

## SEPARABLE QUOTIENTS IN $C_c(X)$ , $C_p(X)$ , AND THEIR DUALS

JERZY KĄKOL AND STEPHEN A. SAXON

(Communicated by Thomas Schlumprecht)

**ABSTRACT.** The quotient problem has a positive solution for the weak and strong duals of  $C_c(X)$  ( $X$  an infinite Tichonov space), for Banach spaces  $C_c(X)$ , and even for barrelled  $C_c(X)$ , but not for barrelled spaces in general. The solution is unknown for general  $C_c(X)$ . A locally convex space is *properly separable* if it has a proper dense  $\aleph_0$ -dimensional subspace. For  $C_c(X)$  quotients, *properly separable* coincides with *infinite-dimensional separable*.  $C_c(X)$  has a properly separable algebra quotient if  $X$  has a compact denumerable set. Relaxing *compact* to *closed*, we obtain the converse as well; likewise for  $C_p(X)$ . And the weak dual of  $C_p(X)$ , which always has an  $\aleph_0$ -dimensional quotient, has no *properly separable* quotient when  $X$  is a P-space of a certain special form  $X = X_\kappa$ .

### 1. INTRODUCTION

Here we assume topological vector spaces (tvs's) and their quotients are Hausdorff and infinite-dimensional. Banach's famous unsolved problem asks: *Does every Banach space admit a separable quotient?* [Popov's] (Our) negative [tvs] (lcs) solution found a [metrizable tvs] (barrelled lcs) without a separable quotient [16, 20]. (By *lcs* we mean *locally convex tvs over the real or complex scalar field*.)

The familiar Banach spaces admit separable quotients, as do all non-Banach Fréchet spaces [6, Satz 2] and  $(LF)$ -spaces [28, Theorem 3]. The non-Banach  $(LF)$ -space  $\varphi$  is an  $\aleph_0$ -dimensional space with the strongest locally convex topology whose only quotients are copies of  $\varphi$  itself. While separable,  $\varphi$  is not *properly separable* by Robertson's *ad hoc* definition [21] (see Abstract). Saxon's answer [26] to her quarter-century-old question proves  $\varphi$  is the *only* non-Banach  $(LF)$ -space without a *properly separable* quotient if and only if every Banach space has a separable quotient.

Her notion, unexpectedly characterized in weak barrelledness terms [26], intrigues the more: When/how may separable *vs.* properly separable quotients exist/differ in an lcs? Section 2 explores differences but finds no lcs without a separable quotient other than the (rather exotic) examples we found earlier [16]. The main (third) section concentrates on the function spaces  $C_c(X)$  and  $C_p(X)$ , where

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Received by the editors May 3, 2016 and, in revised form, June 1, 2016 and June 21, 2016.  
 2010 *Mathematics Subject Classification*. Primary 46A08, 46A30, 54C35.

*Key words and phrases*. (Properly) separable quotients,  $C(X)$ , weak barrelledness, P-spaces. Thanks to Professor Aaron R. Todd for vital discussions/encouragement/prequels.

The first author's research was supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, grant PROMETEO/2013/058, and by GACR grant 16-34860L and RVO 67985840 (Czech Republic).

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Ver al dorso fecha aceptación, 3 julio 2016

**Subject:** PROC 160503-Saxon - Accepted  
**Date:** Sun, 3 Jul 2016 20:16:21 -0400  
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by Jerzy Kakol and Stephen A. Saxon

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# Separable quotients in $C_c(X)$ , $C_p(X)$ , and their duals

Jerzy Kąkol and Stephen A. Saxon

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## 1. Introduction

Here we assume topological vector spaces (tvs's) and their quotients are Hausdorff and infinite-dimensional. Banach's famous unsolved problem asks: *Does every Banach space admit a separable quotient?* [Popov's] ⟨Our⟩ negative [tvs] ⟨lcs⟩ solution found a [metrizable tvs] ⟨barrelled lcs⟩ without a separable quotient [16, 20]. (By *lcs* we mean *locally convex tvs over the real or complex scalar field*.)

The familiar Banach spaces admit separable quotients, as do all non-Banach Fréchet spaces [6, Satz 2] and  $(LF)$ -spaces [28, Theorem 3]. The non-Banach  $(LF)$ -space  $\varphi$  is an  $\aleph_0$ -dimensional space with the strongest locally convex topology whose only quotients are copies of  $\varphi$  itself. While separable,  $\varphi$  is not *properly separable* by Robertson's *ad hoc* definition [21] (see Abstract). Saxon's answer [26] to her quarter-century-old question proves  $\varphi$  is the *only* non-Banach  $(LF)$ -space without a *properly separable* quotient if and only if every Banach space has a separable quotient.

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Kąkol's research was supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, grant PROMETEO/2013/058, and by GACR grant 16-34860L and RVO 67985840 (Czech Republic).

main (third) section concentrates on the function spaces  $C_c(X)$  and  $C_p(X)$ , where the two notions of separable quotient coincide and beg the existential question (we fully answer) for separable *algebra* quotients. Robertson's dual-distinct notion keys new analytic characterizations ([Theorem 23] ⟨Theorem 24⟩):  *$X$  is a  $P$ -space if and only if the weak dual [of  $C_p(X)$ ] ⟨of  $C_c(X)$ ⟩ has no properly separable quotient.* The final section reviews open questions.

## 2. Weak barrelledness motivations, general lcs quotients

Remarkably, dense subspaces of  $GM$ -spaces are barrelled [5]. A Banach space has no separable quotient if and only if its dense subspaces are barrelled [30, 31]. An lcs  $E$  has no separable quotient if and only if its dense subspaces are non- $S_\sigma$  (defined below) [16].  $E$  has no *properly* separable quotient if and only if its dense subspaces are primitive [26].

$S_\sigma$ -spaces are those lcs's covered by increasing sequences of closed proper subspaces. An lcs  $E$  is [inductive] ⟨primitive⟩ if  $\phi$  is continuous whenever  $\{E_n\}_n$  is an increasing covering sequence of subspaces and  $\phi$  is a [seminorm] ⟨linear form⟩ on  $E$  such that each restriction  $\phi|_{E_n}$  is continuous. These notions from the study of weak barrelledness relate as follows:

$$\begin{array}{c} \text{barrelled} \\ \Downarrow \\ \text{non-}S_\sigma \Rightarrow \text{inductive} \Rightarrow \text{primitive.} \end{array}$$

Hence  $GM$ -spaces lack properly separable quotients. Quotients and countable-codimensional subspaces preserve each of the four properties. *Non- $S_\sigma$*  and *primitive* are duality invariant properties, unlike *barrelled* and *inductive*. Under the Mackey topology, *inductive*  $\Leftrightarrow$  *primitive*. Under metrizability, *non- $S_\sigma$*   $\Leftrightarrow$  *primitive* [27, 29].

Easily, *properly* separable quotients and separable quotients coincide for metrizable spaces and for non- $S_\sigma$  spaces, *e.g.*, for all  $C_c(X)$  and  $C_p(X)$ . Our negative barrelled solution [16] now follows from the two paragraphs above; we merely needed a  $GM$ -space that is non- $S_\sigma$ , and such spaces exist (that are even Baire) [5].

An lcs  $E$  is properly separable if and only if  $E$  is separable and  $E' \neq E^*$ , a corollary to the fact that (†) *finite*-codimensional subspaces of separable lcs's are separable [4]. Moreover, (††) *countable*- cannot replace *finite*- [4, 32].

The *separable quotient* analogs of (†) and (††) hold:

**THEOREM 1.** *If an lcs  $E$  has a separable quotient, so do the finite-codimensional subspaces of  $E$ .*

**PROOF.** Immediate from (†) and [23, Theorem 2(b)]. □

**EXAMPLE 2.** *A countable-codimensional subspace  $G$  of a barrelled space  $E$  does not necessarily admit a separable quotient when  $E$  does.*

**PROOF.** Let  $G$  be any non- $S_\sigma$   $GM$ -space and set  $E = G \oplus \varphi$ . □

The *properly separable quotient* story excludes Theorem 1:

**EXAMPLE 3.** *There is a Mackey space  $E$  with dense hyperplane  $H$  such that  $E$  admits a properly separable quotient and  $H$  does not.*

**PROOF.** Let  $(H, \tau)$  be any  $S_\sigma$   $GM$ -space, *e.g.*,  $\varphi$ . By [29, Theorem 3.2],  $H$  is a dense hyperplane of a non-primitive Mackey space  $(E, \mu)$  with  $(H, \mu)' = (H, \tau)'$ .

Thus all the dense subspaces of  $H$  are primitive. This Section's first paragraph assures (i)  $E$  has a properly separable quotient, but (ii)  $H$  does not.  $\square$

### 3. $C_c(X)$ , $C_p(X)$ and their duals

Throughout,  $X$  denotes an infinite completely regular Hausdorff topological space with Stone-Ćech compactification  $\beta X$ . Let  $C(X)$  [resp.,  $C^b(X)$ ] denote the vector space of  $\mathbb{R}$ -valued continuous [resp., and bounded] functions on  $X$ . Let  $C_c(X)$  denote  $C(X)$  endowed with the compact-open topology. For  $A \subset X$  and  $\varepsilon > 0$ , we put  $[A, \varepsilon] = \{f \in C(X) : |f(x)| \leq \varepsilon \text{ for all } x \in A\}$ . Sets of the form  $[K, \varepsilon]$  with  $K$  a compact (resp., finite) subset of  $X$  and  $\varepsilon > 0$  constitute a base of neighborhoods of 0 for  $C_c(X)$  (resp., for the lcs denoted by  $C_p(X)$ ). By  $C_u^b(X)$  we denote the Banach space whose unit ball is  $[X, 1]$ .

For compact  $X$ , Rosenthal [22] implies the Banach space  $c$  is an algebra quotient of  $C_c(X)$  if  $X$  has a denumerable compact subset. We prove the result and its converse for arbitrary  $X$ . We prove  $C_c(X)$  and  $C_p(X)$  have separable algebra quotients if and only if  $X$  has a denumerable closed subset. We prove  $C_c(X)$  admits a separable quotient when  $X$  is non-pseudocompact, or a P-space, or of pointwise countable type.

It is unknown whether  $C_c(X)$  or  $C_p(X)$  always has a separable quotient. Their weak and strong duals do [16]. The dual  $L(X)$  of  $C_p(X)$  given any topology compatible with the dual pairing, e.g., the weak dual  $L_p(X)$  or the Mackey dual  $L_m(X)$ , even has an  $\aleph_0$ -dimensional quotient [9]. When  $L_p(X)$  does not have a *properly* separable quotient,  $C_p(X)$  does, and when  $C_p(X)$  does, so does  $C_c(X)$  [Theorem 23, Corollaries 19, 20]. Example 3 (with the same proof) holds for  $H = L_m(X)$  if and only if  $X$  is a P-space, as we show later. Or, we could combine Theorem 23 with [2] to obtain  $[X \text{ is a non-discrete P-space}] \Leftrightarrow [\text{every dense subspace of } L_m(X) \text{ is primitive and not barrelled}]$ , adding new examples/characterizations to the previous section and recent work [9, 10, 24]. (Many other modern marriages of topology and analysis may be found in [13].)

Now  $X$  is compact if and only if  $C_u^b(X) = C_c(X)$ . Always,  $C_c(X)$  and  $C_p(X)$  are non- $S_\sigma$  [18, II.4.7] since  $C^b(X)$  is dense and  $C_u^b(X)$  is non- $S_\sigma$ . Density derives from a well-known corollary to [11, 3.11(a),(c)]:

LEMMA 4. *If  $K \subset \mathcal{O} \subset X$  with  $K$  compact and  $\mathcal{O}$  open, and  $g \in C(K)$ , then there exists  $f \in C^b(X)$  such that  $\sup\{|f(x)| : x \in X\} = \sup\{|g(x)| : x \in K\}$ ,  $f|_K = g$ , and  $f$  vanishes on  $X \setminus \mathcal{O}$ .*

The next Lemma, likely known, has a simple proof.

LEMMA 5. *The following five statements are equivalent.*

- (1)  $X$  admits a nontrivial convergent sequence.
- (2)  $X$  admits a sequentially compact infinite set.
- (3)  $X$  admits a compact denumerable set.
- (4)  $X$  admits a countably compact denumerable set.
- (5)  $X$  admits a compact metrizable infinite set.

PROOF. Obviously, (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4), and (5)  $\Rightarrow$  (2)  $\Rightarrow$  (1). Moreover, (1)  $\Rightarrow$  (5), since any sequence  $\{x_0, x_1, x_2, x_3, \dots\}$  of distinct points converging to  $x_0$  in  $X$  is clearly homeomorphic to  $\{0, 1, 1/2, 1/3, \dots\}$  in the (metrizable) unit interval  $[0, 1]$ .

Finally, note that  $\neg(1) \Rightarrow \neg(4)$ : Let  $S = \{y_1, y_2, \dots\}$  be an arbitrary sequence of distinct points in  $X$ . By hypothesis  $S$  does not converge to  $y_1$ , so there exists some neighborhood  $N_1$  of  $y_1$  which misses a subsequence  $S_1 = \{x_{11}, x_{12}, \dots\}$  of  $S$ . We inductively find a neighborhood  $N_k$  of  $y_k$  which misses a subsequence  $S_k = \{x_{k1}, x_{k2}, \dots\}$  of  $S_{k-1}$  for  $k = 2, 3, \dots$ . The diagonal sequence  $T = \{x_{11}, x_{22}, \dots\}$  of distinct points in  $S$  has no cluster point in  $S$ , since  $N_k$  is a neighborhood of  $y_k$  that contains at most  $k-1$  points of  $T$  ( $k = 1, 2, \dots$ ). Therefore  $S$  is not countably compact, denying (4).  $\square$

The proof of [22, Corollary 2.6] explicitly uses a form of

LEMMA 6. *If  $X$  is countably compact, at least one of the following two cases holds:*

Case 1.  *$X$  admits a nontrivial convergent sequence.*

Case 2. *The derived set  $X^d$  of all cluster points is infinite and perfect.*

PROOF. If  $X^d$  is finite, then its union with any denumerable set in  $X$  verifies (4), hence (1). If  $X^d$  is not perfect, then there exists  $x_0$  in  $X^d \setminus X^{dd}$ . Let  $V$  be a closed neighborhood of  $x_0$  that misses  $X^d \setminus \{x_0\}$ , let  $x_1, x_2, \dots$  be distinct points in  $V$ , and define  $S := \{x_0, x_1, x_2, \dots\}$ . Since  $S^d \subset V \cap X^d = \{x_0\}$ , the set  $S$  is countably compact by hypothesis; *i.e.*, (4) holds. Then so does (1).  $\square$

In the simplest Case 1 examples,  $X$  is a convergent sequence of distinct points, making  $C_c(X)$  isomorphic to the Banach space  $c$  of convergent scalar sequences, and  $C_p(X)$  isomorphic to  $c$  with the topology induced by  $\mathbb{R}^{\mathbb{N}}$ , both separable.

We can now sketch a proof from [22] of the seminal

THEOREM 7 (Rosenthal). *When  $X$  is compact, the Banach space  $C_c(X)$  has a (separable) quotient isomorphic to either  $c$  or the Hilbert space  $\ell^2$ .*

PROOF. By Lemma 6, there are only two cases possible.

Case 1.  *$X$  contains a sequence  $\{x_n\}_n$  of distinct points converging to some point  $x_0 \neq x_n$  ( $n \in \mathbb{N}$ ).* The linear map  $T : C_c(X) \rightarrow c$  defined by  $f \mapsto (f(x_n))_{n \geq 1}$  is obviously continuous, and is onto the Banach space  $c$  by Lemma 4. Hence the quotient  $C_c(X)/T^{-1}(0)$  is isomorphic to  $c$ .

Case 2.  *$X$  contains a perfect infinite set.* Via the Khinchin inequality (*cf.* [8]) one finds  $\ell^2$  is isomorphic to a subspace of  $L^1[0, 1]$ , and in Case 2,  $L^1[0, 1]$  is isomorphic to a subspace of  $L^1(X, \mathfrak{B}_X, \mu)$ , with  $\mathfrak{B}_X$  the Borel sets in  $X$  and  $\mu$  a nonnegative, finite, regular Borel measure on  $X$ . In turn, the latter space is isomorphic to a subspace of the strong dual  $C_c(X)'_{\beta}$  of  $C_c(X)$ . Therefore the reflexive  $\ell^2$  is a subspace of  $C_c(X)'_{\beta}$  that is  $w^*$ -closed [22, Corollary 1.6, Proposition 1.2]. This yields a quotient of  $C_c(X)$  isomorphic to  $\ell^2$ .  $\square$

Rosenthal recalled on p.180 of [22] that  $\ell^{\infty}$  may be identified with  $C_u^b(\beta\mathbb{N})$ , clearly aware of Corollaries 8, 9 below. Whether he actually observed Corollaries 10, 11 is less clear.

COROLLARY 8 (Rosenthal).  *$C_u^b(X) \approx C_c(\beta X)$  has a separable quotient.*

PROOF. By the Stone-Ćech theorem [33].  $\square$

COROLLARY 9 (Rosenthal). *Some quotient of  $\ell^{\infty}$  is isomorphic to  $\ell^2$ .*

PROOF.  $\ell^\infty = C_u^b(\mathbb{N}) \approx C_c(\beta\mathbb{N})$  and  $(\beta\mathbb{N})^d = \beta\mathbb{N} \setminus \mathbb{N}$  is infinite, perfect. Case 2 of Theorem 7 applies.  $\square$

COROLLARY 10. *If  $X$  has an infinite compact subset  $Y$ , then  $C_c(X)$  has a quotient isomorphic to  $c$  or  $\ell^2$ .*

PROOF. The restriction map  $f \mapsto f|_Y$  from  $C_c(X)$  into  $C_c(Y)$  is clearly linear and continuous. By Lemma 4, it is open. And quotient-taking is transitive.  $\square$

COROLLARY 11. *If  $C_p(X)$  has a separable quotient, then so does  $C_c(X)$ .*

PROOF. Either  $C_c(X) = C_p(X)$ , or Corollary 10 applies.  $\square$

Recall that an lcs  $E$  is a *GM-space* [5] if every linear map  $t : E \rightarrow F$ , where  $F$  is any metrizable lcs and  $t$  has closed graph, is necessarily continuous. Immediately from Mahowald's theorem, every *GM-space* is barrelled. No  $C_u^b(X)$  is a *GM-space*, twice-proved: (i) No metrizable lcs  $F$  is a *GM-space*, since there is always a strictly finer metrizable locally convex topology on  $F$ ; (ii) *GM-spaces* lack properly separable quotients. Moreover,

THEOREM 12. *Neither  $C_c(X)$  nor  $C_p(X)$  is a *GM-space*.*

PROOF. Barrelled  $C_c(X)$  spaces admit (properly) separable quotients [16]. And if  $C_p(X)$  is barrelled, then  $C_p(X) = C_c(X)$  [2].  $\square$

COROLLARY 13. *Always, there exists a discontinuous linear map with closed graph from  $C_c(X)$  into some metrizable lcs.*

From Lemma 4, the closed ideals of  $C_p(X)$  are precisely the spaces

$$\mathfrak{I}_A = \{f \in C(X) : f(x) = 0 \text{ for all } x \in A\}$$

where  $A$  ranges over the closed subsets of  $X$ . These are also the closed ideals of  $C_c(X)$  [7, Theorem 4.10.6]. An *algebra quotient* of  $C_c(X)$  or  $C_p(X)$  is one by a closed ideal, thus preserving vector multiplication. In Rosenthal's Case 1 the quotient is an algebra quotient, since the kernel of  $T$  is  $\mathfrak{I}_A$  with  $A = \{x_0, x_1, x_2, \dots\}$ .

Please recall that  $X$  is *pseudocompact* if  $C(X) = C^b(X)$ . Algebra quotients add to a list [15, Theorem 1.1] of characterizations found jointly with Todd.

THEOREM 14. *When  $X$  is non-pseudocompact,  $C_c(X)$  and  $C_p(X)$  admit separable quotients. In fact, the following seven statements are equivalent.*

- (1)  $X$  is not pseudocompact.
- (2)  $C_c(X)$  contains a copy of  $\mathbb{R}^{\mathbb{N}}$ .
- (3)  $C_p(X)$  contains a copy of  $\mathbb{R}^{\mathbb{N}}$ .
- (4)  $C_c(X)$  admits a quotient isomorphic to  $\mathbb{R}^{\mathbb{N}}$ .
- (5)  $C_p(X)$  admits a quotient isomorphic to  $\mathbb{R}^{\mathbb{N}}$ .
- (6)  $C_c(X)$  admits an algebra quotient isomorphic to  $\mathbb{R}^{\mathbb{N}}$ .
- (7)  $C_p(X)$  admits an algebra quotient isomorphic to  $\mathbb{R}^{\mathbb{N}}$ .

Moreover, if one (and thus each) of (1-7) holds, then  $C_u^b(X)$  admits quotients isomorphic to  $\ell^\infty$  and to  $\ell^2$ .

PROOF. In [14] we showed that (1)  $\Leftrightarrow$  (2), and since the topology on  $\mathbb{R}^{\mathbb{N}}$  is minimal, (2)  $\Rightarrow$  (3). In every lcs, each copy of  $\mathbb{R}^{\mathbb{N}}$  is complemented [19, 2.6.5(iii)], so (2)  $\Rightarrow$  (4) and (3)  $\Rightarrow$  (5).

Now let  $M$  be any closed subspace of  $C_c(X)$  [resp., of  $C_p(X)$ ]. If  $X$  is pseudo-compact, then  $M$  is a closed subspace of  $C_u^b(X)$ . The topology of the Banach space  $C_u^b(X)/M$  is finer than that of  $C_c(X)/M$  [resp., of  $C_p(X)/M$ ], so, by the open mapping theorem, the latter cannot be the non-Banach Fréchet space  $\mathbb{R}^{\mathbb{N}}$ . This shows (the contrapositive of) (4)  $\Rightarrow$  (1) [resp., (5)  $\Rightarrow$  (1)]. Thus (1-5) are equivalent.

[(1)  $\Rightarrow$  (6),(7)]. By definition, the non-pseudocompact  $X$  admits a disjoint sequence  $\{U_n\}_n$  of non-empty open sets that is locally finite; *i.e.*, each point in  $X$  has a neighborhood which meets only finitely many of the  $U_n$ . Choose  $x_n$  in  $U_n$  for each  $n \in \mathbb{N}$  and define the linear map  $T : C(X) \rightarrow \mathbb{R}^{\mathbb{N}}$  so that, for all  $f \in C(X)$ ,

$$T(f) = (f(x_n))_n.$$

$T$  is continuous on  $C_p(X)$ , and thus also on  $C_c(X)$ . By Lemma 4, for each  $n$  there exists  $f_n \in [X, 1]$  with  $f_n(x_n) = 1$  and  $f_n(X \setminus U_n) = \{0\}$ . For an arbitrary scalar sequence  $(a_n)_n$ , the point-wise sum  $\sum_n a_n f_n$  is in  $C(X)$  by local finiteness. If  $K$  is compact in  $X$ , it is countably compact and meets  $U_k$  only for those  $k$  in some finite set  $\sigma \subset \mathbb{N}$ . If  $\varepsilon > 0$  and

$$W = \{(a_n)_n \in \mathbb{R}^{\mathbb{N}} : |a_k| \leq \varepsilon \text{ for each } k \in \sigma\},$$

then for each  $(a_n)_n \in W$  we have  $\sum_n a_n f_n \in [K, \varepsilon]$  with  $T(\sum_n a_n f_n) = (a_n)_n$ . Hence  $T([K, \varepsilon]) \supset W$ , so that  $T$  is an open map both from  $C_c(X)$  and  $C_p(X)$ , and  $T^{-1}(0) = \mathfrak{S}_A$ , where  $A = \{x_1, x_2, \dots\}$  is obviously closed. Thus (6) and (7) hold.

Trivially, (7) and (6) imply (5) and (4), respectively, completing the proof that (1-7) are equivalent.

The  $C_u^b(X)$  case remains. If (1-7) hold, then  $T$  exists as above, and the restriction  $T|_{C_u^b(X)}$  is clearly continuous and open from  $C_u^b(X)$  onto the Banach space  $\ell^\infty$ . Thus  $\ell^\infty$  is a quotient of  $C_u^b(X)$ , as is  $\ell^2$  by Corollary 9 and transitivity.  $\square$

In [15] we proved that  $C_c(X)$  contains a copy of a dense subspace of  $\mathbb{R}^{\mathbb{N}}$  if and only if  $X$  is not Warner bounded. ( $X$  is *Warner bounded* if for every disjoint sequence  $(U_n)_n$  of non-empty open sets in  $X$  there exists a compact  $K \subset X$  such that  $U_n \cap K \neq \emptyset$  for infinitely many  $n \in \mathbb{N}$ .)

LEMMA 15. *Let  $A$  be a closed infinite subset of  $X$ . Then  $C_p(X)/\mathfrak{S}_A$  is isomorphic to a dense subspace of  $C_p(A)$ , itself a dense subspace of the product space  $\mathbb{R}^A$ . If  $A$  is also compact, then  $C_c(X)/\mathfrak{S}_A$  is isomorphic to the Banach space  $C_c(A)$ .*

PROOF. Let  $q$  denote the quotient map. One may use Lemma 4 to see that: (i) in both cases, the map  $q(f) \mapsto f|_A$  is an isomorphism from the quotient onto its image in  $C(A)$ ; (ii) the image is a dense subspace of  $\mathbb{R}^A$  since some  $f$  in  $C(X)$  achieves arbitrarily prescribed values on any given finite subset of  $A$ ; (iii) the map is onto  $C(A)$  when  $A$  is compact.  $\square$

Rosenthal's Banach algebra quotient (Case 1) generalizes, with converse:

THEOREM 16. *Statements (1-5) of Lemma 5 are equivalent to the next four.*

- (6)  $C_c(X)$  admits an algebra quotient isomorphic to  $c$ .
- (7)  $C_c(X)$  admits a separable Banach algebra quotient.
- (8)  $C_c(A)$  is isomorphic to  $c$  for some  $A \subset X$ .
- (9)  $C_c(A)$  is a separable Banach space for some  $A \subset X$ .

PROOF. If  $A$  is closed in  $X$  and  $C_c(X)/\mathfrak{S}_A$  is normable, then for some compact  $K$  the quotient map  $q$  takes  $[K, 1]$  into a bounded set. If we suppose that  $K \not\supseteq A$ , then Lemma 4 provides  $f \in \mathfrak{S}_K \setminus \mathfrak{S}_A$ . But then the span  $\mathcal{L}$  of  $f$  is in  $[K, 1]$  and  $q(f) \neq 0$ , so the unbounded line  $q(\mathcal{L})$  is in  $q([K, 1])$ , a contradiction. Therefore  $K$  must contain the closed set  $A$ , and  $A$  must be compact. We combine this with Lemma 15 to see that (6)  $\Leftrightarrow$  (8) and (7)  $\Leftrightarrow$  (9).

If  $A$  consists of a nontrivial convergent sequence and its limit, then it is clear from Case 1 that  $C_c(A) \approx c$ ; *i.e.*, (1)  $\Rightarrow$  (8). Trivially, (8)  $\Rightarrow$  (9).

Finally, the Krein-Krein criterion [17] merely says (9)  $\Leftrightarrow$  (5).  $\square$

LEMMA 17. *If  $X$  has no closed denumerable sets, then  $C_p(X)$  is not separable.*

PROOF. Let  $f_1, f_2, \dots \in C(X)$  be arbitrary. We desire  $y_1 \neq y_2$  in  $X$  with

$$|f_n(y_1) - f_n(y_2)| \leq 1$$

for all  $n \in \mathbb{N}$ . By hypothesis, every denumerable set has more than one cluster point in  $X$ . Fix a cluster point  $y_1$  in  $X$ . Continuity allows us to choose a strictly decreasing sequence of closed neighborhoods  $V_n$  of  $y_1$  so that each  $f_n(V_n)$  has diameter no larger than 1. We choose  $x_n \in V_n \setminus V_{n+1}$  and let  $y_2$  be a cluster point of  $\{x_n\}_n$  distinct from  $y_1$ . Since all but finitely many of the  $x_k$  are in a given  $V_n$ , this closed set contains the cluster point  $y_2$ . Indeed, then, the displayed inequality holds for each  $n$ .

Lemma 4 provides  $h \in C(X)$  with  $h(y_1) = 5$  and  $h(y_2) = 9$ . If we assume some  $f_n \in h + [\{y_1, y_2\}, 1]$ , we have  $|f_n(y_1) - f_n(y_2)| \geq (9 - 5) - 1 - 1 = 2$ , a contradiction. Thus the arbitrary sequence is not dense in  $C_p(X)$ .  $\square$

THEOREM 18. *The following three statements are equivalent.*

- (1)  $X$  admits a closed denumerable set  $D$ .
- (2)  $C_c(X)$  admits a separable algebra quotient.
- (3)  $C_p(X)$  admits a separable algebra quotient.

PROOF. [(1)  $\Rightarrow$  (2)]. If  $D$  admits a compact infinite subset, the previous Theorem ensures  $c$  is a (separable) algebra quotient of  $C_c(X)$ . If  $D$  has no such subset, then  $C_c(X)/\mathfrak{S}_D = C_p(X)/\mathfrak{S}_D$  is isomorphic to a dense subspace of the metrizable separable  $\mathbb{R}^D$  by Lemma 15. Hence the algebra quotient is separable.

[(2)  $\Rightarrow$  (3)]. If  $C_c(X)/\mathfrak{S}_A$  is separable, so is  $C_p(X)/\mathfrak{S}_A$ .

[(3)  $\Rightarrow$  (1)]. If  $A$  is closed in  $X$  with  $C_p(X)/\mathfrak{S}_A$  separable, then so is  $C_p(A)$  by Lemma 15. Since  $A$  is infinite, (the contrapositive of) Lemma 17 shows  $A$  has a closed denumerable subset  $D$ . Thus  $D$  is closed in  $X$ , and (1) holds.  $\square$

Thus  $C_c(X)$  and  $C_p(X)$  have separable algebra quotients if  $X$  has an infinite closed subset that is metrizable; *e.g.*, if  $X$  is a tvs. Since  $\beta\mathbb{N}$  lacks a closed denumerable set,  $C_c(\beta\mathbb{N})$  and  $C_p(\beta\mathbb{N})$  lack separable algebra quotients, although  $C_c(\beta\mathbb{N})$  contains a copy of  $c$ , as do all Banach spaces of the form  $C_c(X)$ .

If countable intersections of open sets are open,  $X$  is called a *P-space*; then denumerable sets are closed, not compact, so one may apply Theorem 18, not 16:

COROLLARY 19. *If  $X$  is a P-space, then  $C_c(X)$  and  $C_p(X)$  admit separable algebra quotients.*

COROLLARY 20. *Both  $C_c(X)$  and  $C_p(X)$  have properly separable quotients when  $L_m(X)$  does not.*

PROOF. By hypothesis, dense subspaces of  $L_m(X)$  are primitive [26], including  $L_m(X)$ . Therefore  $X$  is a P-space [9, 10] and the previous Corollary applies.  $\square$

$X$  is of *pointwise countable type* (Arkhangel'skii) if every point in  $X$  is in some compact set  $K$  for which there exists a sequence of open sets  $U_n$  in  $X$  with the properties that (i) each  $U_n$  contains  $K$  and (ii) some  $U_n$  is contained in  $U$  whenever  $U$  is an open set containing  $K$ . Obviously,  $X$  is of pointwise countable type if  $X$  is first countable, and conversely when every compact set  $K$  is finite. Only the most extreme P-spaces are of pointwise countable type. Indeed,

THEOREM 21. *Assume  $X$  is of pointwise countable type. The following five statements are equivalent.*

- (1)  $X$  is discrete.
- (2)  $X$  is a P-space.
- (3) No compact set in  $X$  is infinite.
- (4) No compact set in  $X$  is denumerable, and  $X$  is first countable.
- (5)  $C_c(X) = C_p(X) = \mathbb{R}^X$ .

PROOF. Trivially, (1)  $\Rightarrow$  (2). Suppose (2) holds. Then every denumerable set in  $X$  is closed and not compact. Therefore there are no denumerable subsets of compact sets, thus no infinite compact sets in  $X$ ; i.e., (2)  $\Rightarrow$  (3). Since  $X$  is of pointwise countable type, (3)  $\Rightarrow$  (4).

[(4)  $\Rightarrow$  (1)]. Suppose (4) holds and not (1). Then there is some  $x_0 \in X$  such that  $\{x_0\}$  is not open in  $X$ . First countability posits a countable base  $\{V_n\}_n$  of open neighborhoods of  $x_0$ . We may assume each  $V_n \supset V_{n+1}$  and inductively choose distinct points  $x_1, x_2, \dots$  with each  $x_n \in V_n$ . Clearly, this sequence converges to  $x_0$ , and  $\{x_0, x_1, x_2, \dots\}$  is a denumerable compact set in  $X$ , a contradiction of (4); the desired implication follows.

We now have (1) - (4) are equivalent. Since [(1)  $\Rightarrow$  (5)] and [(5)  $\Rightarrow$  (3)] are obvious, the proof is complete.  $\square$

COROLLARY 22. *If  $X$  is of pointwise countable type, then  $C_c(X)$  has a quotient isomorphic to either  $\mathbb{R}^{\mathbb{N}}$ ,  $c$ , or  $\ell^2$ .*

PROOF. Clearly,  $\mathbb{R}^{\mathbb{N}}$  is a quotient of  $\mathbb{R}^X$ . If  $C_c(X) \neq \mathbb{R}^X$ , then  $X$  contains an infinite compact set  $Y$ , and Corollary 10 applies.  $\square$

The weak and strong duals of  $C_c(X)$  have separable quotients [16], but not always *properly* separable quotients (e.g., when  $X$  is discrete). Re-examination of the dual of  $C_p(X)$  adds to the analytic P-space characterizations [2, 9, 10].

THEOREM 23. *The following five assertions are equivalent.*

- (1)  $X$  is a P-space.
- (2)  $L_m(X)$  is primitive.
- (3) Every dense subspace of  $L_m(X)$  is primitive.
- (4) Every dense subspace of  $L_m(X)$  is inductive.
- (5) No quotient of  $L_m(X)$  is properly separable.

PROOF. By [9, Theorem 6], (1)  $\Leftrightarrow$  (2). Always, *inductive*  $\Rightarrow$  *primitive*, and for Mackey spaces, *primitive*  $\Leftrightarrow$  *inductive* [29, box 4 of Fig. 3], so Theorem 3.12 of [29] yields (3)  $\Leftrightarrow$  (4). And [(3)  $\Leftrightarrow$  (5)] is a part of [26, Theorem 1]. Trivially, (3)  $\Rightarrow$  (2). We are left to prove

[(1)  $\Rightarrow$  (5)]. Suppose  $X$  is a P-space and  $L_m(X)$  admits a properly separable quotient. By definition,  $L_m(X)$  contains a closed subspace  $M$  and a sequence  $\{y_n\}_n$  such that  $F = M + \text{sp}\{y_n\}_n$  is a dense proper subspace. In the usual manner, we identify  $X$  with a Hamel basis for  $L(X)$ , and  $C(X)$  with the dual of  $L_m(X)$ . Since the countable union of finite sets is countable, there is a sequence  $\{x_n\}_n \subset X$  with  $F \subset G = M + \text{sp}\{x_n\}_n$ . Assume  $G = L(X)$ , so that  $M$  is countable-codimensional. Now  $L_m(X)$  is primitive [9], and by [29, Theorem 2.4], every subspace between  $M$  and  $L_m(X)$  is closed. Therefore  $F$  is closed, contradicting the fact that  $F$  is supposed to be dense and proper; the assumption is false:  $G \neq L(X)$ , and there exists some  $x \in X \setminus G$ . The Hahn-Banach theorem yields  $f \in C(X)$  with  $f(x) = 1$  and  $f(M) = \{0\}$ . Since  $X$  is a P-space, there is a neighborhood  $\mathcal{O}$  of  $x$  with each  $x_n \notin \mathcal{O}$ . Lemma 4 yields  $h \in C(X)$  with  $h(x) = 1$  and each  $h(x_n) = 0$ . Now the product  $fh \in C(X)$  vanishes on  $G$ , hence on  $F$ , but not at  $x$ , which contradicts density of  $F$ . Our supposition is false, then, which proves [(1)  $\Rightarrow$  (5)].  $\square$

**THEOREM 24.**  *$X$  is a P-space if and only if the weak dual  $C_c(X)'_\sigma$  of  $C_c(X)$  has no properly separable quotient.*

**PROOF.** Always,  $L_p(X)$  is a dense subspace of  $C_c(X)'_\sigma$ , so the latter has a properly separable quotient if the former does. Thus by Theorem 23 and duality invariance,  $C_c(X)'_\sigma$  has a properly separable quotient if  $X$  is *not* a P-space.

Conversely, if  $X$  is a P-space, then all compact sets in  $X$  are finite, so that  $C_c(X)'_\sigma = L_p(X)$  has no properly separable quotient, again by Theorem 23 and duality invariance.  $\square$

Precisely the  $X$  that are P-spaces provide a wealth of simple lcs's  $L_m(X)$  to which the proof of Example 3 applies:

**EXAMPLE 25.** *Suppose  $X$  is a P-space; equivalently, every dense subspace of  $L_m(X)$  is primitive. The  $S_\sigma$  space  $L_m(X)$  [9] dominates a dense hyperplane  $H$  of some non-primitive lcs  $(E, \tau)$  with  $(H, \tau)' = L_m(X)'$  [29, Theorem 3.2]. The non-primitive  $E$  must admit a properly separable quotient, but, via duality invariance, its hyperplane  $H$  cannot (see first paragraph of Section 2).*

We mention some concrete nondiscrete P-spaces. If  $\kappa$  is an infinite cardinal, let  $X_\kappa$  denote the closed interval  $[0, \kappa]$  of ordinals with a finer topology whose open sets are precisely those which either omit  $\kappa$  or contain the closed interval  $[\alpha, \kappa]$  for some ordinal  $\alpha < \kappa$ . Certainly,  $X_\kappa$  is an infinite completely regular Hausdorff space. The *cofinality* of  $\kappa$ , denoted  $\text{cof}(\kappa)$ , is the least cardinality of the cofinal subsets of the well-ordered interval  $[0, \kappa]$ .

By Theorem 18, all  $C_p(X_\kappa)$  and  $C_c(X_\kappa)$  admit separable algebra quotients. Easily,  $[X_\kappa \text{ is a P-space}] \Leftrightarrow [\text{cof}(\kappa) \neq \aleph_0] \Leftrightarrow [X_\kappa \text{ has no infinite compact set}] \Leftrightarrow [X_\kappa \text{ has no denumerable compact set}]$ . By Theorems 23 and 24, then,  $\text{cof}(\kappa)$  determines whether  $C_p(X_\kappa)'_\sigma$  and  $C_c(X_\kappa)'_\sigma$  admit properly separable quotients.

Cofinality is similarly crucial in [25]. If  $E$  is a linear space with infinite Hamel basis  $B$  of size  $|B|$ , define the subspace  $E^{B, |B|}$  of the algebraic dual  $E^*$  by writing

$$E^{B, |B|} = \{f \in E^* : |\{x \in B : f(x) \neq 0\}| < |B|\}.$$

The lcs that  $E$  becomes under the Mackey topology  $\mu(E, E^{B, |B|})$ , denoted  $E_B$ , is never  $\aleph_0$ -barrelled and has dense subspaces of codimension  $|B|$ , the largest possible [25, Theorems 2, 3]. Moreover,

THEOREM 26. *The following seven statements are equivalent.*

- (1)  $\text{cof}(|B|) \neq \aleph_0$ .
- (2)  $E_B$  is primitive.
- (3) [Every]  $\langle$ Some $\rangle$  dense hyperplane in  $E_B$  is nonMackey.
- (4) Every dense proper subspace in  $E_B$  is nonMackey.
- (5) Every dense subspace in  $E_B$  is primitive.
- (6) Every dense subspace in  $E_B$  is inductive.
- (7)  $E_B$  does not admit a properly separable quotient.

PROOF. By [25, Theorems 2, 3], (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3). In particular, the [Every] and  $\langle$ Some $\rangle$  versions of (3) are equivalent.

[(3)  $\Leftrightarrow$  (4)]. If  $F$  is a dense proper subspace in  $E_B$ , there is a hyperplane  $H$  in  $E_B$  with  $F \subset H$ ; if  $F$  is also Mackey, so is  $H$ , routinely, which contrapositively proves [(3)  $\Rightarrow$  (4)]. The converse is trivial.

[(5)  $\Leftrightarrow$  (6)  $\Leftrightarrow$  (7)]. Suppose (5) holds. Then the primitive Mackey space  $E_B$  is inductive [29], which, combined with (5), implies (6) [29, Theorem 3.12]. The converse is obvious, so (5)  $\Leftrightarrow$  (6). Theorem 1(ii) of [26] equates (5) and (7).

[(5)  $\Rightarrow$  (2)]. Trivially. To complete the proof, we show

[(1)  $\Rightarrow$  (7)]. By Theorem 1(iii) of [26],  $E_B$  has a properly separable quotient if and only if there exists a sequence  $\{f_n\}_n \subset E'_B (= E^{B,|B|})$  such that the subspace

$$\text{ez}\{f_n\}_n = \{x \in E : f_n(x) = 0 \text{ for all but finitely many } n \in \mathbb{N}\}$$

is dense and proper in  $E_B$ . Given any  $\{f_n\}_n \subset E'_B$ ,

$$B_0 = \{x \in B : f_n(x) \neq 0 \text{ for some } n \in \mathbb{N}\}$$

is a countable union of sets of size  $< |B|$ , so if (1) holds, then  $|B_0| < |B|$  and all superspaces of  $\text{sp}(B \setminus B_0)$  are closed by definition of  $E'_B$ . In particular,  $\text{ez}\{f_n\}_n$  is closed, and cannot be dense and proper in  $E_B$ ; thus (7) holds.  $\square$

Since each  $E_B$  is  $S_\sigma$ , the process of Examples 3 and 25 applies to precisely those  $E_B = H$  with  $\text{cof}(|B|) \neq \aleph_0$ . Conversely, no other process avails, since

THEOREM 27. *If an lcs  $E$  admits a properly separable quotient, and a dense hyperplane  $H$  does not, then  $H$  is  $S_\sigma$  and  $E$  is not primitive.*

PROOF. Theorem 1 implies  $H$  has a separable quotient  $Q$ . Thus  $H$  must be  $S_\sigma$ , since  $Q$  would otherwise be properly separable, contrary to hypothesis.

Assume  $E$  is primitive. Let  $\{f_n\}_n \subset E'$  with  $Z = \text{ez}\{f_n\}_n$  dense in  $E$ . Let  $\overline{H \cap Z}^H$  and  $\overline{H \cap Z}^E$  denote the closure of  $H \cap Z$  in  $H$  and  $E$ , respectively. Clearly,

$$\text{codim}_H \left( \overline{H \cap Z}^H \right) \leq \text{codim}_E \left( \overline{H \cap Z}^E \right).$$

If  $Z \subset H$ , then by density both codimensions are null. If there exists  $x \in Z \setminus H$ , then  $Z = H \cap Z + \text{sp}x$ , and  $E = \overline{Z} = \overline{H \cap Z}^E + \text{sp}x$ : Both codimensions are  $\leq 1$ . In every case, then,

$$\text{codim}_H \left( \overline{H \cap Z}^H \right) \leq \text{codim}_E \left( \overline{H \cap Z}^E \right) \leq 1.$$

We need Theorem 3.11(a) of [29], which says:

(\*) Let  $F$  be a dense primitive subspace of an lcs  $E$ . Every subspace between  $F$  and  $E$  is primitive if and only if  $\text{ez}\{h_n\}_n = E$  whenever  $\{h_n\}_n \subset E'$  with  $\text{ez}\{h_n\}_n \supset F$ .

By hypothesis, every dense subspace of  $H$  is primitive. If  $G$  is a subspace between  $H \cap Z$  and  $\overline{H \cap Z}^H$ , it has codimension  $\leq 1$  in a dense, hence primitive subspace of  $H$ , and therefore  $G$  itself is primitive [29, Theorem 2.9]. Application of (\*) to the primitive subspace  $H \cap Z$ , dense in  $\overline{H \cap Z}^H$ , yields the fact that  $Z = \text{ez}\{f_n\} \supset \overline{H \cap Z}^H$ . Consequently,  $\overline{H \cap Z}^H = H \cap Z$  and

$$\text{codim}_E(Z) \leq \text{codim}_H(H \cap Z) + \text{codim}_E(H) = \text{codim}_H(\overline{H \cap Z}^H) + 1 \leq 2.$$

Therefore every subspace between the dense  $Z$  and the primitive  $E$  has codimension  $\leq 2$  and is, itself, primitive. Now (\*) implies  $Z = \text{ez}\{f_n\}_n = E$ . Hence there is no  $\{f_n\}_n \subset E'$  with  $\text{ez}\{f_n\}_n$  dense and proper in  $E$ ; i.e.,  $E$  has no properly separable quotient, a contradiction of hypothesis. We must conclude the assumption is false;  $E$  is not primitive.  $\square$

#### 4. Remaining questions

Despite the strong dual solution [1], the Banach problem remains. Analogs implicate P-spaces and weak barrelledness. We have clear answers as to when

- the strong and weak duals of  $C_c(X)$  have separable quotients (always [16])
- the weak dual of  $C_c(X)$  has a properly separable quotient (when  $X$  is not a P-space, Theorem 24)
- the weak dual of  $C_p(X)$  has a separable quotient (always [9]) or a properly separable quotient (when  $X$  is not a P-space, Theorem 23)
- proper  $(LF)$ -spaces have separable quotients (always [28]) or properly separable quotients (almost always [26])
- non-normable Fréchet spaces have separable quotients (always [6, Satz 2])
- $GM$ -spaces have properly separable quotients (never) or separable quotients (when they are  $S_\sigma$  [16])
- $C_c(X)$ ,  $C_p(X)$  have separable algebra quotients (Theorem 18)
- barrelled  $C_c(X)$ ,  $C_p(X)$  have separable quotients (always [16]).

**Q1.** Must arbitrary  $C_c(X)$  have separable quotients? (Rosenthal and Theorem 18 leave only the case where  $X$  is countably compact and not compact.)

**Q2.** If  $C_c(X)$  has separable quotients, must  $C_p(X)$ ? (See Corollary 11.)

**Q3.** If  $X$  is compact, must  $C_p(X)$  have separable quotients?

So far, only certain  $GM$ -spaces have been shown to lack separable quotients [16]. Are there Schwartz spaces or nuclear spaces, for example, that lack separable quotients? Our answer is positive, albeit pedestrian: Any non-trivial variety  $\mathcal{V}$  of lcs's, the nuclear and Schwartz varieties included, must contain the smallest non-trivial variety  $\mathcal{W}$  of all lcs's having their weak topology [3]. Let  $E$  be a non- $S_\sigma$   $GM$ -space. Then  $(E, \sigma(E, E'))$  is in  $\mathcal{W} \subset \mathcal{V}$  and does not admit separable quotients.

**Q4.** Does some Schwartz or nuclear space not in  $\mathcal{W}$  lack separable quotients? (Both varieties have separable universal generators not in  $\mathcal{W}$  [12].)

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