theory and measure-theoretic properties of $\ell_1$, we show that the following assertions are equivalent for a Banach space $E$: (i) $E$ does not contain an isomorphic copy of $\ell_1$, (ii) every real-valued sequentially continuous map on the unit ball $B_w$ with the weak topology is continuous, (iii) $B_w$ is a $k_{\mathbb{R}}$-space, (iv) $B_w$ is an Ascoli space.

We also prove that a Fréchet lcs $F$ does not contain an isomorphic copy of $\ell_1$ iff each closed and convex bounded subset of $F$ is Ascoli in the weak topology. Moreover we show that a Banach space $E$ in the weak topology is Ascoli iff $E$ is finite-dimensional. We supplement the last result by showing that a Fréchet lcs $F$ which is a quojection is Ascoli in the weak topology iff $F$ is either finite-dimensional or isomorphic to $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**1. Introduction.** Several topological properties of function spaces have been intensively studied for many years (see for instance [1, 17, 20] and references therein). In particular, various topological properties generalizing metrizability attracted a lot of attention. Let us mention, for example, the Fréchet–Urysohn property, sequentiality, the $k$-space property and the $k_{\mathbb{R}}$-space property (all relevant definitions are given in Section 2 below). It is well known that

$$\text{metric} \Rightarrow \text{Fréchet–Urysohn} \Rightarrow \text{sequential} \Rightarrow k\text{-space} \Rightarrow k_{\mathbb{R}}\text{-space},$$

and none of these implications is reversible (see [9, 21]).

2010 *Mathematics Subject Classification*: Primary 46A04, 46B25, 54C35; Secondary 28C15.

*Key words and phrases*: Banach space, Fréchet space, weak topology, Ascoli property.

Received 4 May 2015.

Published online 19 May 2016.

**Referencia al Prometeo en página 137, ver dorso**

DOI: 10.4064/sm8289-4-2016  

For topological spaces $X$ and $Y$, we denote by $C_k(X,Y)$ the space $C(X,Y)$ of all continuous functions from $X$ into $Y$ endowed with the compact-open topology. For $\mathbb{I}=[0,1]$, Pol [30] proved the following remarkable result.

Theorem 1.1 ([30]). *Let $X$ be a first countable paracompact space. Then the space $C_k(X,\mathbb{I})$ is a $k$-space if and only if $X$ is the topological sum of a locally compact Lindelöf space $L$ and a discrete space $D$.*

Theorem 1.1 easily implies the following result noticed in [13]; see also [19], where McCoy proved that for a first countable paracompact $X$ the space $C_k(X)$ is a $k$-space if and only if $X$ is hemicompact.

Corollary 1.2. *For a metric space $X$, the space $C_k(X)$ is a $k$-space if and only if $C_k(X)$ is a Polish space if and only if $X$ is a Polish locally compact space.*

Note also that by a result of Pytkeev [29], for a topological space $X$ the space $C_k(X)$ is a $k$-space if and only if it is Fréchet–Urysohn. For a metrizable space $X$ and the doubleton $\mathbf{2}=\{0,1\}$, the topological properties of the space $C_k(X,\mathbf{2})$ are thoroughly studied in [13].

For a topological space $X$, denote by $\psi: X\times C_k(X)\to\mathbb{R}$, $\psi(x,f):=f(x)$, the evaluation map. Recall that a subset $\mathcal{K}$ of $C_k(X)$ is *evenly continuous* if the restriction of $\psi$ onto $X\times\mathcal{K}$ is jointly continuous, i.e. for any $x\in X$, each $f\in\mathcal{K}$ and every neighborhood $O_{f(x)}\subset Y$ of $f(x)$ there exist neighborhoods $U_f\subset\mathcal{K}$ of $f$ and $O_x\subset X$ of $x$ such that $U_f(O_x):=\{g(y): g\in U_f,\ y\in O_x\}\subset O_{f(x)}$.

Following [3], a Tychonoff (Hausdorff) space $X$ is called an *Ascoli space* if each compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous. In other words, $X$ is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [20, p. 45].

It is easy to see that a space $X$ is Ascoli if and only if the canonical valuation map $X\hookrightarrow C_k(C_k(X))$ is an embedding (see [3]). By Ascoli's theorem [9, 3.4.20], each $k$-space is Ascoli. Moreover, Noble [23] proved that any $k_{\mathbb{R}}$-space is Ascoli. We have the implication

$$k_{\mathbb{R}}\text{-space} \;\Rightarrow\; \text{Ascoli},$$

and this implication is not reversible [3].

The aforementioned results motivate the following general question.

Question 1.3. *For which spaces $X$ and $Y$ is the space $C_k(X,Y)$ Ascoli?*

Below we present the following partial answer to this question.

Theorem 1.4. *For a metrizable space $X$, $C_k(X)$ is Ascoli if and only if $C_k(X)$ is a $k_{\mathbb{R}}$-space if and only if $X$ is locally compact.*

Corson [7] started a systematic study of various topological properties of the weak topology of Banach spaces. The famous Kaplansky Theorem states that a normed space $E$ in the weak topology has countable tightness; for further results see [8, 15]. Schlüchtermann and Wheeler [33] showed that an infinite-dimensional Banach space is never a $k$-space in the weak topology. We strengthen this result as follows.

Theorem 1.5. *A Banach space $E$ in the weak topology is Ascoli if and only if $E$ is finite-dimensional.*

Below we generalize Theorem 1.5 to an interesting class of Fréchet locally convex spaces, i.e. metrizable and complete locally convex spaces (lcs). We say that a Fréchet lcs $E$ is a *quojection* if it is isomorphic to the projective limit of a sequence of Banach spaces with surjective linking maps or, equivalently, if every quotient of $E$ which admits a continuous norm is a Banach space (see [4]). Obviously a countable product of Banach spaces is a quojection. Moscatelli [22] gave examples of quojections which are not isomorphic to countable products of Banach spaces.

Theorem 1.6. *Let a Fréchet lcs $E$ be a quojection. Then $E$ in the weak topology is Ascoli if and only if $E$ is either finite-dimensional or isomorphic to the product $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.*

Since every Fréchet lcs $C_k(X)$ is a quojection (see the survey [5]), Theorem 1.6 yields the following

Corollary 1.7. *For a Fréchet lcs $C_k(X)$, the space $C_k(X)$ in the weak topology is Ascoli if and only if $X$ is countable and discrete.*

Let $E$ be a Banach space; denote by $B_w$ the closed unit ball $B = B_E$ in $E$ endowed with the weak topology of $E$. Schlüchtermann and Wheeler [33] showed that some topological properties of $B_w$ are closely related to the isomorphic structure of $E$:

Theorem 1.8 ([33]). *The following conditions for a Banach space $E$ are equivalent:*

(a) *$B_w$ is Fréchet–Urysohn;*
(b) *$B_w$ is sequential;*
(c) *$B_w$ is a $k$-space;*
(d) *$E$ contains no isomorphic copy of $\ell_1$.*

Therefore it seems to be natural to verify whether there exists a Banach space $E$ containing a copy of $\ell_1$ and such that $B_w$ is Ascoli or a $k_{\mathbb{R}}$-space. We answer this question in the negative, by proving the following extension of Theorem 1.8.

Theorem 1.9. *Let $E$ be a Banach space and $B_w$ its closed unit ball with the weak topology. Then the following assertions are equivalent:*

(i) *$B_w$ is an Ascoli space;*
(ii) *$B_w$ is a $k_{\mathbb{R}}$-space;*
(iii) *every sequentially continuous real-valued map on $B_w$ is continuous;*
(iv) *$E$ does not contain a copy of $\ell_1$.*

The proof of (i)⇒(iv), given in Proposition 4.5 below, uses basic properties of stochastically independent measurable functions. We also present a result related to (ii): for Banach spaces containing an isomorphic copy of $\ell_1$ we provide, in a sense, a canonical example of a sequentially continuous but not continuous function on $B_w$. Our construction builds on measure-theoretic properties of $\ell_1$-sequences of continuous functions (see Example 5.2 below).

For Fréchet lcs we supplement Theorem 1.8 by proving the following theorem.

Theorem 1.10. *For a Fréchet lcs $E$ the following conditions are equivalent:*

(i) *$E$ contains no isomorphic copy of $\ell_1$;*
(ii) *each closed and convex bounded subset of $E$ is Ascoli in the weak topology.*

Theorems 1.9–1.10 heavily depend on our result stating that the closed unit ball $B$ of $\ell_1$ in the weak topology is not an Ascoli space (Proposition 4.1).

**2. The Ascoli property for function spaces. Proof of Theorem 1.4.** We start from the definitions of the following well-known notions. A topological space $X$ is called

- *Fréchet–Urysohn* if for any cluster point $a \in X$ of a subset $A \subset X$ there is a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ which converges to $a$;
- *sequential* if for each non-closed subset $A \subset X$ there is a sequence $\{a_n\}_{n\in\mathbb{N}} \subset A$ converging to some point $a \in \bar{A} \setminus A$;
- a *$k$-space* if for each non-closed subset $A \subset X$ there is a compact subset $K \subset X$ such that $A \cap K$ is not closed in $K$;
- a *$k_{\mathbb{R}}$-space* if a real-valued function $f$ on $X$ is continuous if and only if its restriction $f|_K$ to any compact subset $K$ of $X$ is continuous.

Recall that the family of subsets

$$[C; \epsilon] := \{f \in C_k(X) : |f(x)| < \epsilon \ \forall x \in C\},$$

where $C$ is a compact subset of $X$ and $\epsilon > 0$, forms a basis of open neighborhoods at the zero function $\mathbf{0} \in C_k(X)$. Below we give a simple sufficient condition for a space $X$ not to be Ascoli.

PROPOSITION 2.1. *Assume a Tychonoff space $X$ admits a family $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of $X$, a subset $A = \{a_i : i \in I\} \subset X$ and a point $z \in X$ such that*

(i) $a_i \in U_i$ *for every* $i \in I$;
(ii) $\big|\{i \in I : C \cap U_i \neq \emptyset\}\big| < \infty$ *for each compact subset $C$ of $X$;*
(iii) $z$ *is a cluster point of $A$.*

*Then $X$ is not an Ascoli space.*

*Proof.* For every $i \in I$, take a continuous function $f_i : X \to [0,1]$ such that $f_i(a_i) = 1$ and $f_i(X \setminus U_i) = \{0\}$. Set $\mathcal{K} := \{f_i : i \in I\} \cup \{\mathbf{0}\}$.

We claim that $\mathcal{K}$ is a compact subset of $C_k(X)$ and $\mathbf{0}$ is a unique cluster point of $\mathcal{K}$. Indeed, let $C$ be a compact subset of $X$ and $\epsilon > 0$. By (ii), the set $J := \{i \in I : C \cap U_i \neq \emptyset\}$ is finite. So, if $i \notin J$, then $f_i(C) = \{0\}$. Hence $f_i \in [C; \epsilon]$ for every $i \in I \setminus J$. This means that $\mathcal{K}$ is a compact set with the unique cluster point $\mathbf{0}$.

We show that $\mathcal{K}$ is not evenly continuous considering $\mathbf{0}$, $z$ and $O = (-1/2, 1/2)$. By the above claim, any neighborhood $U_{\mathbf{0}} \subset \mathcal{K}$ of $\mathbf{0}$ contains almost all functions $f_i$, and, by (iii), any neighborhood $O_z$ of $z$ contains infinitely many points $a_i$. So, there is $m \in I$ such that $f_m \in U_{\mathbf{0}}$ and $a_m \in O_z$. Since $f_m(a_m) = 1$, we see that $U_{\mathbf{0}}(O_z) \not\subset O$. Hence $\mathcal{K}$ is not evenly continuous. Thus $X$ is not Ascoli. ∎

The next corollary follows also from [3, Proposition 5.11(1)].

COROLLARY 2.2. *Let $X$ be a Tychonoff space with a unique cluster point $z$ and such that every compact subspace of $X$ is finite. Then $X$ is not an Ascoli space.*

*Proof.* Since every $x \in X$, $x \neq z$, is isolated, we set $I = A = X \setminus \{z\}$ and $U_x = \{x\}$ for $x \in A$. Now Proposition 2.1 applies. ∎

The proof of the next proposition is a modification of the proof of an assertion in [30, Section 5].

PROPOSITION 2.3. *Let $X$ be a first countable paracompact space. If $X$ is not locally compact, then $C_k(X)$ contains a countable family $\mathcal{U} = \{U_s\}_{s\in\mathbb{N}}$ of open subsets in $C_k(X)$ and a countable subset $A = \{a_s\}_{s\in\mathbb{N}} \subset C_k(X, \mathbb{I})$ such that*

(i) $a_s \in U_s$ *for every* $s \in \mathbb{N}$;
(ii) *if $K \subset C_k(X)$ is compact, then $\{s \in \mathbb{N} : U_s \cap K \neq \emptyset\}$ is finite;*
(iii) *the zero function $\mathbf{0}$ is a cluster point of $A$.*

*In particular, the spaces $C_k(X)$ and $C_k(X, \mathbb{I})$ are not Ascoli.*

*Proof.* Suppose for a contradiction that $X$ is not locally compact and let $x_0 \in X$ be a point with no compact neighborhood. Take open bases $\{V_i'\}_{i\in\mathbb{N}}$

and $\{W_i\}_{i\in\mathbb{N}}$ at $x_0$ such that

$$V_i' \supset \overline{W}_i \supset W_i \supset \overline{V}_{i+1}', \quad \forall i \in \mathbb{N}.$$

Set $P_i' := \overline{V}_i' \setminus V_{i+1}'$ for $i \in \mathbb{N}$. Since none of the sets $V_i'$ is compact, there exists a sequence $k_1 < k_2 < \cdots$ such that $P_{k_i}'$ is not compact and $k_{i+1} > k_i + 1$. Set $P_i = P_{k_i}'$ and $V_i = V_{k_i}'$. Then $\{P_i\}_{i\in\mathbb{N}}$ is a sequence of closed, non-compact subsets of $X$, $\{V_i\}_{i\in\mathbb{N}}$ is a decreasing open base at $x_0$ and

$$P_i \subset \overline{V}_i \setminus \overline{W}_{k_i+1} \quad \text{and} \quad \overline{V}_{i+1} \subset W_{k_i+1}. \tag{2.1}$$

Fix $i \in \mathbb{N}$. Since $P_i$ is not compact, by [9, 3.1.23] there is a one-to-one sequence $\{x_{j,i}\}_{j\in\mathbb{N}} \subset P_i$ which is discrete and closed in $X$. Now the paracompactness of $X$ and (2.1) imply that there exists an open sequence $\{V_{j,i}\}_{j\in\mathbb{N}}$ such that

$$x_{j,i} \in V_{j,i}, \quad V_{j,i} \cap \overline{V}_{i+1} = \emptyset, \forall j \in \mathbb{N}, \quad \{V_{j,i}\}_{j\in\mathbb{N}} \text{ is discrete in } X. \tag{2.2}$$

For every $p, q \in \mathbb{N}$ such that $1 \le p < q$, choose continuous functions $f_{q,p} : X \to [0,1]$ such that

$$\begin{aligned} &f_{q,p}(x_{q,p}) = 1, \quad f_{q,p}(x_{q,q}) = 0, \quad f_{q,p}(x_0) = 1/p, \\ &f_{q,p}(x) \le 1/p \quad \text{for } x \notin V_{q,p}. \end{aligned} \tag{2.3}$$

Set $A := \{f_{q,p} : 1 \le p < q < \infty\}$ and $\mathcal{U} = \{U_{q,p} : 1 \le p < q < \infty\}$, where $U_{q,p}$ is the set of all functions $h \in C_k(X)$ satisfying

$$|h(x_{q,p}) - 1| < \frac{1}{4^{p+q}}, \quad \left| h(x_0) - \frac{1}{p} \right| < \frac{1}{4^{p+q}}, \quad |h(x_{q,q})| < \frac{1}{4^{p+q}}. \tag{2.4}$$

Let us show that $A$ and $\mathcal{U}$ are as desired. Clearly, (i) holds. Let us prove (ii).

Fix a compact subset $K$ of $C_k(X)$. Let us first observe that

$$\exists p_0 \in \mathbb{N} : \quad \text{if } p \ge p_0 \text{ and } q > p, \text{ then } U_{q,p} \cap K = \emptyset. \tag{2.5}$$

Indeed, otherwise we would find sequences $p_1 < q_1 < p_2 < q_2 < \cdots$ and $h_{q_i,p_i} \in U_{q_i,p_i} \cap K$. Set

$$Z_1 := \{x_{q_i,p_i} : i \in \mathbb{N}\} \cup \{x_0\}.$$

From (2.1) it follows that $Z_1$ is compact, and thus, by the Ascoli Theorem [9, 3.4.20], there exists $r > 10$ such that if $z', z'' \in Z_1 \cap \overline{V}_r$ and $f \in K$, then $|f(z') - f(z'')| < 1/3$. But since $10 < r \le p_r < q_r$ we obtain $x_0, x_{q_r,p_r} \in Z_1 \cap \overline{V}_r$. Hence, by (2.4), we have

$$|h_{q_r,p_r}(x_{q_r,p_r}) - h_{q_r,p_r}(x_0)| > \left(1 - \frac{1}{4^{20}}\right) - \left(\frac{1}{p_r} + \frac{1}{4^{20}}\right) > \frac{1}{3}.$$

Since $h_{q_r,p_r} \in K$, we get a contradiction.

We shall now prove that

$$\exists q_0 \in \mathbb{N} : \quad \text{if } q \ge q_0 \text{ and } 1 \le p < p_0, \text{ then } U_{q,p} \cap K = \emptyset, \tag{2.6}$$

where $p_0$ is defined in (2.5). Indeed, set

$$Z_2 := \{x_{j,i} : 1 \le j \le i < \infty\} \cup \{x_0\}.$$

Then $Z_2$ is compact by (2.1). Again by the Ascoli Theorem, there exists $q_0 \in \mathbb{N}$ such that for $z', z'' \in Z_2 \cap \overline{V}_{q_0}$ and $f \in K$ we have $|f(z') - f(z'')| < 1/(4p_0)$. Then $q_0$ satisfies (2.6), since otherwise there would exist $q \ge q_0$ and $1 \le p < p_0$ such that $U_{q,p} \cap K \ne \emptyset$. Fix $h_{q,p} \in U_{q,p} \cap K$. Then $x_0, x_{q,q} \in Z_2 \cap \overline{V}_{q_0}$, and by (2.3) and (2.4),

$$|h_{q,p}(x_{q,q}) - h_{q,p}(x_0)| > \left(\frac{1}{p} - \frac{1}{4^{p+q}}\right) - \frac{1}{4^{p+q}} > \frac{1}{3p} > \frac{1}{4p_0},$$

which gives a contradiction. Now (2.5) and (2.6) immediately imply (ii).

Now we prove (iii). Fix a compact subset $Z \subset X$ and $\epsilon > 0$. Choose $p_0$ such that $1/p_0 < \epsilon$. By (2.2), we can find $j_0 \in \mathbb{N}$ such that $Z \cap V_{j,p_0} = \emptyset$ for every $j \ge j_0$. Take $q_0 = p_0 + j_0$. Then $f_{q_0,p_0} \in A$, and for $z \in Z$ we have $z \notin V_{q_0,p_0}$, and thus, in accordance with (2.3), $f_{q_0,p_0}(z) \le 1/p_0 < \epsilon$. Thus $f_{q_0,p_0} \in [Z; \epsilon]$.

Finally, $C_k(X)$ and $C_k(X, \mathbb{I})$ are not Ascoli by Proposition 2.1. ∎

The next corollary proved by R. Pol solves Problem 6.8 in [3].

COROLLARY 2.4 ([31]). *For a separable metrizable space $X$, $C_k(X)$ is Ascoli if and only if $X$ is locally compact.*

*Proof.* If $C_k(X)$ is Ascoli, then $X$ is locally compact by Proposition 2.3. Conversely, if $X$ is a separable metrizable locally compact space, then $C_k(X)$ is even a Polish space. ∎

Recall that a family $\mathcal{N}$ of subsets of a topological space $X$ is called a *network* in $X$ if, whenever $x \in U$ with $U$ open in $X$, then $x \in N \subset U$ for some $N \in \mathcal{N}$. A space $X$ is called a *$\sigma$-space* if it is regular and has a $\sigma$-locally finite network. Any metrizable space is a $\sigma$-space by the Nagata–Smirnov Metrization Theorem.

Now Theorem 1.4 follows from the following theorem in which the equivalence of (i) and (ii) is well-known.

THEOREM 2.5. *Let $X$ be a first countable paracompact $\sigma$-space. Then the following assertions are equivalent:*

(i) *$X$ is a locally compact metrizable space;*
(ii) *$X = \bigoplus_{i\in\kappa} X_i$, where all $X_i$ are separable metrizable locally compact spaces;*
(iii) *$C_k(X)$ is a $k_{\mathbb{R}}$-space;*
(iv) *$C_k(X)$ is an Ascoli space;*
(v) *$C_k(X, \mathbb{I})$ is a $k_{\mathbb{R}}$-space;*
(vi) *$C_k(X, \mathbb{I})$ is an Ascoli space.*

*In cases* (i)–(vi), *the spaces* $C_k(X)$ *and* $C_k(X,\mathbb{I})$ *are the products of families of Polish spaces.*

*Proof.* (i)⇒(ii) follows from [9, 5.1.27].

(ii)⇒(iii),(v): If $X = \bigoplus_{i\in\kappa} X_i$, then

$$C_k(X) = \prod_{i\in\kappa} C_k(X_i) \quad \text{and} \quad C_k(X,\mathbb{I}) = \prod_{i\in\kappa} C_k(X_i,\mathbb{I}),$$

where all the spaces $C_k(X_i)$ and $C_k(X_i,\mathbb{I})$ are Polish (see Corollary 1.2). So $C_k(X)$ and $C_k(X,\mathbb{I})$ are $k_{\mathbb{R}}$-spaces by [24, Theorem 5.6].

(iii)⇒(iv) and (v)⇒(vi) follow from [23]. The implications (iv)⇒(i) and (vi)⇒(i) follow from Proposition 2.3 and the fact that any locally compact $\sigma$-space is metrizable by [25]. ■

Note that Theorem 2.5 holds true for first countable stratifiable spaces since any stratifiable space is a paracompact $\sigma$-space (see [16, Theorems 5.7 and 5.9]).

**3. Proofs of Theorems 1.5 and 1.6.** Following Arhangel'skii [1, II.2], we say that a topological space $X$ has *countable fan tightness at a point* $x \in X$ if for any sets $A_n \subset X$, $n \in \mathbb{N}$, with $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$ there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\bigcup_{n\in\mathbb{N}} F_n}$; and $X$ has *countable fan tightness* if it has countable fan tightness at each point. Clearly, if $X$ has countable fan tightness, then it also has countable tightness.

For a topological space $X$ we denote by $C_p(X)$ the space $C(X)$ endowed with the topology of pointwise convergence.

For a lcs $E$, denote by $E'$ the dual space of $E$. The space $E$ endowed with the weak topology $\sigma(E,E')$ is denoted by $E_w$. The closure of a subset $A \subset E$ in $\sigma(E,E')$ is denoted by $\overline{A}^w$. If $E$ is a metrizable lcs, then $X := (E', \sigma(E',E))$ is $\sigma$-compact by the Alaoglu–Bourbaki Theorem. Since $E_w$ embeds into $C_p(X)$, Theorem II.2.2 of [1] immediately implies the following result noticed in [15].

FACT 3.1 ([15]). *If* $E$ *is a metrizable lcs, then* $E_w$ *has countable fan tightness.*

Denote the unit sphere of a normed space $E$ by $S_E$. Theorem 1.5 immediately follows from the next proposition.

PROPOSITION 3.2. *Let* $E$ *be a normed space. Then* $E$ *with the weak topology is Ascoli if and only if it is finite-dimensional.*

*Proof.* We show that $E_w$ is not Ascoli for any infinite-dimensional normed space $E$.

For every $n \in \mathbb{N}$, let $A_n$ be a countable subset of $nS$ such that $0 \in \overline{A_n}^w$ (see [10, Exercise 3.46] and Fact 3.1). Now Fact 3.1 implies that there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, such that $0 \in \overline{\bigcup_{n\in\mathbb{N}} F_n}$. Set $A := \bigcup_{n\in\mathbb{N}} F_n$. Using the Hahn–Banach Theorem, for every $n \in \mathbb{N}$ and each $a \in F_n$ take a weakly open neighborhood $U_a$ of $a$ such that

$$U_a \cap (n - 1/2)B = \emptyset. \tag{3.1}$$

Let us show that the family $\mathcal{U} = \{U_a : a \in A\}$, the set $A$ and the zero 0 satisfy conditions (i)–(iii) of Proposition 2.1. Clearly, (i) and (iii) hold. To check (ii), let $C$ be a compact subset of $E_w$. Then $C \subset mB$ for some $m \in \mathbb{N}$, and (3.1) implies that the set

$$\{a \in A : U_a \cap C \neq \emptyset\} \subset \bigcup_{n\le m} F_n$$

is finite. Finally, Proposition 2.1 implies that $E_w$ is not Ascoli. ∎

We also need the following

Proposition 3.3. *Let $p : X \to Y$ be an open continuous map of a topological space $X$ onto a regular space $Y$. If $X$ is Ascoli, then so is $Y$.*

*Proof.* Let $\mathcal{K}$ be a compact subset of $C_k(Y)$. We have to show that $\mathcal{K}$ is evenly continuous. Denote by $p^* : C_k(Y) \to C_k(X)$, $p^*(h) := h(p(x))$, the adjoined continuous map.

Fix $y_0 \in Y$, $h_0 \in \mathcal{K}$ and an open neighborhood $O_{z_0}$ of the point $z_0 := h_0(y_0)$. Set $f := p^*(h_0) \in C_k(X)$ and take any preimage $x_0$ of $y_0$, so $p(x_0) = y_0$. Since $p^*(\mathcal{K})$ is a compact subspace of $C_k(X)$ it is evenly continuous. Hence we can find neighborhoods $U_f \subset p^*(\mathcal{K})$ of $f$ and $O_{x_0} \subset X$ of $x_0$ such that $U_f(O_{x_0}) \subset O_{z_0}$. Set $U_{h_0} := \mathcal{K} \cap (p^*)^{-1}(U_f)$ and $O_{y_0} := p(O_{x_0})$ (which is a neighborhood of $y_0$ as $p$ is open). For every $h \in U_{h_0}$ and each $y \in O_{y_0}$, take $x \in O_{x_0}$ with $p(x) = y$, so we obtain

$$h(y) = h(p(x)) = p^*(h)(x) \in O_{z_0}.$$

Thus $\mathcal{K}$ is evenly continuous, and therefore $Y$ is Ascoli. ∎

*Proof of Theorem 1.6.* Assume that $E$ is infinite-dimensional. By Proposition 3.2 the space $E$ is not normed. Let $(p_n)_n$ be a sequence of continuous seminorms providing the topology of $E$. For each $n \in \mathbb{N}$, let $E_n := E/p_n^{-1}(0)$ be the quotient endowed with the norm topology $p_n^* : [x] \mapsto p_n(x)$, where $[x]$ is the equivalence class of $x$ in $E$. Since $E$ is a quojection, the quotient $E_n$ with the original quotient topology is a Banach space by [4, Proposition 3].

By Proposition 3.3 the space $E_n$ endowed with the weak topology is Ascoli, so we apply Proposition 3.2 to deduce that each $E_n$ is finite-dimensional. On the other hand, $E$ embeds into the product $\prod_n E_n$. So $E$, being complete, is isomorphic to a closed subspace of $\mathbb{K}^{\mathbb{N}}$. Thus $E$ is also isomorphic to $\mathbb{K}^{\mathbb{N}}$ by [27, Corollary 2.6.5]. ∎

*Proof of Corollary 1.7.* By Theorem 1.6 the space $C_k(X)$ is isomorphic to $\mathbb{R}^{\mathbb{N}}$, and since $\mathbb{R}^{\mathbb{N}}$ does not admit a weaker locally convex topology (see [27, Corollary 2.6.5]), $C_k(X) = C_p(X) = \mathbb{R}^{\mathbb{N}}$. Thus $X$ is countable and discrete. The converse assertion is trivial. ∎

We do not know whether there exists a Fréchet space $E$ such that $E_w$ is an Ascoli non-metrizable space.

REMARK 3.4. The first example of a non-distinguished Fréchet space (so also not a quojection) was given by Grothendieck and Köthe, and it was the Köthe echelon space $\lambda_1(A)$ of order 1 for the Köthe matrix $A = (a_n)_n$ defined on $\mathbb{N}\times\mathbb{N}$ by $a_n(i,j) := j$ if $i < n$ and $a_n(i,j) = 1$ otherwise (see [5] for more references). We do not know however if this space with the weak topology is an Ascoli space.

**4. Proof of Theorem 1.9.** To prove Theorem 1.9 we need the following key proposition, which proves, among other things, that the unit ball $B_{\ell_1}$ in the weak topology is not Ascoli. In particular, since the $k$-space property is inherited by closed subspaces, this also shows that any Banach space $E$ whose weak unit ball $B_w$ is a $k$-space contains no isomorphic copy of $\ell_1$, i.e. the proposition proves (c)⇒(d) in Schlüchtermann–Wheeler's Theorem 1.8. A sequence $\{x_i\}_{i\in\mathbb{N}} \subset E$ is called *trivial* if there is $n \in \mathbb{N}$ such that $x_i = x_n$ for all $i > n$.

PROPOSITION 4.1. *Let $E = \ell_1$ and $B_w$ its closed unit ball in the weak topology. Then there is a countable subset $A$ of $S_{\ell_1}$ and a family $\mathcal{U} = \{U_a : a \in A\}$ of weakly open subsets of the unit ball $B$ such that*

(1) $a \in U_a$ *for any* $a \in A$;
(2) $\operatorname{dist}(U_a, U_b) \geq 1/5$ *for any distinct* $a, b \in A$;
(3) *the zero* $0$ *is the unique cluster point of* $A$;
(4) $|\{a \in A : C \cap U_a \neq \emptyset\}| < \infty$ *for every weakly compact subset* $C$ *of* $B$;
(5) $\overline{A}^{w} = A \cup \{0\}$ *and every weakly compact subset of* $\overline{A}^{w}$ *is finite;*
(6) *$A$ contains a sequence which is equivalent to the unit basis of* $\ell_1$;
(7) *$A$ has no non-trivial weakly fundamental subsequence;*
(8) *the countable space $\overline{A}^{w}$ and $B_w$ are not Ascoli.*

*Proof.* Let $\{(e_i, e_i^*) : i \in \mathbb{N}\}$ be the standard biorthogonal basis in $\ell_1 \times \ell_1' = \ell_1 \times \ell_\infty$. Following [15], set $\Omega := \{(m,n) \in \mathbb{N}\times\mathbb{N} : m < n\}$ and

$$A := \left\{ a_{m,n} := \frac{1}{2}(e_m - e_n) : (m,n) \in \Omega \right\} \subset S_{\ell_1}.$$

For every $(m,n)\in\Omega$, define the following weak neighborhood of $a_{m,n}$:

$$U_{m,n} := \left\{ x\in B : |\langle e^*_m, a_{m,n}-x\rangle| < \frac{1}{10} \text{ and } |\langle e^*_n, a_{m,n}-x\rangle| < \frac{1}{10}\right\}$$
$$= \left\{ x=(x_i)\in B : \left|\frac{1}{2}-x_m\right| < \frac{1}{10} \text{ and } \left|\frac{1}{2}+x_n\right| < \frac{1}{10}\right\}.$$

Then (1) holds trivially. Let us check (2). For every $k\notin\{m,n\}$ and each $x=(x_i)\in U_{m,n}$, one has

$$|x_k| \le \|x\| - |x_m| - |x_n| < 1 - \left(\frac{1}{2}-\frac{1}{10}\right) - \left(\frac{1}{2}-\frac{1}{10}\right) = \frac{1}{5}.$$

So, if $(m,n)\neq(k,l)$ and $x=(x_i)\in U_{m,n}$, we obtain either

$$\left|\frac{1}{2}-x_k\right| > \frac{1}{2}-\frac{1}{5} = \frac{3}{10} \text{ if } k\notin\{m,n\}, \quad \text{or} \quad \left|\frac{1}{2}+x_l\right| > \frac{3}{10} \text{ if } l\notin\{m,n\}.$$

Hence $\mathrm{dist}(U_{m,n}, U_{k,l}) \ge 3/10 - 1/10 = 1/5$ for all $(m,n)\neq(k,l)$. This proves (2). In particular, every point of $A$ is weakly isolated.

To prove (3) we note first that $0\in\overline{A}^w$ by [15, Lemma 3.2]. We provide a proof of this result to keep the paper self-contained. Let $U$ be a neighborhood of 0 of the canonical form

$$U = \{x\in\ell_1 : |\langle\chi_k, x\rangle| < \epsilon, \text{ where } \chi_k = (\chi_k(i))_{i\in\mathbb{N}}\in S_{\ell_\infty} \text{ for } 1\le k\le s\}.$$

Let $I$ be an infinite subset of $\mathbb{N}$ such that, for every $1\le k\le s$, either $\chi_k(i)>0$ for all $i\in I$, or $\chi_k(i)=0$ for all $i\in I$, or $\chi_k(i)<0$ for all $i\in I$. Take a natural number $N>1/\epsilon$. Since $I$ is infinite, by induction, one can find $(m,n)\in\Omega$ satisfying the following condition: for every $1\le k\le s$ there is $0<t_k\le N$ such that

$$\text{(4.1)}\qquad \frac{t_k-1}{N} \le \min\{|\chi_k(m)|, |\chi_k(n)|\} \le \max\{|\chi_k(m)|, |\chi_k(n)|\} \le \frac{t_k}{N}.$$

Then, by the construction of $I$, we obtain

$$|\langle\chi_k, a_{m,n}\rangle| < 1/N < \epsilon \quad \text{for every } 1\le k\le s.$$

Thus $a_{m,n}\in U$, and hence $0\in\overline{A}^w$.

Now fix a non-zero $z=(z_i)\in\ell_1$ and consider the following three cases.

(a) There is $z_i\notin\{-1/2,0,1/2\}$, so $z\notin A$. Set

$$\epsilon := \frac{1}{2}\min\left\{|z_i|, \left|z_i-\frac{1}{2}\right|, \left|z_i+\frac{1}{2}\right|\right\}, \qquad U := \{x\in\ell_1 : |\langle e^*_i, z-x\rangle| < \epsilon\}.$$

Clearly, $U\cap A=\emptyset$ and $z\notin\overline{A}^w$.

(b) Assume that $z\notin A$ and $z_i\in\{-1/2,0,1/2\}$ for every $i\in\mathbb{N}$. So there are distinct $i$ and $j$ such that $z_i=z_j\in\{-1/2,1/2\}$. Set

$$U := \{x\in\ell_1 : |\langle e^*_i + e^*_j, z-x\rangle| < 1/10\}.$$

By the definition of $A$, we obtain $U\cap A=\emptyset$, and hence $z\notin\overline{A}^w$.

(c) Assume that $z \in A$. Then $z$ is not a cluster point of $A$ because it is weakly isolated.

Now (a)–(c) prove (3). Let us prove (4). Fix a weakly compact subset $C$ of $\ell_1$. Assuming that $C \cap U_a \neq \emptyset$ for an infinite subset $J \subset A$ we choose $x_j \in C \cap U_j$ for every $j \in J$. Since $\ell_1$ has the Schur property, $C$ is also compact in the norm topology of $\ell_1$. So we can assume that $x_j$ converges to some $x_\infty \in C$ in the norm topology. But this contradicts (2), proving (4).

(5) immediately follows from (3) and (4).

(6): Clearly, the sequence $\{a_{1,i}\}_{i>1} \subset A$ is equivalent to the unit basis of $\ell_1$.

(7): Assuming the converse let $\{a_{m_i,n_i}\}_{i\in\mathbb{N}}$ be a faithfully indexed weakly fundamental subsequence of $A$. Then only the next two cases are possible.

Case 1: *There is $k \in \mathbb{N}$ and $i_1 < i_2 < \cdots$ such that $k = m_{i_1} = m_{i_2} = \cdots$.* Passing to a subsequence we can assume that $m_1 = m_2 = \cdots = k$ and $k < n_1 < n_2 < \cdots$. Set

$$\chi := (\chi_j)_{j\in\mathbb{N}} \in \ell_\infty, \quad \text{where} \quad \chi_j = \begin{cases} -1 & \text{if } j \in \{n_2, n_4, \dots\}, \\ 0 & \text{if } j \notin \{n_2, n_4, \dots\}. \end{cases}$$

Then $\chi \in S_{\ell_\infty}$ and

$$\langle \chi, a_{k,n_{2s}} - a_{k,n_{2s+1}} \rangle = \frac{1}{2}, \quad \forall s \in \mathbb{N}.$$

Thus the sequence $\{a_{m_i,n_i}\}_{i\in\mathbb{N}}$ is not fundamental, a contradiction.

Case 2: *$m_i \to \infty$ and $n_i \to \infty$.* Passing to a subsequence if necessary, we can assume that

$$m_1 < n_1 < m_2 < n_2 < \cdots .$$

Defining $\chi \in S_{\ell_\infty}$ as in Case 1, we obtain

$$\langle \chi, a_{m_{2s},n_{2s}} - a_{m_{2s+1},n_{2s+1}} \rangle = \frac{1}{2}, \quad \forall s \in \mathbb{N}.$$

Thus the sequence $\{a_{m_i,n_i}\}_{i\in\mathbb{N}}$ is not weakly fundamental in this case either.

Therefore $A$ does not have a weakly fundamental subsequence.

(8): The space $\overline{A}^w$ is not Ascoli by (5) and Corollary 2.2, and $B_w$ is not Ascoli by (1)–(4) and Proposition 2.1. ∎

Recall that a (normalized) sequence $(x_n)$ in a Banach space $E$ is said to be equivalent to the standard basis of $\ell_1$, or simply to be an $\ell_1$-*sequence*, if for some $\theta > 0$,

$$\Big\| \sum_{i=1}^n c_i x_i \Big\| \geq \theta \sum_{i=1}^n |c_i|$$

for any natural number $n$ and any scalars $c_i \in \mathbb{R}$. We also call such a sequence a $\theta$-$\ell_1$-sequence if we want to specify the constant in the definition.

We need some measure-theoretic preparations. Let $(T,\Sigma,\mu)$ be a probability space. Measurable functions $g_n : T \to \mathbb{R}$ are said to be *stochastically independent with respect to* $\mu$ if

$$\mu\Big(\bigcap_{n\le k} g_n^{-1}(B_n)\Big) = \prod_{n\le k} \mu(g_n^{-1}(B_n))$$

for every $k$ and any Borel sets $B_n \subseteq \mathbb{R}$; see e.g. Fremlin [11, 272] for basic facts concerning independence. Recall (see [11, 272Q]) that if integrable functions $f, g : T \to \mathbb{R}$ are independent with respect to $\mu$, then $\int_T f \cdot g\, d\mu = (\int_T f\, d\mu)\cdot(\int_T g\, d\mu)$.

Lemma 4.2. *Let $(T,\Sigma,\mu)$ and $(S,\Theta,\nu)$ be probability spaces and let $\Phi : T \to S$ be a measurable mapping such that $\Phi[\mu] = \nu$, that is $\mu(\Phi^{-1}(E)) = \nu(E)$ for every $E \in \Theta$. If $(p_n)_n$ be a sequence of measurable functions $S \to \mathbb{R}$ which is stochastically independent with respect to $\nu$, then the functions $g_n = p_n \circ \Phi$ are stochastically independent with respect to $\mu$.*

Lemma 4.2 is standard and follows for instance from [11, Theorem 272G].

In the proof of the crucial Proposition 4.5 we essentially use the following version of the Riemann–Lebesgue lemma, which is mentioned in Talagrand's [35, p. 3].

Theorem 4.3. *Let $(T,\Sigma,\mu)$ be any probability space and let $(g_n)_n$ be a stochastically independent uniformly bounded sequence of measurable functions $T \to \mathbb{R}$ with $\int_T g_n\, d\mu = 0$ for every $n$. Then*

$$\lim_{n\to\infty} \int_T f \cdot g_n\, d\mu = 0$$

*for every bounded measurable function $f : T \to \mathbb{R}$.*

Finally, let us recall the following fact (see e.g. [35, 1-2-5]).

Lemma 4.4. *Let $\Phi$ be a continuous surjection of a compact space $K$ onto a compact space $L$. If $\lambda$ is a regular probability Borel measure on $L$ then there exists a regular probability Borel measure $\mu$ on $K$ such that $\Phi[\mu] = \lambda$, that is, $\mu(\Phi^{-1}(B)) = \lambda(B)$ for every Borel set $B \subseteq L$.*

Proposition 4.5. *If a Banach space $E$ contains an isomorphic copy of $\ell_1$, then $B_w$ is not an Ascoli space.*

*Proof.* The proof is in four steps.

Step 1. Since the Hilbert cube $H = [0,1]^{\mathbb{N}}$ is separable, one can find a continuous function $\Phi_0$ from the discrete space $\mathbb{N}$ onto a dense subset of $H$. By [9, Theorem 3.6.1], we can extend $\Phi_0$ to a continuous map $\Phi : \beta\mathbb{N} \to H$. As $\Phi_0(\mathbb{N})$ is dense in $H$, we obtain $\Phi(\beta\mathbb{N}) = H$. Let $\pi_n : H \to [-1,1]$ be the projection onto the $n$th coordinate, and let $\lambda = \prod_n \mathrm{m}_n$ be the product of the normalized Lebesgue measures $\mathrm{m}_n$ on $[-1,1]$. Then the sequence $(\pi_n)$ is stochastically independent with respect to $\lambda$ and

$$\int_H \pi_n\,d\lambda = \int_H \pi_n\pi_m\,d\lambda = 0, \qquad \int_H \pi_n^2\,d\lambda = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{3} \tag{4.2}$$

for all $n,m\in\mathbb{N}$ with $n\neq m$. Moreover, $(\pi_n)_n$ is a 1-$\ell_1$-sequence in $C(H)$. Indeed, for every $n\in\mathbb{N}$ and any scalars $c_1,\dots,c_n\in\mathbb{R}$, set

$$x := (\mathrm{sign}(c_1),\dots,\mathrm{sign}(c_n),0,\dots)\in H.$$

Then $\sum_{i\le n} c_i\pi_i(x) = \sum_{i\le n}|c_i|$. Thus $(\pi_n)$ is a 1-$\ell_1$-sequence in $C(H)$.

Step 2. Let $\mu$ be a measure on $\beta\mathbb{N}$ such that $\Phi[\mu]=\lambda$ (see Lemma 4.4). Set $g_n := \pi_n\circ\Phi$ for every $n\in\mathbb{N}$. Then the sequence $(g_n)$ is stochastically independent with respect to $\mu$ by Lemma 4.2. As $\Phi$ is surjective, $(g_n)$ is also a 1-$\ell_1$-sequence in $C(\beta\mathbb{N})$.

Step 3. Let $Y$ be a subspace of $E$ isomorphic to $\ell_1$ and let $T_1 : Y\to\ell_1$ be an isomorphism. For every $n\in\mathbb{N}$ choose $x_n\in Y$ such that $T_1(x_n)=e_n$, where $(e_n)$ is the standard coordinate basis in $\ell_1$. In turn, as $(g_n)$ is a 1-$\ell_1$-sequence in $C(\beta\mathbb{N})$, there is an isometric embedding $T_2:\ell_1\to C(\beta\mathbb{N})$, sending $e_n$ to $g_n$.

As the space $C(\beta\mathbb{N})$ is 1-injective, the operator $T=T_2\circ T_1 : Y\to C(\beta\mathbb{N})$ can be extended to an operator $\widetilde{T} : E\to C(\beta\mathbb{N})$ having the same norm (cf. [10, Proposition 5.10]).

Step 4. Set $d:=\sup\{\|x_n\|_E : n\in\mathbb{N}\}$ and $\gamma:=\sup\{\|\widetilde{T}(x)\| : x\in dB_E\}$. Let $h_{m,n}=(g_m-g_n)/2$ for $n,m\in\mathbb{N}, n>m$, and set

$$V_{m,n}=\Big\{f\in\gamma B_{C(\beta\mathbb{N})} : \Big|\int_{\beta\mathbb{N}} f\cdot g_i\,d\mu\Big| > 1/4 \text{ for } i=m,n\Big\}.$$

Denote by $T^+$ the map $\widetilde{T}$ from $E_w$ into $C_w(\beta\mathbb{N})$. Clearly, $T^+$ is also continuous. Finally we set

$$A := \{a_{m,n} := (x_m-x_n)/2 : 1\le m<n\},$$
$$\mathcal{U} := \{U_{m,n} := (T^+)^{-1}(V_{m,n})\cap dB_E : 1\le m<n\}.$$

Now the following claim finishes the proof.

Claim. *The ball $dB_E$ is not Ascoli in the weak topology.*

To prove the claim it is enough to check (i)–(iii) of Proposition 2.1 for the set $A$ and the family $\mathcal{U}$.

(i): To show that $a_{m,n}\in U_{m,n}$ it is enough to prove that $h_{m,n}\in V_{m,n}$. But this follows from (4.2) since

$$2\int_{\beta\mathbb{N}} h_{m,n}\cdot g_n\,d\mu = \int_{\beta\mathbb{N}} g_m\cdot g_n\,d\mu - \int_{\beta\mathbb{N}} g_n^2\,d\mu = -\frac{2}{3} = -2\int_{\beta\mathbb{N}} h_{m,n}\cdot g_m\,d\mu.$$

(iii): The zero function $\mathbf{0}$ is the weak cluster point of $A$ by Proposition 4.1.

Let us check (ii), i.e. if $C \subseteq dB_E$ is weakly compact, then $C$ can meet only a finite number of $U_{m,n}$'s. Suppose otherwise: let $x_i \in C \cap U_{m_i,n_i}$, where the pairs $(m_i, n_i)$ are distinct. As $m_i < n_i$ we may also assume that $n_i \neq n_{i'}$ for $i \neq i'$. Since $C$ is weakly compact, it is Fréchet–Urysohn by the Eberlein–Shmul'yan theorem [10, 3.109]. So we can further assume that the $x_i$ converge weakly to some $x \in C$. Then also the functions $f_i := T^+(x_i) \in V_{m_i,n_i}$ converge weakly to $f := T^+(x) \in T^+(C) \subset \gamma B_{C(\beta\mathbb{N})}$, and they are uniformly bounded on $\beta\mathbb{N}$ and $f_i \to f$ pointwise.

Take $0 < \delta < 1/16(1 + \gamma + 2\gamma^2)$. By Theorem 4.3, there is $N_1 \in \mathbb{N}$ such that $|\int_{\beta\mathbb{N}} f \cdot g_{n_i}\, d\mu| < \delta$ for all $i > N_1$. By the classical Egorov theorem, $f_i$ converge almost uniformly to $f$, i.e. there is $B \subseteq \beta\mathbb{N}$ such that $\mu(\beta\mathbb{N} \setminus B) < \delta$ and $f_i$ converge uniformly to $f$ on $B$. Take $N_2 > N_1$ such that $|f_i - f| < \delta$ on $B$ for all $i > N_2$. Taking into account that $|h| \leq \gamma$ for each $h \in \gamma B_{C(\beta\mathbb{N})}$, for every $i > N_2$ we obtain

$$\begin{aligned}\Big|\int_{\beta\mathbb{N}} f_i \cdot g_{n_i}\, d\mu\Big| &\leq \Big|\int_{\beta\mathbb{N}} f_i \cdot g_{n_i}\, d\mu - \int_{\beta\mathbb{N}} f \cdot g_{n_i}\, d\mu\Big| + \Big|\int_{\beta\mathbb{N}} f \cdot g_{n_i}\, d\mu\Big| \\ &\leq \int_{\beta\mathbb{N}} |f_i - f| \cdot |g_{n_i}|\, d\mu + \delta \leq \int_B + \int_{\beta\mathbb{N}\setminus B} + \delta \\ &\leq \gamma \cdot \delta + 2\gamma^2 \cdot \delta + \delta = \delta(1 + \gamma + 2\gamma^2) < 1/16.\end{aligned}$$

On the other hand, $f_i \in V_{m_i,n_i}$ implies $|\int_{\beta\mathbb{N}} f_i \cdot g_{n_i}\, d\mu| > 1/4$. This contradiction proves the claim. ∎

To prove Theorem 1.9 we also need the following simple lemma.

LEMMA 4.6. *Let $E$ be a Banach space and let $B_w$ denote the unit ball of $E$ equipped with the weak topology. For any function $f : B_w \to \mathbb{R}$ the following are equivalent:*

(i) *$f$ is sequentially continuous on $B_w$;*
(ii) *$f$ is continuous on every compact subset of $B_w$.*

*Proof.* Let $f$ be sequentially continuous on $B_w$ and let $C$ be a compact subset of $B_w$. For any closed set $H \subseteq \mathbb{R}$, the set $F = f^{-1}(H) \cap C$ is sequentially closed in $C$. Hence $F$ is closed in $C$, since $C$, as a weakly compact set, has the Fréchet–Urysohn property by the Eberlein–Shmul'yan theorem.

We have checked that (i) implies (ii); the reverse implication is obvious. ∎

*Proof of Theorem 1.9.* (i)⇒(iv) follows from Proposition 4.5. Theorem 1.8 implies (iv)⇒(iii). (iii)⇒(ii) follows from Lemma 4.6. Finally, (ii)⇒(i) holds by [23]. ∎

Remark 4.7. Banakh [2] provided a different proof of the equivalence (i)$\Leftrightarrow$(iv) in Theorem 1.9.

**5. On weakly sequentially continuous functions on the unit ball.** Let $E$ be a Banach space containing an isomorphic copy of $\ell_1$ and let $B_w$ denote the unit ball in $E$ equipped with the weak topology. It follows from Theorem 1.9 that $B_w$ is not a $k_{\mathbb{R}}$-space, which, in view of Lemma 4.6, is equivalent to saying that there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous. We show below that such a function can be defined, in a sense, effectively by means of measure-theoretic properties of $\ell_1$-sequences of continuous functions.

Proposition 5.1. *Let $K$ be a compact space and let $(g_n)$ be a normalized $\theta$-$\ell_1$-sequence in the Banach space $C(K)$. Then there exists a regular probability measure $\mu$ on $K$ such that*

$$\int_K |g_n - g_k|\, d\mu \ge \theta/2 \qquad \text{whenever } n \neq k.$$

*Proof.* Suppose that $(g_n)$ is $\theta$-equivalent to the standard basis $(e_n)$ in $\ell_1$. Set

$$H = \overline{\operatorname{conv}}(\{|g_n - g_k| : n \neq k\}) \subseteq C(K).$$

Note that it is enough to check that $\|h\| \ge \theta/2$ for all $h \in H$ since in that case, by the separation theorem, there is a norm-one $\mu \in C(K)^*$ such that $\int_K h\, d\mu \ge \theta/2$ for every $h \in H$. As $h \ge 0$ for $h \in H$, we can then replace the signed measure $\mu$ by its variation $|\mu|$.

In turn, the fact that $\|h\| \ge \theta/2$ for $h \in H$ is a consequence of the following.

Claim. *Suppose that $n_i \neq k_i$ for $i \le p$. Then for any convex coefficients $\alpha_1, \ldots, \alpha_p$,*

$$\Big\|\sum_{i=1}^{p} \alpha_i |g_{n_i} - g_{k_i}|\Big\| \ge \theta/2.$$

We shall verify the claim in two steps.

Step 1. There is $E \subseteq \{1, \ldots, p\}$ such that

$$\Big\|\sum_{i\in E} \alpha_i (e_{n_i} - e_{k_i})\Big\| \ge 1/2.$$

Indeed, if $L$ denotes the Cantor set $\{-1, 1\}^{\mathbb{N}}$, then the projections $\pi_n : L \to \{-1, 1\}$ form a sequence in $C(L)$ which is a 1-$\ell_1$-sequence, so we have an isometric embedding $T : \ell_1 \to C(L)$, where $Te_n = \pi_n$ for every $n \in \mathbb{N}$.

Write $\lambda$ for the standard product measure on $L$. We calculate directly that $\int_K |\pi_n - \pi_k|\, d\lambda = 1$ for $n \neq k$ and therefore

$$\Big\|\sum_{i=1}^{p} \alpha_i |\pi_{n_i} - \pi_{k_i}|\Big\| \ge \int_L \sum_{i=1}^{p} \alpha_i |\pi_{n_i} - \pi_{k_i}|\, d\lambda = 1.$$

Hence there is $t \in L$ such that $\sum_{i=1}^{p} \alpha_i |\pi_{n_i}(t) - \pi_{k_i}(t)| \geq 1$. Examining the signs of summands we conclude that for some set $E \subseteq \{1, \ldots, p\}$ we have

$$\Big| \sum_{i \in E} \alpha_i (\pi_{n_i}(t) - \pi_{k_i}(t)) \Big| \geq 1/2.$$

This implies that

$$\begin{aligned} \Big\| \sum_{i \in E} \alpha_i (e_{n_i} - e_{k_i}) \Big\| &= \Big\| \sum_{i \in E} \alpha_i (Te_{n_i} - Te_{k_i}) \Big\| \\ &\geq \Big| \sum_{i \in E} \alpha_i (\pi_{n_i}(t) - \pi_{k_i}(t)) \Big| \geq 1/2. \end{aligned}$$

STEP 2. Taking a set $E$ from Step 1 we conclude that

$$\Big\| \sum_{i=1}^{p} \alpha_i |g_{n_i} - g_{k_i}| \Big\| \geq \Big\| \sum_{i \in E} \alpha_i (g_{n_i} - g_{k_i}) \Big\| \geq \theta \cdot \Big\| \sum_{i \in E} \alpha_i (e_{n_i} - e_{k_i}) \Big\| \geq \theta/2.$$

This verifies the claim, and thus the proof is complete. ∎

EXAMPLE 5.2. *Suppose that $E$ is a Banach space containing an isomorphic copy of $\ell_1$. Then there is a function $\Phi : B_w \to \mathbb{R}$ which is sequentially continuous but not continuous.*

*Proof.* Let $K$ denote the dual unit ball $B_{E^*}$ equipped with the weak$^*$ topology. Write $Ix$ for the function on $K$ given by $Ix(x^*) = x^*(x)$ for $x^* \in K$. Then $I : E \to C(K)$ is an isometric embedding.

Since $E$ contains a copy of $\ell_1$, there is a normalized sequence $(x_n)$ in $E$ which is a $\theta$-$\ell_1$-sequence for some $\theta > 0$. Then the functions $g_n = Ix_n$ form a $\theta$-$\ell_1$-sequence in $C(K)$. By Proposition 5.1 there is a probability measure $\mu$ on $K$ such that $\int_K |g_n - g_k| \, d\mu \geq \theta/2$ whenever $n \neq k$.

Define a function $\Phi$ on $E$ by $\Phi(x) = \int_K |Ix| \, d\mu$. If $y_j \to y$ weakly in $E$ then $Iy_j \to Iy$ weakly in $C(K)$, i.e. $(Iy_j)_j$ is a uniformly bounded sequence converging pointwise to $Iy$. Consequently, $\Phi(y_j) \to \Phi(y)$ by the Lebesgue dominated convergence theorem. Thus $\Phi$ is sequentially continuous.

We now check that $\Phi$ is not weakly continuous at 0 on $B_w$. Consider a basic weak neighborhood of $0 \in B_w$ of the form

$$V = \{x \in B_w : |x_j^*(x)| < \varepsilon \text{ for } j = 1, \ldots, r\}.$$

Then there is an infinite set $N \subseteq \mathbb{N}$ such that $(x_j^*(x_n))_{n \in N}$ is a convergent sequence for every $j \leq r$. Hence there are $n \neq k$ such that $|x_j^*(x_n - x_k)| < \varepsilon$ for every $j \leq r$, which means that $(x_n - x_k)/2 \in V$. On the other hand, $\Phi((x_n - x_k)/2) \geq \theta/4$, which demonstrates that $\Phi$ is not continuous at 0. ∎

**6. Proof of Theorem 1.10 and final questions.** In order to prove Theorem 1.10 we need the following two results also of independent interest.

Proposition 6.1 ([14]). *Let $E$ be a metrizable lcs. Then every bounded subset of $E$ is Fréchet–Urysohn in the weak topology of $E$ if and only if every bounded sequence in $E$ has a Cauchy subsequence in the weak topology of $E$.*

Proposition 6.2 ([32]). *Let $E$ be a complete lcs such that every bounded set in $E$ is metrizable. Then $E$ does not contain a copy of $\ell_1$ if and only if every bounded sequence in $E$ has a Cauchy subsequence in the weak topology of $E$.*

*Proof of Theorem 1.10.* (i)⇒(ii): By Propositions 6.1 and 6.2 every bounded set $A$ in $E$ is even Fréchet–Urysohn in the weak topology of $E$. The converse implication (ii)⇒(i) follows from Theorem 1.9. ∎

We complete the paper with a few open questions. By Proposition 4.1, there is a countable (hence Lindelöf) non-Ascoli space $A$. So $A$ is homeomorphic to a closed subspace of some $\mathbb{R}^\kappa$. As $\mathbb{R}^\kappa$ is a $k_{\mathbb{R}}$-space, we see that a $k_{\mathbb{R}}$-space may contain a countable closed non-Ascoli subspace. So the $k_{\mathbb{R}}$-space property and the Ascoli property are not preserved in general by closed subspaces.

Question 6.3. *Let $X$ be an Ascoli space such that every closed subspace of $X$ is Ascoli. Is $X$ a $k$-space?*

Arhangel'skii [9, 3.12.15] proved that a topological space $X$ is a hereditarily $k$-space if and only if $X$ is Fréchet–Urysohn.

Question 6.4. *Let $X$ be a hereditarily Ascoli space. Is $X$ Fréchet–Urysohn?*

Let $E = C_p(\omega_1) = \mathbb{R}^{\omega_1}$. Then the lcs $E$ is a $k_{\mathbb{R}}$-space by [24, Theorem 5.6] and is not a $k$-space by [18, Problem 7.J(b)]. So the $k_{\mathbb{R}}$-space property and the Ascoli property are not equivalent to the $k$-space property for $C_p$-spaces (see the Pytkeev and Gerlits–Nagy theorem [1, II.3.7]).

Question 6.5. *For which Tychonoff spaces $X$ is the space $C_p(X)$ Ascoli (or a $k_{\mathbb{R}}$-space)?*

It is well-known (see [1, III.1.2]) that, for a compact space $K$, the space $C_p(K)$ is a $k$-space if and only if $K$ is scattered. Below we generalize this result.

Proposition 6.6. *Let $K$ be a compact space. Then $C_p(K)$ is a $k_{\mathbb{R}}$-space if and only if $K$ is scattered.*

*Proof.* If $K$ is scattered, then $C_p(K)$ is Fréchet–Urysohn, and we are done (see [1, Theorem III.1.2]). Now assume that $K$ is not scattered. Then there is a continuous map $p$ from $K$ onto $[0,1]$ by [34, 8.5.4]. Let $\lambda$ be the

Lebesgue measure on $[0,1]$. Take a measure $\mu$ on $K$ such that $p[\mu]=\lambda$ (see Lemma 4.4). Note that the measure $\mu$ vanishes on points. If we define

$$\Psi(g)=\int_X \frac{|g|}{|g|+1}\,d\mu,$$

then $\Psi$ is easily seen to be sequentially continuous on $C_p(K)$ by the Lebesgue theorem. This implies that $\Psi$ is continuous on every compact subset $\mathcal{K}$ of $C_p(X)$ (recall that $\mathcal{K}$ is Fréchet–Urysohn, [1, Theorem III.3.6]). On the other hand, it is easy to construct a family $\mathfrak{G}$ of functions $g: K\to[0,1]$ such that $\int_K g\,d\mu\ge 1/2$ and the zero function lies in the pointwise closure of $\mathfrak{G}$ (see [1, Theorem II.3.5]). This means that $\Psi$ is not continuous on $C_p(K)$. ∎

Remark 6.7. Let $\kappa$ be a cardinal number endowed with the discrete topology. Then $C_p(\kappa)=\mathbb{R}^\kappa$ is a $k_{\mathbb{R}}$-space by [24]. Recall also that in a model of set theory without weakly inaccessible cardinals, any sequentially continuous function on $\mathbb{R}^\kappa$ is in fact continuous (see [28] for further references).

Theorem 1.4 and Proposition 6.6 motivate the following problem.

Question 6.8. *Does there exist $X$ such that $C_k(X)$ or $C_p(X)$ is Ascoli but is not a $k_{\mathbb{R}}$-space?*

For a Tychonoff space $X$ denote by $L(X)$ (respectively, $F(X)$ and $A(X)$) the free locally convex space (respectively, the free or the free abelian topological group) over $X$.

Question 6.9. *Let $L(X)$ ($F(X)$ or $A(X)$) be an Ascoli space. Is $X$ Ascoli?*

Question 6.10. *For which metrizable spaces $X$, are the groups $F(X)$ and $A(X)$ Ascoli?*

In [12] the first named author proved that the free lcs $L(X)$ over a Tychonoff space $X$ is a $k$-space if and only if $X$ is a discrete countable space.

Question 6.11. *Let $L(X)$ be an Ascoli space. Is $X$ a discrete countable space?*

We do not know the answer even if "Ascoli" is replaced by a stronger assumption "$L(X)$ is a $k_{\mathbb{R}}$-space" (see [12, Question 3.6]).

**Acknowledgments.** The second named author was supported by the Center for Advanced Studies in Mathematics at Ben-Gurion University of the Negev, by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2013/058 and by GAČR project 16-34860L and RVO: 67985840.

The third named author was partially supported by NCN grant 2013/11 /B/ST1/03596 (2014-2017).

The authors are greatly indebted to Professor R. Pol who sent to T. Banakh and the first named author a solution of [3, Problem 6.8] (see Corollary 2.4). In [31], R. Pol noticed that it can be shown that the space $C_k(M)$, where $M$ is the countable metric fan, contains a closed countable non-Ascoli subspace using ideas from [30]. Using this fact and stratifiability of metric spaces, for a separable metric space $X$, R. Pol proved that the space $C_k(X)$ is Ascoli if and only if $X$ is locally compact. We provide another proof of a more general result by modifying the proof in [30, Section 5] (see Proposition 2.3).

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