Journal of Functional Analysis 272 (2017) 1876–1891

Contents lists available at ScienceDirect

Journal of Functional Analysis

www.elsevier.com/locate/jfa

A separable Fréchet space of almost universal disposition

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ARTICLE INFO

Article history:
Received 6 April 2016
Accepted 26 September 2016
Available online 6 October 2016
Communicated by G. Schechtman

MSC:
46A61
46A04
46B20
46M40

Keywords:
Graded Fréchet space
The Gurariĭ space
Universality

ABSTRACT

The Gurariĭ space is the unique separable Banach space $\mathbb{G}$ which is of almost universal disposition for finite-dimensional Banach spaces, which means that for every $\varepsilon > 0$, for all finite-dimensional normed spaces $E \subseteq F$, for every isometric embedding $e\colon E \to \mathbb{G}$ there exists an ε-isometric embedding $f\colon F \to \mathbb{G}$ such that $f \restriction E = e$. We show that $\mathbb{G}^{\mathbb{N}}$ with a special sequence of semi-norms is of almost universal disposition for finite-dimensional graded Fréchet spaces. The construction relies heavily on the universal operator on the Gurariĭ space, recently constructed by Garbulińska-Węgrzyn and the third author. In addition, we consider a non-graded sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$ with which the space $\mathbb{G}^{\mathbb{N}}$ is of almost universal disposition for finite-dimensional Fréchet spaces with a fixed sequence of semi-norms. In both cases, this yields in par-

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[1] The second named author was supported by Generalitat Valenciana, Conselleria d'Educació, Cultura i Esport, Spain, Grant PROMETEO/2015/058 and by the GAČR project 16-34860L and RVO: 67985840.
[2] The third author was supported by the GAČR project 16-34860L and RVO: 67985840 and by the NCN grant 2011/03/B/ST1/00419.

http://dx.doi.org/10.1016/j.jfa.2016.09.019

ticular that $\mathbb{G}^{\mathbb{N}}$ is universal in the class of all separable Fréchet spaces.

1. Introduction

A remarkable result of Banach and Mazur [14] states that the separable Banach space $\mathcal{C}[0,1]$ is universal for separable Banach spaces. The above theorem has been extended by Mazur and Orlicz to the class of separable Fréchet spaces, i.e. metrizable and complete locally convex spaces: They proved that the separable Fréchet space $\mathcal{C}(\mathbb{R})$ is universal in the class of all separable Fréchet spaces, see again [14]. An essential progress of the research around the Banach–Mazur theorem is due to Gurariĭ. The Gurariĭ space constructed by Gurariĭ [8] in 1965, is the separable Banach space $\mathbb{G}$ of "almost universal disposition for finite-dimensional spaces" that is:

(G) For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every isometric embedding $e\colon E \to \mathbb{G}$ there exists an ε-isometric embedding $f\colon F \to \mathbb{G}$ such that $f \restriction E = e$.

Moreover, if Y is any other separable Banach space fitting this definition, then there exists a linear isomorphism $u\colon \mathbb{G} \to Y$ with $\|u\| \cdot \|u^{-1}\|$ arbitrarily close to 1. Lusky [11] proved that all separable Banach spaces of almost universal disposition are isometric, see also [10] for a simpler proof. Recall, that a linear operator $f\colon E \to F$ between Banach spaces E and F is an *ε-isometric embedding* for $\varepsilon > 0$ if

$$(1+\varepsilon)^{-1} \cdot \|x\| \le \|f(x)\| \le (1+\varepsilon) \cdot \|x\|, \ x \in E \setminus \{0\}.$$

Recall also that Gurariĭ has already observed in [8] that no separable Banach space E is of universal disposition, i.e. satisfies condition (G) with removing ε.

Being motivated by recent developments in the theory of Fréchet spaces we will study the concrete separable Fréchet space $\mathbb{G}^{\mathbb{N}}$ endowed with the product topology generated by two natural sequences of semi-norms, where $\mathbb{G}$ is the Gurariĭ space. We prove that $\mathbb{G}^{\mathbb{N}}$ is universal in the class of all separable Fréchet spaces although (as we show) that there is no separable Fréchet space which is of universal disposition for finite-dimensional Fréchet spaces. Our main results state however that $\mathbb{G}^{\mathbb{N}}$ is the unique separable (graded) Fréchet space which is of almost universal disposition for finite-dimensional (graded) Fréchet spaces. These results can be found in Sections 3 and 4 for the graded case and in Section 5 for the non-graded case.

2. Preliminaries

Recall that a *Fréchet space* is a metrizable locally convex linear topological space which is complete with respect to its canonical uniformity. It is well-known that a complete separated locally convex topological vector space is a Fréchet space if and only if it satisfies the following condition: There exists a sequence $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ of semi-norms in X such that $B_i(v,r) = \{x \in X\colon \max_{n\le i} \|x-v\|_n < r\}$, where $i \in \mathbb{N}$, $v \in X$, $r > 0$, generate the topology of X. Fréchet spaces have played an important role in functional analysis for a very long time. Many vector spaces of holomorphic, differentiable or continuous functions which appear in analysis and its applications carry a natural Fréchet topology. We refer the reader to an excellent survey of some recent developments in the theory of Fréchet spaces and of their duals, see [2]. We refer also to [14] for other fundamental facts and concepts related with Fréchet spaces.

In this paper Fréchet spaces E are considered with a fixed sequence of semi-norms. In the case of an increasing sequence, i.e. if $\|\cdot\|_1 \le \|\cdot\|_2 \le \ldots$, we call (following Vogt [15]) E endowed with this sequence a *graded Fréchet space*.

Graded Fréchet spaces have been studied in the context of the inverse function theorem of Nash and Moser, see [9], and in the context of tame Fréchet spaces, see e.g. [15,16, 13]. Note however that the concept of the category of graded Fréchet spaces considered in [5] differs from the notion of graded Fréchet spaces used in this article. For a recent application of graded Fréchet spaces, we refer the interested reader to [4].

In this section we construct an increasing sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$ under which $\mathbb{G}^{\mathbb{N}}$ is a graded Fréchet space of almost universal disposition for finite-dimensional Fréchet spaces. Our construction uses the following result being a special case of Theorem 6.5 in [3]; property (2) below appears also as condition (‡) on page 765 in [3]. It turns out that condition (2) determines π uniquely up to linear isometries, although this fact was not proved in [3].

Theorem 2.1. *There exists a non-expansive linear operator $\pi\colon \mathbb{G} \to \mathbb{G}$ with the following properties.*

(1) *For every non-expansive operator $T\colon X \to \mathbb{G}$ with X a separable Banach space, there exists an isometric embedding $i\colon X \to \mathbb{G}$ such that $T = \pi \circ i$.*
(2) *For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every non-expansive linear operator $T\colon F \to \mathbb{G}$, for every isometric embedding $e\colon E \to \mathbb{G}$ such that $T \restriction E = \pi \circ e$, there exists an ε-isometric embedding $f\colon F \to \mathbb{G}$ such that*

$$\|f \restriction E - e\| \le \varepsilon \quad \text{and} \quad \|\pi \circ f - T\| \le \varepsilon.$$

Furthermore, $\ker \pi$ *is linearly isometric to* $\mathbb{G}$.

The last proposition may be applied to obtain the following useful

Proposition 2.2. *The operator π from Theorem 2.1 is a projection and it satisfies the following condition:*

(2′) *For every $\varepsilon > 0$, for every finite-dimensional normed spaces $E \subseteq F$, for every non-expansive linear operator $T\colon E \to \mathbb{G}$, for every isometric embedding $e\colon E \to \mathbb{G}$ such that $T \restriction E = \pi \circ e$, there exists an ε-isometric embedding $f\colon F \to \mathbb{G}$ such that*

$$f \restriction E = e \quad \text{and} \quad \pi \circ f = T.$$

Note that Property (2′) implies (2) but in the following proof, we use (2) to prove (2′).

Proof. Taking $T = \mathrm{id}_{\mathbb{G}}$ in Theorem 2.1, we see that π is a projection. From now on, we shall identify the range of π with a suitable subspace of its domain. In order to show (2′), we need to use the following fact.

Claim 2.3. *Let F be a finite-dimensional normed space and let $S = \{v_0, \dots, v_n\}$ be a linear basis of F. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of linear operators $f, g\colon F \to X$ into an arbitrary Banach space X, the following implication holds:*

$$\max_{v \in S} \|f(v) - g(v)\| \le \delta \implies \|f - g\| \le \varepsilon.$$

Note that for every fixed S, there is a $C > 0$ for which we can define $\delta = C\varepsilon$. Indeed, since the norm of F is equivalent to the ℓ_1-norm of the coefficients with respect to the basis S, there is a $C_1 > 0$ so that for $0 \neq v = \sum_{i=0}^{n} \lambda_i v_i$, one gets

$$\frac{\|f(v) - g(v)\|}{\|v\|} \le \sum_{i=0}^{n} \frac{|\lambda_i|}{\|v\|} \max_{v \in S} \|f(v) - g(v)\| \le C_1 \max_{v \in S} \|f(v) - g(v)\|.$$

Then $C := \frac{1}{C_1}$ is the desired constant. For a detailed proof of the above claim we refer the reader to the proof of [6, Theorem 2.7].

Now assume that $E \subseteq F$ and $S \cap E$ is a linear basis of E. Fix $\varepsilon > 0$ and let $\delta > 0$ be as mentioned in the claim. We apply property (2) from Theorem 2.1 with δ instead of ε. This provides a map $f\colon F \to \mathbb{G}$. Define f' so that $f' \restriction E = e$ and $f'(v) = f(v)$ for every $v \in S \setminus E$. These conditions specify f' uniquely and by the claim we have that f' is a 2ε-isometric embedding. Furthermore, we have that $\|\pi \circ f' - T\| \le 2\varepsilon$. We have proved that for every $r > 0$ there is an r-isometric embedding $f\colon F \to \mathbb{G}$ extending e and such that $\|\pi \circ f - T\| \le r$. Let us use this fact for $r = \delta$, where δ and ε are as before. We obtain a δ-isometric embedding $f\colon F \to \mathbb{G}$ extending e and satisfying $\|\pi \circ f - T\| \le \delta$. Fix $v \in S \setminus E$. Then the vector $w_v = \pi(f(v)) - T(v)$ has norm $\le \delta$. Define $f'\colon F \to \mathbb{G}$ so that $f' \restriction E = f$ and

$$f'(v) = f(v) - w_v, \qquad v \in S \setminus E.$$

Note that $\|f'(v) - f(v)\| \leq \delta$ for $v \in S$, therefore $\|f' - f\| \leq \varepsilon$. Finally, $\pi(f'(v)) = \pi(f(v)) - \pi(f(v)) + \pi(T(v)) = T(v)$. It follows that $\pi \circ f' = T$. $\square$

Now we are ready to construct a graded sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$. We start by defining $\|\cdot\|_1$ to be the $\mathbb{G}$-norm of the first component, i.e., $\|x\|_1 := \|x_1\|_{\mathbb{G}}$. Having in mind the last two results mentioned above, we conclude that there is a norm $\|\cdot\|_2$ on the product $\mathbb{G} \times \mathbb{G} \cong (\operatorname{im} \pi) \times (\ker \pi)$ satisfying

$$\|x\|_{\mathbb{G}} = \|(x, y)\|_1 \leq \|(x, y)\|_2$$

since $x = \pi(x, y)$ and π is non-expansive. In addition, $(\mathbb{G}^2, \|\cdot\|_2)$ is isometric to $\mathbb{G}$. We identify $\mathbb{G} \times \mathbb{G}$ with $\mathbb{G}$ and use the notation π_2 for π in this case to stress that we consider it as mapping from $\mathbb{G}^2$ to $\mathbb{G}$. Inductively, for all $n \in \mathbb{N}$, we get a norm $\|\cdot\|_n$ on $\mathbb{G}^n$ satisfying $\|x\|_1 \leq \cdots \leq \|x\|_{n-1} \leq \|x\|_n$ for all $x \in \mathbb{G}^n$ and $(\mathbb{G}^n, \|\cdot\|_n)$ is isometric to $\mathbb{G}$. Therefore this construction provides an increasing sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$ as claimed. We use the notation $\pi_n \colon \mathbb{G}^n \to \mathbb{G}^{n-1}$ for the universal operator π if we want to stress that we consider it as projection from $\mathbb{G}^n$ to $\mathbb{G}^{n-1}$, i.e., a projection onto the first $n-1$ components.

In order to formulate a condition similar to (G) for Fréchet spaces we need to define the corresponding concept of ε-isometries for Fréchet spaces.

Definition 2.4. Let $(X, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ and $(Y, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ be Fréchet spaces with fixed sequences of semi-norms. A mapping $f \colon X \to Y$ is called an *ε-isometric embedding* iff it is an embedding and

$$(1+\varepsilon)^{-1}\|x\|_i \leq \|f(x)\|_i \leq (1+\varepsilon)\|x\|_i \tag{1}$$

holds for all $i \in \mathbb{N}$ and all $x \in X$. The mapping f is called an *isometric embedding* iff

$$\|f(x)\|_i = \|x\|_i \tag{2}$$

holds for all $i \in \mathbb{N}$ and all $x \in X$.

Now we are ready to formulate the analogue of condition (G) for Fréchet spaces.

Definition 2.5. A (graded) Fréchet space E is *of almost universal disposition for finite-dimensional (graded) Fréchet spaces* if for all $\varepsilon > 0$ and for all finite-dimensional (graded) Fréchet spaces $X \subseteq Y$ with an isometric embedding $f_0 \colon X \to E$ there exists an ε-isometric embedding $f \colon Y \to E$ satisfying $f \restriction X = f_0$.

3. A graded Fréchet space of almost universal disposition

We show that the space $\mathbb{G}^{\mathbb{N}}$ equipped with the graded sequence $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ of semi-norms defined above (coming from the universal operator π) is of almost universal disposition for finite-dimensional graded Fréchet spaces. We need the following

Lemma 3.1. *Let $(X, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ and $(Y, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ be Fréchet spaces with fixed semi-norms and $\iota\colon X \to Y$ an isometric embedding. Then for all $i \in \mathbb{N}$ the mapping*

$$\iota_i\colon X/\ker\|\cdot\|_i \to Y/\ker\|\cdot\|_i, \bar{x} \mapsto \overline{\iota(x)}$$

is a well-defined isometric embedding. Moreover, the diagram

$$\begin{array}{ccc} \prod X/\ker\|\cdot\|_i & \xhookrightarrow{\prod \iota_i} & \prod Y/\ker\|\cdot\|_i \\ \big\uparrow & & \big\uparrow \\ X & \xhookrightarrow{\quad\iota\quad} & Y \end{array}$$

is commutative.

Proof. As ι is an isometric embedding, we have $\|\iota(x)\|_i = \|x\|_i$. Hence $x \in \ker\|\cdot\|_i$ iff $\iota(x) \in \ker\|\cdot\|_i$, i.e. ι_i is well-defined. By definition we have

$$\|\iota_i(\bar{x})\|_i = \left\|\overline{\iota(x)}\right\|_i = \|\iota(x)\|_i = \|x\|_i = \|\bar{x}\|_i$$

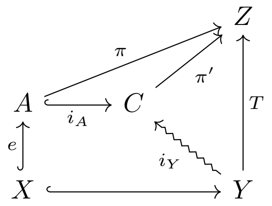
and hence ι_i is an isometric embedding. The commutativity of the diagram directly follows from the definition of ι_i. $\square$

In addition, we also need the following technical lemma which is similar in spirit to the construction given in [12, p. 137].

Lemma 3.2. *Let $X \subseteq Y$ and A be finite-dimensional Banach spaces, Z a Banach space, $e\colon X \to A$ an isometric embedding, $T\colon Y \to Z$ a linear operator with $\|T\| \le r$, $r > 1$, and $\pi\colon A \to Z$ a non-expansive operator such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{\pi} & Z \\ {\scriptstyle e}\big\uparrow & & \big\uparrow{\scriptstyle T} \\ X & \hookrightarrow & Y \end{array}$$

commutes. There exists a finite-dimensional Banach space C, an isometric embedding $i_A\colon A \to C$, an $(r-1)$-isometric embedding $i_Y\colon Y \to C$ and a non-expansive operator $\pi'\colon C \to Z$ such that the diagram



is commutative.

Proof. We define $C := (A \oplus Y)/\Delta$ with $\Delta = \{(e(x), -x) \colon x \in X\}$ equipped with the norm

$$\|(a, y)\| = \inf_{x \in X} \{\|a - e(x)\|_A + r\|x + y\|_Y\}.$$

First we show that i_A is an isometry. For $a \in A$ we obtain

$$\begin{aligned}\|i_A(a)\| &= \inf_{x \in X} \{\|a - e(x)\|_A + r\|x\|_Y\} \\ &= \inf_{x \in X} \{\|a - e(x)\|_A + \|e(x)\|_A + (r-1)\|x\|_Y\} \\ &\geq \inf_{x \in X} \{\|a\|_A + (r-1)\|x\|_Y\} = \|a\|_A\end{aligned}$$

and, by setting $x = 0$, $\|i_A(a)\| \leq \|a\|_A$. For $y \in Y$ we get

$$\|i_Y(y)\| = \|(0, y)\| = \inf_{x \in X} \{\| - e(x)\|_A + r\|x + y\|_Y\} \leq r\|y\|_Y$$

by setting $x = 0$, and

$$\begin{aligned}\|i_Y(y)\| &= \inf_{x \in X} \{\| - e(x)\|_A + r\|x + y\|_Y\} \\ &= \inf_{x \in X} \{\| - x\|_Y + \|y + x\|_Y + (r-1)\|x + y\|_Y\} \\ &\geq \inf_{x \in X} \{\|y\|_Y + (r-1)\|x + y\|_Y\} \geq \|y\|_Y \geq \frac{1}{r}\|y\|_Y,\end{aligned}$$

again using the triangle inequality. The linear operator $\pi' \colon A \to Z$ can be defined as

$$\begin{aligned}\pi'((a, y)) &= \pi(a - e(x)) + T(x + y) = \pi(a) - \pi(e(x)) + T(x) + T(y) \\ &= \pi(a) + T(y).\end{aligned}$$

It satisfies

$$\begin{aligned}\|\pi'((a, y))\|_Z &= \|\pi'((a - e(x), x + y))\|_Z \\ &\leq \|\pi(a - e(x))\|_Z + \|T(x + y)\|_Z \\ &\leq \|a - e(x)\|_A + r\|x + y\|_Y\end{aligned}$$

for all $x \in X$ and hence $\|\pi'((a, y))\| \leq \|(a, y)\|$. $\square$

Theorem 3.3. *The space $\mathbb{G}^{\mathbb{N}}$ equipped with the graded sequence*

$$\{\|\cdot\|_n\}_{n\in\mathbb{N}}$$

of semi-norms defined above is of almost universal disposition for finite-dimensional graded Fréchet spaces.

Proof. Given finite-dimensional graded Fréchet spaces $X \subseteq Y$, an isometric embedding $f\colon X \to \mathbb{G}^{\mathbb{N}}$ and $\varepsilon > 0$, we choose a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ satisfying $\prod_{i=1}^{\infty}(1+\varepsilon_i) < 1+\varepsilon$. We will use the notation $X_i := X/\ker\|\cdot\|_i$, $Y_i := Y/\ker\|\cdot\|_i$. By $f_i\colon X_i \to \mathbb{G}^i$ we denote the mapping induced by the isometric embedding $f\colon X \to \mathbb{G}^{\mathbb{N}}$.

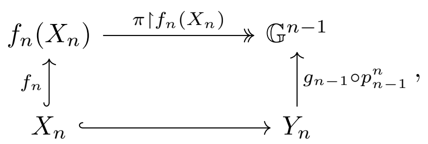
As a first step, we can use that $X_1 \subseteq Y_1$ are finite-dimensional Banach spaces and $f_1\colon X_1 \to \mathbb{G}$ is an isometric embedding to obtain an ε_1-isometric embedding $g_1\colon Y_1 \to \mathbb{G}$ which extends f_1.

Now assume that we already have constructed a δ-isometric embedding

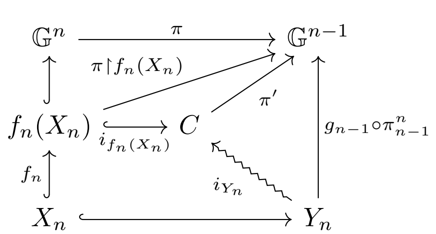
$$g_{n-1}\colon Y_{n-1} \to \mathbb{G}^{n-1},$$

where $1+\delta = \prod_{i=1}^{n-1}(1+\varepsilon_i)$, extending $f_{n-1}\colon X_{n-1} \to \mathbb{G}^{n-1}$.

We consider the diagram



where $p_{n-1}^n\colon Y_n \to Y_{n-1}$ denotes the canonical projection. Note that it is commutative since $\pi \circ f_n = f_{n-1}$. From Lemma 3.2 we deduce the existence of a finite-dimensional Banach space C such that the diagram



commutes and π' is non-expansive. From Proposition 2.2, we may now conclude that there is an ε_n-isometric embedding $\tilde{g}_n\colon C \to \mathbb{G}^n$ which extends both f_n and π'. Hence $g_n = \tilde{g}_n \circ \iota_{Y_n}$ is an δ'-isometric embedding, where $1+\delta' = \prod_{i=1}^{n}(1+\varepsilon_i)$, which extends both f_n and g_{n-1}.

Therefore by induction we get an ε-isometric embedding $g\colon Y \to \mathbb{G}^{\mathbb{N}}$ extending the embedding $f\colon X \to \mathbb{G}^{\mathbb{N}}$. $\square$

4. Uniqueness and universality

The aim of this section is to prove universality and the following uniqueness result for the space $\mathbb{G}^{\mathbb{N}}$.

Proposition 4.1. *Let E and F be separable graded Fréchet spaces which are of almost universal disposition for finite-dimensional graded Fréchet spaces, $\varepsilon > 0$, $X \subseteq E$ a finite-dimensional subspace and $f\colon X \to F$ an ε-isometric embedding. Then there exists a bijective isometry $h\colon E \to F$ satisfying $\|h(x) - f(x)\|_i \leq 4\varepsilon\|x\|_i$ for all $x \in X$ and all $i \in \mathbb{N}$.*

Note that by setting $X = \{0\}$, we obtain the following

Corollary 4.2. *The space $\mathbb{G}^{\mathbb{N}}$ is up to isometries the unique separable graded Fréchet space of almost universal disposition for finite-dimensional graded Fréchet spaces.*

First we need to show some additional lemmata used for the proof. The proofs of these lemmata and of Proposition 4.1 follow the lines of the corresponding proofs in the Banach space case, see [7,10]. Let $(X, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ and $(Y, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ be Fréchet spaces with a fixed sequence of semi-norms and $f\colon X \to Y$ a linear mapping with the property that for all $i \in \mathbb{N}$ there exists a constant $C_i > 0$ such that $\|f(x)\|_i \leq C_i\|x\|_i$ holds for all $x \in X$. We can assume this since we only want to consider mappings which are isometries or at least ε-isometries.

We can now define

$$\|f\|_i = \sup_{\|x\|_i=1} \|f(x)\|_i$$

for all $i \in \mathbb{N}$. Note that the above condition on f is stronger than the continuity of f.

We need the following Lemma motivated by [7, Lemma 2.2] for Banach spaces.

Lemma 4.3. *Let $(X, \{\|\cdot\|_{X,i}\}_{i\in\mathbb{N}})$ and $(Y, \{\|\cdot\|_{Y,i}\}_{i\in\mathbb{N}})$ be finite-dimensional graded Fréchet spaces and let $\varepsilon > 0$ and $f\colon X \to Y$ be an ε-isometric embedding. There exists a finite-dimensional graded Fréchet space Z and isometric embeddings $\iota\colon X \to Z$ and $j\colon Y \to Z$ such that $\|j \circ f - \iota\|_i \leq \varepsilon$ holds for all $i \in \mathbb{N}$.*

Proof. We set $Z = X \oplus Y$ equipped with the semi-norms

$$\|(x, y)\|_i = \inf\{\|u\|_{X,i} + \|v\|_{Y,i} + \varepsilon\|w\|_{X,i} \colon x = u + w,$$
$$y = v - f(w), u, w \in X, v \in Y\}.$$

We have

$$\|j(f(x)) - \iota(x)\|_i = \|(x, -f(x))\|_i \leq \varepsilon\|x\|_{X,i}$$

by taking $u = 0$, $v = 0$ and $w = x$. Hence $\|j \circ f - \iota\|_i \leq \varepsilon$ for all $i \in \mathbb{N}$. From

$$\frac{1}{1+\varepsilon} \geq 1 - \varepsilon \Leftrightarrow 1 \geq 1 - \varepsilon^2$$

we may deduce $\|f(x)\|_{Y,i} \geq (1+\varepsilon)^{-1}\|x\|_{X,i} \geq (1-\varepsilon)\|x\|_{X,i}$ which we will need in the following. Now we show that ι is an isometric embedding. For $x \in X$ we have

$$\|\iota(x)\|_i = \|(x, 0)\|_i \leq \|x\|_{X,i}.$$

Setting $x = u + w$ and $0 = v - f(w)$, we obtain

$$\begin{aligned}\|u\|_{X,i} + \|v\|_{Y,i} + \varepsilon\|w\|_{X,i} &= \|u\|_{X,i} + \|f(w)\|_{Y,i} + \varepsilon\|w\|_{X,i} \\ &\geq \|u\|_{X,i} + (1-\varepsilon)\|w\|_{X,i} + \varepsilon\|w\|_{X,i} \\ &= \|u\|_{X,i} + \|w\|_{X,i} \geq \|u + w\|_{X,i} = \|x\|_{X,i}\end{aligned}$$

and hence $\|\iota(x)\|_i = \|(x, 0)\|_i \geq \|x\|_{X,i}$. Therefore ι is an isometric embedding. Analogously to above, we have $\|j(y)\|_i = \|(0, y)\|_i \leq \|y\|_{Y,i}$. Setting $0 = u+w$ and $y = v-f(w)$, we obtain

$$\begin{aligned}\|u\|_{X,i} + \|v\|_{Y,i} + \varepsilon\|w\|_{X,i} &= \|w\|_{X,i} + \|v\|_{Y,i} + \varepsilon\|w\|_{X,i} \\ &= (1+\varepsilon)\|w\|_{X,i} + \|v\|_{Y,i} \\ &\geq \|f(w)\|_{Y,i} + \|v\|_{Y,i} \\ &\geq \|f(w) - v\|_{Y,i} = \|y\|_{Y,i}\end{aligned}$$

and hence $\|j(y)\|_i = \|(0, y)\|_i \geq \|y\|_{Y,i}$, i.e. j is also an isometric embedding. □

We need also the next

Lemma 4.4. *Let E be a graded Fréchet space which is of almost universal disposition for finite-dimensional graded Fréchet spaces, $X \subseteq E$ a finite-dimensional subspace and $\varepsilon > 0$. Given a finite-dimensional graded Fréchet space Y and an ε-isometry $f : X \to Y$, for all $\delta > 0$ there exists a δ-isometry $g : Y \to E$ with the property $\|g \circ f - \mathrm{id}_X\|_{X,i} < 2\varepsilon$ for all $i \in \mathbb{N}$.*

Proof. Choose $0 < \delta' < \min\{\delta, 1\}$. By Lemma 4.3 there exists a finite-dimensional Fréchet space Z and isometric embeddings $\iota : X \to Z$ and $j : Y \to Z$ such that $\|j \circ f - \iota\|_i \leq \varepsilon$ for all $i \in \mathbb{N}$. As E is of almost universal disposition for finite-dimensional Fréchet spaces, there exists a δ'-isometric embedding $h : Z \to E$ with the property $h \circ \iota \restriction X = \mathrm{id}_X$. Setting $g = h \circ j$, we get that g is a δ-isometric embedding as it is the composition of a δ-isometric embedding with an isometric embedding. Additionally for $x \in X$ we obtain

$$\|g(f(x)) - x\|_i = \|h(j(f(x))) - h(\iota(x))\|_i \le (1+\delta')\|j(f(x)) - \iota(x)\|_i$$
$$\le \varepsilon(1+\delta')\|x\|_i < 2\varepsilon\|x\|_i$$

for all $i \in \mathbb{N}$. □

Now we can use these results to show that $\mathbb{G}^{\mathbb{N}}$ is, up to isometry, uniquely determined by the property of being of almost universal disposition for finite-dimensional graded Fréchet spaces. Recall that a graded Fréchet space is a Fréchet space E with a fixed sequence of semi-norm $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ satisfying $\|x\|_1 \le \|x\|_2 \le \ldots$ for all $x \in X$.

Proof of Proposition 4.1. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a fixed sequence of positive real numbers satisfying a decay condition which will be specified later in the proof.

We pick $\varepsilon_0 = \varepsilon$ and set $X_0 = X$, $Y_0 = Y$ and $f_0 = f$. By assumption the mapping $f_0 \colon X_0 \to Y_0$ is an ε_0-isometric embedding.

Now assume that X_i, Y_i and f_i have already been constructed for $i \le n$ and that also the mappings g_i have been constructed for $i < n$. Using Lemma 4.4 we obtain an ε_{n+1}-isometric embedding $g_n \colon Y_n \to X_{n+1}$ satisfying

$$\|g_n(f_n(x)) - x\|_i \le 2\varepsilon_n\|x\|_i \tag{3}$$

for all $i \in \mathbb{N}$. Here the space X_{n+1} is defined as an appropriately enlarged $g_n[Y_n]$ such that $Y_{n-1} \subseteq Y_n$ and $\bigcup_{n\in\mathbb{N}} X_n$ is dense in E. Again by using Lemma 4.4 we get an ε_{n+1}-isometric embedding $f_{n+1} \colon X_{n+1} \to Y_{n+1}$ where Y_{n+1} is chosen analogously to X_{n+1}. This mapping satisfies

$$\|f_{n+1}(g_n(y)) - y\|_i \le 2\varepsilon_{n+1}\|y\|_i \tag{4}$$

for all $i \in \mathbb{N}$.

Now for fixed n and $x \in X_n$ we get

$$\|f_{n+1}(g_n(f_n(x))) - f_n(x)\|_i \le 2\varepsilon_{n+1}\|f_n(x)\|_i \le 2\varepsilon_{n+1}(1+\varepsilon_n)\|x\|_i$$

and

$$\|f_{n+1}(g_n(f_n(x))) - f_{n+1}(x)\|_i \le (1+\varepsilon_{n+1})\|g_n(f_n(x)) - x\|_i$$
$$\le 2\varepsilon_n(1+\varepsilon_{n+1})\|x\|_i.$$

Using the triangle inequality, we obtain

$$\|f_{n+1}(x) - f_n(x)\|_i \le (\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1})2\|x\|_i$$

from the inequalities above. Now we assume that

$$\varepsilon_0 + 2\varepsilon_0\varepsilon_1 + \varepsilon_1 + \sum_{n=1}^{\infty}(\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1}) < 2\varepsilon$$

which implies that $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in F.

For $x \in \bigcup_{n\in\mathbb{N}} X_n$ we define $h(x) = \lim_{n\geq m} f_n(x)$ where m is chosen such that $x \in X_m$. Then h is an ε_n-isometry for all $n \in \mathbb{N}$ and hence an isometry which can be uniquely extended to an isometry on E, which will be denoted by h as well. From the inequalities above we deduce

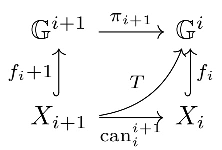
$$\|f(x) - h(x)\|_i \leq 2\sum_{n=0}^{\infty}(\varepsilon_n + 2\varepsilon_n\varepsilon_{n+1} + \varepsilon_{n+1}) < 4\varepsilon.$$

In order to show that h is bijective, we repeat the above procedure to show that $\{g_n(y)\}_{n\geq m}$ is a Cauchy sequence for all $y \in Y_m$. Again we may obtain an isometry $g_\infty\colon F \to E$. From the conditions (3) and (4) we may conclude $g_\infty \circ h = \mathrm{id}_E$ and $h \circ g_\infty = \mathrm{id}_F$. □

We conclude this section by showing that every separable graded Fréchet space can be embedded isometrically into $\mathbb{G}^{\mathbb{N}}$.

Theorem 4.5. *The space $\mathbb{G}^{\mathbb{N}}$ is universal for separable graded Fréchet spaces.*

Proof. Let X be a separable graded Fréchet space and $\{\|\cdot\|_i\}_{i\in\mathbb{N}}$ its fixed sequence of semi-norms. For all $i \in \mathbb{N}$ we denote by X_i the normed space $X/\ker\|\cdot\|_i$ equipped with the norm $\|\cdot\|_i$ and by $\widetilde{X_i}$ its completion. From the universality of $\mathbb{G}$ we may deduce the existence of an isometric embedding $f_1\colon \widetilde{X}_1 \to \mathbb{G}$. Now assume that we have an isometric embedding $f_i\colon \widetilde{X}_i \to \mathbb{G}^i$. Note that since $\|x\|_i \leq \|x\|_{i+1}$ for all $x \in X$, the composition T of f_i with the canonical mapping $\mathrm{can}_i^{i+1}\colon X_{i+1} \to X_i$ is non-expansive. As $\mathbb{G}^{i+1}$, equipped with the semi-norm $\|\cdot\|_{i+1}$, is isometric to $\mathbb{G}$, we can use property (1) of the universal projection π_{i+1} given in Theorem 2.1 to find an isometric embedding $f_{i+1}\colon \widetilde{X}_{i+1} \to \mathbb{G}^{i+1}$ so that the diagram



is commutative. Hence,

$$f\colon X \to \mathbb{G}^{\mathbb{N}}, \quad x \mapsto ((f_i(\mathrm{can}_i(x)))_i)_{i\in\mathbb{N}}$$

is an isometric embedding. □

5. Final remarks

Note that by [14, Proposition V.5.4] the space $\mathcal{C}(\mathbb{R})$ is universal for all separable Fréchet spaces. The following shows that, like in the case of $\mathcal{C}([0,1])$ for Banach spaces, the space $\mathcal{C}(\mathbb{R})$ is not of almost universal disposition for finite-dimensional Fréchet spaces.

Proposition 5.1. *Let X be a metrizable hemicompact space, i.e., there is a sequence of compact subsets of X so that every compact subset of X is contained in one of the sets in this sequence, and $\{K_i\}_{i\in\mathbb{N}}$ be a sequence of compact sets satisfying $K_i \subseteq K_{i+1}$ and $\bigcup_{i\in\mathbb{N}} K_i = X$. The space $\mathcal{C}(X)$ equipped with the sequence of semi-norms $\|\cdot\|_i$ where $\|f\|_i = \sup_{x\in K_i} |f(x)|$ is not of almost universal disposition for finite-dimensional Fréchet spaces.*

Proof. Assume for a contradiction that $\mathcal{C}(X)$ is of almost universal disposition for finite-dimensional Fréchet spaces. By Corollary 4.2 we deduce that $\mathcal{C}(X) \cong \mathbb{G}^{\mathbb{N}}$ holds isometrically. Hence for all $i \in \mathbb{N}$ we obtain $\mathcal{C}(X)/\ker\|\cdot\|_i \cong \mathbb{G}$ isometrically. Observe that $\mathcal{C}(X)/\ker\|\cdot\|_i = \mathcal{C}(K_i)$. This shows that the space $\mathcal{C}(K_i)$ is a Gurariĭ space, a contradiction with [1, Corollary 5.4]. □

Proposition 5.2. *There is no separable Fréchet space which is of universal disposition for finite-dimensional Fréchet spaces.*

Proof. Assume, for a contradiction, that F is a separable Fréchet space of universal disposition for finite-dimensional Fréchet spaces. Hence F is also of almost universal disposition for finite-dimensional Fréchet spaces, and hence by Proposition 4.1 isometrically isomorphic to $\mathbb{G}^{\mathbb{N}}$. Therefore it is sufficient to show that $\mathbb{G}^{\mathbb{N}}$ is not of universal disposition.

Let $X \subseteq Y$ be finite-dimensional Banach spaces. Setting $\|\cdot\|_{X,i} := \|\cdot\|_X$ and $\|\cdot\|_{Y,i} := \|\cdot\|_Y$ for all $i \in \mathbb{N}$, we obtain two finite-dimensional Fréchet spaces. Assume, for a contradiction, that $\mathbb{G}^{\mathbb{N}}$ is of universal disposition for finite-dimensional Fréchet spaces. Given an isometric embedding $f\colon (X, \|\cdot\|_X) \to \mathbb{G}$ the product mapping

$$X \to \mathbb{G}^{\mathbb{N}}, x \mapsto \{f(x)\}_{i\in\mathbb{N}}$$

is an isometric embedding of Fréchet spaces. Hence there would exist an isometric extension $g\colon Y \to \mathbb{G}^{\mathbb{N}}$. Therefore also the mapping $Y \to \mathbb{G}, y \mapsto (g(y))_1$ is an isometry since $\mathbb{G} = \mathbb{G}^{\mathbb{N}}/\ker\|\cdot\|_1$ and it extends f. This would mean that $\mathbb{G}$ is a separable Banach space of universal disposition for finite-dimensional Banach spaces, in contradiction to [6, Proposition 5.1]. □

We conclude the paper with the construction of a sequence of semi-norms on $\mathbb{G}^{\mathbb{N}}$ under which it is of almost universal disposition for Fréchet spaces with a fixed but not necessarily increasing sequence of semi-norms.

For this we can use the semi-norms coming from the coordinates, namely, for each $n \in \mathbb{N}$ define $\|x\|'_n = \|x(n)\|_{\mathbb{G}}$, $x \in \mathbb{G}^{\mathbb{N}}$, where $\|\cdot\|_{\mathbb{G}}$ is the norm of the Gurariĭ space.

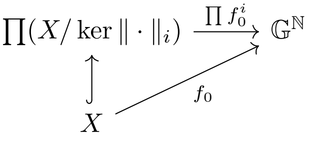
In order to shorten the notation, we denote by $X_i := X/\ker\|\cdot\|_i$ equipped with the norm $\|\cdot\|_i$ and by Y_i the corresponding quotient of Y.

In order to show that $\mathbb{G}^{\mathbb{N}}$ with these semi-norms is of almost universal disposition, we need the following

Lemma 5.3. *Let $f_0 \colon X \to \mathbb{G}^{\mathbb{N}}$ be an (ε-)isometric embedding. For all $i \in \mathbb{N}$ the mapping*

$$f_0^i \colon X_i \to \mathbb{G}, \bar{x} \mapsto (f_0(x))_i$$

is an (ε-)isometric embedding and the diagram



is commutative.

Proof. First we show that

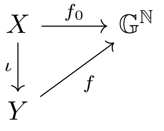
$$f_0^i \colon X_i \to \mathbb{G}, \bar{x} \mapsto (f_0(x))_i$$

is well-defined. Let $x_1, x_2 \in X$ such that $\|x_1 - x_2\|_i = 0$. Then

$$\|(f_0(x_1))_i - (f_0(x_2))_i\| = \|(f_0(x_1 - x_2))_i\| \leq (1+\varepsilon)\|x_1 - x_2\|_i = 0$$

and hence $(f_0(x_1))_i = (f_0(x_2))_i$, i.e. the mapping is well-defined. It is by definition an (ε-)isometric embedding. Finally note that the identity $(f_0(x))_i = f_0^i(\bar{x}^{\|\cdot\|_i})$ for $i \in \mathbb{N}$ follows analogously. $\square$

Proposition 5.4. *Let $(X, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ and $(Y, \{\|\cdot\|_i\}_{i\in\mathbb{N}})$ be finite-dimensional Fréchet spaces with fixed semi-norms, $\varepsilon > 0$, $\iota \colon X \to Y$ and $f_0 \colon X \to \mathbb{G}^{\mathbb{N}}$ isometric embeddings. Then there is an ε-isometric embedding $f \colon Y \to \mathbb{G}^{\mathbb{N}}$ such that the diagram*



is commutative.

Proof. As X and Y are finite-dimensional, the quotient spaces X_i and Y_i are finite-dimensional Banach spaces for all $i \in \mathbb{N}$. By Lemma 3.1 the mappings $\iota_i \colon X_i \to Y_i$ are isometric embeddings. The same is true for the mappings $f_0^i \colon X_i \to \mathbb{G}$ by Lemma 5.3. As the Gurariĭ space is of almost universal disposition for finite-dimensional Banach spaces, there is an ε-isometric embedding $f_i \colon Y_i \to \mathbb{G}$ making the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{f_0^i} & \mathbb{G} \\ {\scriptstyle \iota_i}\downarrow & \nearrow_{f_i} & \\ Y_i & & \end{array}$$

commutative. For $y \in Y$, we set $f(y) = \{f_i(\overline{y}^{\|\cdot\|_i})\}_{i\in\mathbb{N}}$, i.e. f is defined so that

$$\begin{array}{ccc} \prod Y_i & \xrightarrow{\prod f_i} & \mathbb{G}^{\mathbb{N}} \\ \uparrow & \nearrow_{f} & \\ Y & & \end{array}$$

is commutative. Since f_i is an ε-isometric embedding for all $i \in \mathbb{N}$, we get

$$\|f(y)\|_i' = \|(f(y))_i\| = \|f_i(\overline{y}^{\|\cdot\|_i})\| \leq (1+\varepsilon)\|\bar{y}\|_i = (1+\varepsilon)\|y\|_i$$

and by an analogous computation $\|f(y)\|_i' \geq (1+\varepsilon)^{-1}\|y\|_i$, i.e. f is an ε-isometric embedding. Now let $x \in X$, we have

$$(f(\iota(x)))_i = f_i\left(\overline{\iota(x)}^{\|\cdot\|_i}\right) = f_i(\iota_i(\bar{x})) = f_0^i(\bar{x}) = (f_0(x))_i.$$

Hence $f(\iota(x)) = f_0(x)$, i.e. $f \restriction X = f_0$. $\square$

Proposition 5.5. *The space $\mathbb{G}^{\mathbb{N}}$ equipped with the sequence $\{\|\cdot\|_n'\}_{n\in\mathbb{N}}$ of semi-norms is universal for the category of separable Fréchet spaces. In addition it is, up to isometries, the unique separable Fréchet space with a fixed sequence of semi-norms which is of almost universal disposition for finite-dimensional Fréchet spaces.*

We skip the proof of uniqueness and universality as these proofs follow the lines of the corresponding ones for graded Fréchet spaces.

Remark 5.6. Note that in both cases $\mathbb{G}^{\mathbb{N}}/\ker\|\cdot\|_n = \mathbb{G}^n$. Therefore all neighbourhoods of zero contain straight lines. This means in other words that there is no continuous norm on the space $\mathbb{G}^{\mathbb{N}}$ equipped with either of the sequences of semi-norms.

Acknowledgment

The authors wish to thank the anonymous referee for helpful comments and suggestions.

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