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Free locally convex spaces with a small base

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Abstract The paper studies the free locally convex space $L(X)$ over a Tychonoff space X. Since for infinite X the space $L(X)$ is never metrizable (even not Fréchet-Urysohn), a possible applicable generalized metric property for $L(X)$ is welcome. We propose a concept (essentially weaker than first-countability) which is known under the name a $\mathfrak{G}$-base. A space X has a $\mathfrak{G}$-base if for every $x \in X$ there is a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods at x such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$, where $\alpha = (\alpha(n))_{n\in\mathbb{N}} \le \beta = (\beta(n))_{n\in\mathbb{N}}$ if $\alpha(n) \le \beta(n)$ for all $n \in \mathbb{N}$. We show that if X is an Ascoli σ-compact space, then $L(X)$ has a $\mathfrak{G}$-base if and only if X admits an Ascoli uniformity $\mathcal{U}$ with a $\mathfrak{G}$-base. We prove that if X is a σ-compact Ascoli space of $\mathbb{N}^\mathbb{N}$-uniformly compact type, then $L(X)$ has a $\mathfrak{G}$-base. As an application we show: (1) if X is a metrizable space, then $L(X)$ has a $\mathfrak{G}$-base if and only if X is σ-compact, and (2) if X is a countable Ascoli space, then $L(X)$ has a $\mathfrak{G}$-base if and only if X has a $\mathfrak{G}$-base.

Keywords Free locally convex space · $\mathfrak{G}$-base · $C_k(X)$ · Compact resolution

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1 Introduction

The class of free locally convex spaces $L(X)$ over a (Tychonoff) space X is one of the most important classes in the category of locally convex spaces and continuous operators. This class was introduced by Markov [22] and intensively studied over the last half-century, see for example [1,12,15,25,27]. Recall that the *free locally convex space* $L(X)$ over a space X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $i : X \to L(X)$ such that every continuous mapping f from X to a locally convex space E gives rise to a unique continuous linear operator $\bar{f} : L(X) \to E$ with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is unique.

It is well-known that $L(X)$ is metrizable if and only if X is finite. Moreover, $L(X)$ is a k-space if and only if X is a countable discrete space, see [14]. Therefore, seeking for concrete objects $L(X)$ carrying some *small base* at zero might be interesting for specialist both from topology and functional analysis.

One of such possible concepts extending metrizability is related with locally convex spaces having a $\mathfrak{G}$-base. Following [19], a topological space X has a $\mathfrak{G}$*-base at a point* $x \in X$ if it has a base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of neighborhoods at x such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \le \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$, where $\alpha = (\alpha(n))_{n\in\mathbb{N}} \le \beta = (\beta(n))_{n\in\mathbb{N}}$ if $\alpha(n) \le \beta(n)$ for all $n \in \mathbb{N}$; X has a $\mathfrak{G}$-base if it has a $\mathfrak{G}$-base at each point $x \in X$.

Originally, the concept of a $\mathfrak{G}$-base has been formally introduced in [11] in the realm of locally convex spaces for studying (DF)-spaces, $C(X)$-spaces and spaces in the class $\mathfrak{G}$ in the sense of Cascales and Orihuela, see [20]. Every quasibarrelled locally convex space with a $\mathfrak{G}$-base has countable tightness both in the original and the weak topology, respectively; each precompact set in a locally convex space with a $\mathfrak{G}$-base is metrizable, see again [20]. It is easy to see that every metrizable group has a $\mathfrak{G}$-base at the identity. Topological groups with a $\mathfrak{G}$-base are thoroughly studied in [19], see also [4,16,18].

Being motivated by several results of the above type (see [20] also for a long list of references), the authors in [19] posed the following general problem:

Problem 1.1 [19] For which spaces X the free locally convex space $L(X)$ has a $\mathfrak{G}$-base?

For a space X we denote by $C_p(X)$ and $C_k(X)$ the space $C(X)$ of all continuous real-valued functions on X endowed with the pointwise topology τ_p and the compact-open topology τ_k, respectively. Recall that a space X is called an *Ascoli space* if every compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous [2]. In other words, X is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [23, p. 45]. Using a deep result of Uspenskiĭ [27], for a wide class of topological spaces X we show that Problem 1.1 can be reformulated in the term of function spaces $C(X)$.

Theorem 1.2 *Let X be a Dieudonné complete Ascoli space* (in particular, X is a paracompact k-space or a metrizable space). *Then $L(X)$ has a $\mathfrak{G}$-base if and only if $C_k(X)$ has a compact resolution swallowing compact subsets.*

Recall that a family $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact subsets of a space Z is called a *compact resolution* if $\mathcal{K}$ covers Z and $K_\alpha \subseteq K_\beta$ whenever $\alpha \le \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$. Following Christensen [5], we say that $\mathcal{K}$ *swallows compact sets of* Z if for every compact subset K of Z there is an $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq K_\alpha$. The importance of this concept follows from the following deep result of Christensen: *A metrizable and separable space Z is Polish if and only if Z has a compact resolution swallowing compact sets.* Consequently, since $C_k(X)$ is Polish if X is locally compact metrizable and separable, by Theorem 1.2 the space $L(X)$ has a $\mathfrak{G}$-base. These results and Theorem 1.2 motivate the following question:

Problem 1.3 For which spaces X, the space $C_k(X)$ has a compact resolution (swallowing compact sets)?

This problem is of independent interest because (see for example [20, Theorem 9.9]) $C_k(X)$ has a compact resolution if and only if $C_k(X)$ is K-analytic, i.e. $C_k(X)$ is the image under an upper semi-continuous compact-valued map defined in $\mathbb{N}^{\mathbb{N}}$; the same result holds for $C_p(X)$, see [26]. Moreover, Tkachuk proved in [26] that $C_p(X)$ has a compact resolution swallowing compact sets if and only if X is a countable discrete space.

Christensen had already proved the following result (see also Corollary 2.3 below): *If X is a separable metrizable space, then $C_k(X)$ has a compact resolution if and only if X* is σ-compact. Below we strengthen this result by showing that under the same assumption on X the space $C_k(X)$ has even a compact resolution swallowing compact sets, see Corollary 2.10 below. These results motivate the question: *For which σ-compact spaces X the space $C_k(X)$has a compact resolution (swallowing compact sets)*? The aforementioned results explain our study of functions spaces with compact resolutions, see Sect. 2.

In Sect. 3 we prove Theorem 1.2 and obtain the following partial answers to Problem 1.1.

Theorem 1.4 *Let X be an Ascoli σ-compact space. Then $L(X)$ has a $\mathfrak{G}$-base if and only if X admits an Ascoli uniformity $\mathcal{U}$ with a $\mathfrak{G}$-base.*

Theorem 1.4 needs a new concept which is stronger than to be an Ascoli space.

Definition 1.5 A uniformity $\mathcal{U}$ on a space X is said to be *Ascoli* if $\mathcal{U}$ is admissible and any compact subset K of $C_k(X)$ is uniformly equicontinuous with respect to $\mathcal{U}$, i.e. for every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ for every $f \in K$ and each $(x, y) \in U$. We say that X is a *uniformly Ascoli space* if X has an Ascoli uniformity.

We provide also a sufficient condition on a σ-compact space X for which $L(X)$ has a $\mathfrak{G}$-base. This approach requires some additional concept.

Definition 1.6 A topological space X is a space of $\mathbb{N}^{\mathbb{N}}$- *uniformly compact type* if for every compact subset K of X the set $\Delta_K = \{(x, x) \in X \times X : x \in K\}$ has an $\mathbb{N}^{\mathbb{N}}$-decreasing base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of open neighborhoods in $X \times X$, i.e. for every open neighborhood U of Δ_K there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Delta_K \subseteq U_\alpha \subseteq U$.

Theorem 1.7 *Let X be a σ-compact Ascoli space. If X is of $\mathbb{N}^{\mathbb{N}}$-uniformly compact type, then $L(X)$ has a $\mathfrak{G}$-base.*

It is easy to show, see Proposition 2.7 below, that if X is a metrizable space or a countable space such that every point $x \in X$ has a $\mathfrak{G}$-base, then X is of $\mathbb{N}^{\mathbb{N}}$-uniformly compact type. So Theorem 1.7 with Corollary 2.10 imply

Corollary 1.8 *If X is a metrizable space, then $L(X)$ has a $\mathfrak{G}$-base if and only if X is σ-compact.*

In particular, the space $L(\mathbb{Q})$ has a $\mathfrak{G}$-base.

Corollary 1.9 *If X is a countable Ascoli space, then $L(X)$ has a $\mathfrak{G}$-base if and only if X has a $\mathfrak{G}$-base.*

Note that Corollaries 1.8 and 1.9 are proved independently in [3] using different methods.

2 Compact resolutions in function spaces

Recall that a subset A of a topological space X is called *functionally bounded* if every continuous function $f \in C(X)$ is bounded on A. Recall also that a *resolution* $\mathcal{A} = \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in X is a cover of X such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. If all A_α are functionally bounded, the resolution $\mathcal{A}$ is called *functionally bounded*. Note also that by a result of Calbrix, see [20, Theorem 9.7], if $C_p(X)$ is analytic, then X is σ-compact. Recall that a space Z is called *analytic* if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$.

We shall use the following fact, see Corollary 9.1 of [20]. Recall that a (Tychonoff) space X is *cosmic* if it is a continuous image of a separable metric space.

Fact 2.1 *Let X be a cosmic space. Then $C_p(X)$ has a functionally bounded resolution if and only if X is σ-compact. Consequently, if $C_k(X)$ has a compact resolution, then X is σ-compact.*

Proposition 2.2 *Let X be a paracompact first countable space such that $C_k(X)$ is an angelic space. Then X is Lindelöf.*

Proof If X is not Lindelöf, the space $C_k(X)$ contains a closed subset A homeomorphic to $\mathbb{N}^{\omega_1}$ by Lemma 1 of [24]. But the space $\mathbb{N}^{\omega_1}$ is not angelic, so $C_k(X)$ is also not angelic. This contradiction shows that X must be Lindelöf. □

This yields the following

Corollary 2.3 *Let X be a metrizable space. If $C_k(X)$ has a functionally bounded resolution, then X is σ-compact.*

Proof Proposition 9.6 of [20] implies that $C_p(X)$ is angelic. By [13, Theorem, page 31], the space $C_k(X)$ is also angelic. Now Proposition 2.2 implies that X is Lindelöf. So being metrizable, the space X is separable, and hence X is a cosmic space. Thus X is σ-compact by Fact 2.1. □

The next proposition completes Proposition 2.2.

Proposition 2.4 *Let X be a paracompact Čech-complete space. Then $C_k(X)$ is an angelic space if and only if X is Lindelöf.*

Proof Assume that $C_k(X)$ is angelic. By a result of Frolík [6, 5.5.9], there is a perfect map f from X onto a complete metrizable space Y. Suppose that X is not Lindelöf. Then, since f is perfect, Y is also not Lindelöf by [6, Theorem 3.8.9]. As $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$ by [6, Theorem 3.7.2], the space $C_k(Y)$ embeds into $C_k(X)$, and hence $C_k(Y)$ is also angelic. Now Proposition 2.2 implies that Y is Lindelöf, a contradiction. Thus X is a Lindelöf space.

Conversely, let X be Lindelöf. Then X has a compact resolution swallowing compact sets, see the proof of Proposition 4.7 in [17]. So $C_p(X)$ is angelic by Example 4.1 and Theorem 4.5 of [20]. Therefore $C_k(X)$ is angelic by [13, Theorem, page 31]. □

Recall that, for a space X, the family of sets of the form

$$[K, \varepsilon] := \{f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon)\}$$

where K is a compact subset of X, is a base of the compact-open topology τ_k on $C(X)$. Denote by $\delta : X \to C_k(C_k(X))$ the canonical map defined by

$$\delta(x)(f) := f(x), \quad \forall x \in X, \ \forall f \in C(X).$$

Proposition 2.5 *Let X be an Ascoli space. If $C_k(X)$ has a compact resolution swallowing compact sets, then X has an Ascoli uniformity $\mathcal{U}$ with a $\mathfrak{G}$-base.*

Proof Let $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a compact resolution swallowing compact sets of $C_k(X)$. Then, by [9], the space $C_k\big(C_k(X)\big)$ has a $\mathfrak{G}$-base $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $V_\alpha := \big[K_\alpha, \alpha(1)^{-1}\big]$. Since X is Ascoli space, the canonical map $\delta : X \to C_k\big(C_k(X)\big)$ is an embedding by Corollary 5.8 of [2]. For every $\alpha \in \mathbb{N}^\mathbb{N}$, define

$$U_\alpha := \{(x, y) \in X \times X : \ \delta(x) - \delta(y) \in V_\alpha\},$$

and let $\mathcal{U}$ be the uniformity on X induced from $C_k\big(C_k(X)\big)$. Clearly, $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base for $\mathcal{U}$ and $\mathcal{U}$ is admissible. We show that $\mathcal{U}$ is also Ascoli.

Fix a compact subset K of $C_k(X)$ and $\varepsilon > 0$. Take $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq K_\alpha$ and $\alpha(1) > 1/\varepsilon$. Now for every $(x, y) \in U_\alpha$ and each $f \in K$, we obtain

$$|f(x) - f(y)| = \big|\big(\delta(x) - \delta(y)\big)(f)\big| < \frac{1}{\alpha(1)} < \varepsilon,$$

Thus $\mathcal{U}$ is an Ascoli uniformity. □

We shall use the following encoding operation of elements of $\mathbb{N}^\mathbb{N}$. We encode each $\alpha \in \mathbb{N}^\mathbb{N}$ into a sequence $\{\alpha_i\}_{i\in\omega}$ of elements of $\mathbb{N}^\mathbb{N}$ as follows. Consider an arbitrary decomposition of $\mathbb{N}$ onto a disjoint family $\{N_i\}_{i\in\omega}$ of infinite sets, where $N_i = \{n_{k,i}\}_{k\in\mathbb{N}}$. Now for $\alpha = \big(\alpha(n)\big)_{n\in\mathbb{N}}$ and $i \in \omega$, we set $\alpha_i = \big(\alpha_i(k)\big)_{k\in\mathbb{N}}$, where $\alpha_i(k) := \alpha(n_{k,i})$ for every $k \in \mathbb{N}$. Conversely, for every sequence $\{\alpha_i\}_{i\in\omega}$ of elements of $\mathbb{N}^\mathbb{N}$ we define $\alpha = \big(\alpha(n)\big)_{n\in\mathbb{N}}$ setting $\alpha(n) := \alpha_i(k)$ if $n = n_{k,i}$.

For a subset A of a set S, a subset B of $S \times S$ and $(a, b) \in B$, we define

$$\Delta_A := \{(a, a) \in S \times S : a \in A\} \quad \text{and} \quad B(a) := \{s \in S : (a, s) \in B\}.$$

Next definition generalizes the classical notion of spaces of pointwise countable type (due to Arhangel'skii) and also Definition 1.6.

Definition 2.6 Let I be an ordered set and X a topological space. The space X is a space of

(i) *I-compact type* if every compact subset K of X has an decreasing I-base $\{U_i : i \in I\}$ of open neighborhoods, i.e. $U_i \subseteq U_j$ for all $i \geq j$ and for every open neighborhood U of K there is $i \in I$ such that $K \subseteq U_i \subseteq U$;
(ii) *I-pointwise countable type* if for every x in X there exists a compact set K which has a decreasing I-base of open sets;
(iii) *I-uniformly compact type* if for every compact subset K of X the set Δ_K has an I-decreasing base $\{U_i : i \in I\}$ of open neighborhoods in $X \times X$, i.e. for every open neighborhood U of Δ_K there is $i \in I$ such that $\Delta_K \subseteq U_i \subseteq U$.

As usual a decreasing $\mathbb{N}^\mathbb{N}$-base of a subset A of X is called a $\mathfrak{G}$- *base* of A. Next proposition provides possible two cases when X is of $\mathbb{N}^\mathbb{N}$-uniformly compact type.

Proposition 2.7 *A Tychonoff space X is of $\mathbb{N}^\mathbb{N}$-uniformly compact type if one of the following conditions holds:*

(i) *X is a metrizable space;*
(ii) *X is a countable space such that every point $x \in X$ has a $\mathfrak{G}$-base.*

Proof (i) For every compact subset K of X, the compact subset Δ_K of the metrizable space $X \times X$ has a decreasing base $\{V_n : n \in \mathbb{N}\}$. Then the family $\{U_\alpha = V_{\alpha(1)} : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base of K.

(ii) Let $\{x_n : n \in \mathbb{N}\}$ be an enumeration of X and $\{U_{\alpha,x_n} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base at x_n. Fix a compact subset K of X. If K is finite, then clearly the family

$$\left\{ \bigcup_{x_n \in K} U_{\alpha,x_n} \times U_{\alpha,x_n} : \alpha \in \mathbb{N}^{\mathbb{N}} \right\}$$

is a $\mathfrak{G}$-base at Δ_K. Assume that K is infinite. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$ with the encoding (α_n), we define

$$U_\alpha := \bigcup \{U_{\alpha_n,x_n} \times U_{\alpha_n,x_n} : x_n \in K\}.$$

We claim that the family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base at Δ_K. Indeed, fix an open neighborhood U of Δ_K. For every $x_n \in K$ take $\alpha_n \in \mathbb{N}^{\mathbb{N}}$ such that $U_{\alpha_n,x_n} \times U_{\alpha_n,x_n} \subseteq U$. Now if $\alpha \in \mathbb{N}^{\mathbb{N}}$ is built by the sequence (α_n), we obtain $K \subseteq U_\alpha \subseteq U$. Thus $\mathcal{U}$ is a $\mathfrak{G}$-base at Δ_K. □

We shall use the following fact which is proved in the "if" part of the Ascoli theorem [6, 3.4.20].

Fact 2.8 *Let X be a space and A be an evenly continuous* (in particular, equicontinuous) *pointwise bounded subset of $C_k(X)$. Then the closure $\bar{A}$ of A in τ_k is a compact equicontinuous subset of $C_k(X)$.*

If an Ascoli space X is additionally σ-compact, we can reverse Proposition 2.5.

Proposition 2.9 *Let X be a σ-compact space. Then the space $C_k(X)$ has a compact resolution swallowing compact sets if one of the following conditions holds:*

(i) *X has an Ascoli uniformity $\mathcal{U}$ with a $\mathfrak{G}$-base;*
(ii) *X is an Ascoli space of $\mathbb{N}^{\mathbb{N}}$-uniformly compact type.*

Proof Let $X = \bigcup_{n\in\mathbb{N}} C_n$ be the union of an increasing sequence $\{C_n\}_{n\in\mathbb{N}}$ of compact subsets. For the case (i), let $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base for the Ascoli uniformity $\mathcal{U}$. For the case (ii), for every $n \in \mathbb{N}$, let $\{U_{\alpha,n} : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ be a $\mathfrak{G}$-base of Δ_{C_n}. For every $\alpha \in \mathbb{N}$ with the encoding $\{\alpha_k\}_{k\in\omega}$, we define

$$A_\alpha := \bigcap_{k\in\mathbb{N}} \{f \in C(X) : |f(x)| \le \alpha_0(k) \quad \forall x \in C_k\},$$

$$B_\alpha := \bigcap_{n\in\mathbb{N}} \left\{ f \in C(X) : |f(x) - f(y)| \le \frac{1}{n} \quad \forall (x, y) \in U_{\alpha_n} \right\}, \quad \text{for case (i)},$$

$$B_\alpha := \bigcap_{n\in\mathbb{N}} \left\{ f \in C(X) : |f(x) - f(y)| \le \frac{1}{n} \quad \forall (x, y) \in U_{\alpha_n,n} \right\}, \quad \text{for case (ii)},$$

and set $K_\alpha := A_\alpha \cap B_\alpha$. Clearly, K_α is closed in the compact-open topology τ_k and $K_\alpha \subseteq K_\beta$ for every $\alpha \le \beta$. Fix $\alpha \in \mathbb{N}^{\mathbb{N}}$. By construction, K_α is pointwise bounded. We check that the set K_α is equicontinuous. We distinguish between cases (i) and (ii).

Case (i). Given $\varepsilon > 0$ take $n \in \mathbb{N}$ such that $n > 1/\varepsilon$. Then for every $f \in K_\alpha$, by the definition of B_α, we obtain $|f(x) - f(y)| \le \frac{1}{n} < \varepsilon$ whenever $(x, y) \in U_{\alpha_n}$. So K_α is equicontinuous.

Case (ii). Fix $x \in X$, so $x \in C_l$ for some $l \in \mathbb{N}$. Given $\varepsilon > 0$ take $n > l$ such that $n > 1/\varepsilon$. Then for every $f \in K_\alpha$, by the definition of B_α, we obtain $|f(x) - f(y)| \le \frac{1}{n} < \varepsilon$ whenever $(x, y) \in U_{\alpha_n,n}$. Since $U_{\alpha_n,n}(x)$ is an open neighborhood of x, the set K_α is equicontinuous.

Now in both cases (i) and (ii), Fact 2.8 implies that K_α is a compact subset of $C_k(X)$.

Let us show that the family $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows the compact sets of $C_k(X)$. Fix a compact subset K of $C_k(X)$. Since X is an Ascoli space, K is pointwise bounded and equicontinuous. Define $\alpha_0 = \big(\alpha_0(k)\big)_{k\in\mathbb{N}}$ as follows: for every $k \in \mathbb{N}$, set

$$\alpha_0(k) := \big[\sup\{|f(x)| : x \in C_k,\ f \in K\}\big] + 1,$$

where $[t]$ is the integral part of a real number t. Again we distinguish between cases (i) and (ii).

Case (i). Since $\mathcal{U}$ is Ascoli, for every $n \in \mathbb{N}$, take $\alpha_n \in \mathbb{N}^\mathbb{N}$ such that $|f(x) - f(y)| \le 1/n$ for every $f \in K$ and each $(x, y) \in U_{\alpha_n}$. If $\alpha \in \mathbb{N}^\mathbb{N}$ is built by the above procedure we obtain $K \subseteq K_\alpha$.

Case (ii). Fix $n \in \mathbb{N}$. For every $x \in C_n$ take an open neighborhood U_x of x such that $|f(x) - f(y)| \le 1/2n$ for every $f \in K$ and each $y \in U_x$. Set $W := \bigcup_{x\in C_n} U_x \times U_x$. Then for every $(z, y) \in W$ there is $x \in C_n$ such that $(z, y) \in U_x \times U_x$ and hence

$$|f(z) - f(y)| \le |f(x) - f(z)| + |f(x) - f(y)| \le \frac{1}{n}, \quad \text{for every } f \in K.$$

Since X is of $\mathbb{N}^\mathbb{N}$-uniformly compact type, we choose $\alpha_n \in \mathbb{N}^\mathbb{N}$ such that $\Delta_{C_n} \subseteq U_{\alpha_n,n} \subseteq W$. If $\alpha \in \mathbb{N}^\mathbb{N}$ is built by the sequence (α_n), we obtain $K \subseteq K_\alpha$.

Also now in both cases (i) and (ii) the family $\mathcal{K}$ swallows the compact sets of $C_k(X)$. □

As a corollary we obtain the following strengthening of Christensen's theorem.

Corollary 2.10 *For a metrizable space X, $C_k(X)$ has a compact resolution swallowing the compact sets of $C_k(X)$ if and only if X is σ-compact.*

Proof If $C_k(X)$ has a compact resolution swallowing the compact sets of $C_k(X)$, then X is σ-compact by Corollary 2.3. The converse assertion follows from Propositions 2.7 and 2.9. □

In particular, the space $C_k(\mathbb{Q})$ has a compact resolution swallowing its compact sets.

We conclude this section with the following Christensen's type result.

Proposition 2.11 *The following assertions are equivalent.*

(i) *$C_k(X)$ is analytic.*
(ii) *$C_k(X)$ is K-analytic and X is σ-compact.*
(iii) *$C_k(X)$ has a compact resolution and X is σ-compact.*

If additionally X is first countable, the above conditions are equivalent to

(iv) *X is metrizable and σ-compact.*

Proof First we prove the following claim using some ideas from [10] strongly motivated by Ferrando's Theorem 1 of [7].

Claim. If X is a σ-compact space, then $C_p(X)$ *admits a stronger metrizable locally convex topology.* Indeed, let $X = \bigcup_{n=1}^\infty K_n$, where K_n is a compact subset of X and $K_n \subseteq K_{n+1}$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, define

$$V_n := \left\{ f \in C(X) : \sup_{x\in K_n} |f(x)| \le \frac{1}{n} \right\}. \tag{1}$$

Clearly, $V_{n+1} \subseteq V_n$ and $\bigcap_{n=1}^{\infty} V_n = \{0\}$, where 0 stands for the identically null function on X. Note that the sets V_n are absorbing since if $g \in C(X)$, then there is $k \in \mathbb{N}$ such that $\sup_{x \in K_n} |g(x)| \leq k$, so that $g \in knV_n$. Moreover, if

$$U = \left\{ f \in C(X) : \max_{1 \leq i \leq n} |f(x_i)| < \epsilon \right\}$$

and $p \in \mathbb{N}$ is chosen so that $x_i \in V_p$ for $1 \leq i \leq n$ and $p^{-1} < \epsilon$, then $V_p \subseteq U$ and clearly $V_{2n} \subseteq 2^{-1} V_n$ for each $n \in \mathbb{N}$. This shows that $\{V_n : n \in \mathbb{N}\}$ is a base of neighborhoods of the origin of a locally convex topology on $C(X)$ stronger than the pointwise topology. The claim is proved.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear; note that if $C_k(X)$ is analytic, then Calbrix's result, see [20, Theorem 9.7], implies that X is σ-compact. (iii) $\Rightarrow$ (i): If X is σ-compact, the claim and (1) show that the space $C(X)$ admits a metrizable locally convex topology weaker (or equal) to the compact-open topology τ_k. As $C_k(X)$ has a compact resolution (by assumption), we apply Talagrand's result, see [20, Proposition 6.3], to deduce that $C_k(X)$ is analytic. If X first countable, (i) is equivalent to (iv) by [23, Theorem 5.7.5]. □

3 Proofs of Theorems 1.2, 1.4 and 1.7

It is well-known that the dual space of $C_k(X)$ is the space $M_c(X)$ of all regular Borel measures on X with compact support. Denote by τ_e the topology on $M_c(X)$ of uniform convergence on the equicontinuous pointwise bounded subsets of $C(X)$. For $A \subseteq C_k(X)$ and $B \subseteq M_c(X)$, we set as usual

$$A^\circ = \{\mu \in M_c(X) : |\mu(f)| \leq 1 \ \forall f \in A\}, \text{ and}$$
$$B^\circ = \{f \in C_k(X) : |\mu(f)| \leq 1 \ \forall \mu \in B\}.$$

Proposition 3.1 *Let X be an Ascoli space. Then $(M_c(X), \tau_e)$ has a $\mathfrak{G}$-base if and only if $C_k(X)$ has a compact resolution swallowing compact subsets of $C_k(X)$.*

Proof Assume that $C_k(X)$ has a compact resolution swallowing compact subsets of $C_k(X)$. Let $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a compact resolution swallowing compact sets of $C_k(X)$. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $U_\alpha := K_\alpha^\circ$. We show that the family $\mathcal{U} := \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base in $(M_c(X), \tau_e)$. Indeed, every U_α is a neighborhood of zero in τ_e because K_α is equicontinuous. Now let U be a neighborhood of zero in $(M_c(X), \tau_e)$. Take an equicontinuous pointwise bounded subset A of $C(X)$ such that $A^\circ \subseteq U$. By Fact 2.8, the closure K of A in $C_k(X)$ is compact. So there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq K_\alpha$. Clearly, $U_\alpha = K_\alpha^\circ \subseteq A^\circ \subseteq U$. Thus $\mathcal{U}$ is a base of τ_e.

Conversely, let $(M_c(X), \tau_e)$ have a $\mathfrak{G}$-base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $C_\alpha := U_\alpha^\circ$. We show that the family $\mathcal{C} := \{C_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution in $C_k(X)$ swallowing the compact sets.

Clearly, if $\alpha \leq \beta$ then $C_\alpha \subseteq C_\beta$. Since $M_c(X)$ is the dual space of $C_k(X)$, every C_α is closed in $C_k(X)$. To show that C_α is compact in $C_k(X)$, take an absolutely convex neighborhood V of zero in $M_c(X)$ such that $\overline{V} \subseteq U_\alpha$ and choose an equicontinuous pointwise bounded subset A of $C(X)$ such that $A^\circ \subseteq V$. Clearly, the absolutely convex hull $\mathrm{acx}(A)$ of A is also an equicontinuous pointwise bounded subset of $C(X)$. So, by Fact 2.8, the closure $K := \overline{\mathrm{acx}(A)}^{\tau_k}$ of $\mathrm{acx}(A)$ in the compact-open topology τ_k is a compact equicontinuous subset of $C_k(X)$. Since the bounded convex subsets of $C(X)$ in τ_k and $\sigma\big(C(X), M_c(X)\big)$ are the same, the Bipolar theorem implies that $K = K^{\circ\circ}$. As

$$C_\alpha \subseteq \overline{V}^\circ \subseteq A^{\circ\circ} = K^{\circ\circ} = K$$

we obtain that C_α is compact.

Let C be a compact subset of $C_k(X)$. Since X is Ascoli, C is equicontinuous and clearly pointwise bounded. Take $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $U_\alpha \subseteq C^\circ$. Then

$$C \subseteq C^{\circ\circ} \subseteq U_\alpha^\circ = C_\alpha.$$

Thus the family $\mathcal{C}$ swallows the compact sets of $C_k(X)$. □

For a space X we denote by μX the Dieudonné completion of X. Note that any paracompact space is Dieudonné complete, see [6, 8.5.13(d)]. Now Theorem 1.2 immediately follows from the following more general result.

Theorem 3.2 *Let X be a Tychonoff space such that μX is an Ascoli space. Then $L(X)$ has a $\mathfrak{G}$-base if and only if $C_k(\mu X)$ has a compact resolution swallowing compact subsets. In this case the space $C_k(\mu X)$ is Lindelöf.*

Proof The space $L(X)$ has a $\mathfrak{G}$-base if and only if its (Raikov) completion $\overline{L(X)}$ has a $\mathfrak{G}$-base, see Proposition 2.7 of [19]. It is known that $\overline{L(X)}$ is $(M_c(\mu X), \tau_e)$, see Theorem 5 of [27]. Now Proposition 3.1 applies. To prove the last assertion we note that the space $C_k(\mu X)$ is K-analytic by Theorem 9.9 of [20]. So $C_k(\mu X)$ is Lindelöf by Proposition 3.13 of [20]. □

We do not know whether the condition on X to be an Ascoli space is essential in Theorem 1.2.

Question 4.18 in [19] asks whether for a k-space the existence of a $\mathfrak{G}$-base on $L(X)$ implies that also $C_k(C_k(X))$ has a $\mathfrak{G}$-base. By Ferrando–Kąkol theorem [10] (see also [19, Theorem 4.9]), the space $C_k(X)$ has a compact resolution swallowing compact subsets if and only if $C_k(C_k(X))$ has a $\mathfrak{G}$-base. Combining this result with Theorem 1.2 we obtain a partial answer to [19, Question 4.18].

Corollary 3.3 *Let X be a Dieudonné complete Ascoli space. Then $L(X)$ has a $\mathfrak{G}$-base if and only if the space $C_k(C_k(X))$ has a $\mathfrak{G}$-base.*

Corollary 1.9 follows from the next result.

Corollary 3.4 *If X is a countable Ascoli space, then the following assertions are equivalent:*

(i) *$L(X)$ has a $\mathfrak{G}$-base;*
(ii) *$C_k(X)$ has a compact resolution swallowing the compact sets of $C_k(X)$;*
(iii) *X has a $\mathfrak{G}$-base.*

Proof (i)⇔(ii) follows from Theorem 1.2 (recall that any countable space being Lindelöf is Dieudonné complete). (i) ⇒ (iii) follows from the fact that X is a subspace of $L(X)$. (iii) ⇒ (ii) follows from Propositions 2.7 and 2.9. □

Note that Ferrando in [8] gives a direct proof of the implication (iii) ⇒ (ii) in Corollary 3.4.

We provide another necessary condition for a space X to have the space $L(X)$ with a $\mathfrak{G}$-base.

Proposition 3.5 *If $L(X)$ has a $\mathfrak{G}$-base, then every precompact set in $L(X)$* (hence also in *X*) *is metrizable.*

Proof Note that in every locally convex space E with a $\mathfrak{G}$-base every precompact set is metrizable, see [20, Theorem 11.1]. We conclude the proof by noticing that X embeds into $L(X)$. □

Recall that a topological space X is a k_ω-space (an $\mathcal{MK}_\omega$-space) if X is the inductive limit of a countable family of compact (compact and metrizable) subsets. We proved in [16] that $L(X)$ has a $\mathfrak{G}$-base for every $\mathcal{MK}_\omega$-space X. Combining this result with Proposition 3.5 we obtain

Corollary 3.6 *Let X be a k_ω-space. Then $L(X)$ has $\mathfrak{G}$-base if and only if X is an $\mathcal{MK}_\omega$-space.*

Remark 3.7 In [19, Question 4.19] we ask whether the existence of a $\mathfrak{G}$-base in the free abelian group $A(X)$ over a space X implies that $L(X)$ has also a $\mathfrak{G}$-base. Let X be a discrete space. Then it is clear that $A(X)$ being discrete has a $\mathfrak{G}$-base. In [21] it is shown that if X is of cardinality $\geq \mathfrak{c}$, then $L(X)$ does not have a $\mathfrak{G}$-base. This answers Question 4.19 of [19] in the negative. Our Corollary 1.8 implies a stronger result: for every uncountable discrete space X, the space $L(X)$ does not have a $\mathfrak{G}$-base.

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