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Topological properties of function spaces over ordinals

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Abstract A topological space X is said to be an *Ascoli space* if any compact subset $\mathcal{K}$ of $C_k(Y)$ is evenly continuous. This definition is motivated by the classical Ascoli theorem. We study the $k_{\mathbb{R}}$-property and the Ascoli property of $C_p(\kappa)$ and $C_k(\kappa)$ over ordinals κ. We prove that $C_p(\kappa)$ is always an Ascoli space, while $C_p(\kappa)$ is a $k_{\mathbb{R}}$-space iff the cofinality of κ is countable. In particular, this provides the first C_p-example of an Ascoli space which is not a $k_{\mathbb{R}}$-space, namely $C_p(\omega_1)$. We show that $C_k(\kappa)$ is Ascoli iff $\mathrm{cf}(\kappa)$ is countable iff $C_k(\kappa)$ is metrizable.

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1 Introduction

The study of topological properties of function spaces is quite an active area of research attracting specialists both from topology and functional analysis, see for example [1,3,6,9,13] and references therein. In the following diagram we select the most important compact type properties generalizing metrizability

$$\text{metric} \Longrightarrow \begin{array}{c}\text{Fréchet–}\\ \text{Urysohn}\end{array} \Longrightarrow k\text{-space} \Longrightarrow k_{\mathbb{R}}\text{-space} \Longrightarrow \text{Ascoli},$$

and note that none of these implications is reversible, see [3,4] (all relevant definitions are given in the next section).

For a Tychonoff topological space X, we denote by $C_k(X)$ and $C_p(X)$ the space $C(X)$ of all continuous real-valued functions on X endowed with the compact-open topology and the topology of pointwise convergence, respectively.

It is well-known that $C_p(X)$ is metrizable if and only if X is countable. Pytkeev, Gerlitz and Nagy (see Sect. 3 of [1]) characterized spaces X for which $C_p(X)$ is Fréchet–Urysohn or a k-space (these properties coincide for spaces of the form $C_p(X)$). The authors in [5] obtained some sufficient conditions on X for which the space $C_p(X)$ is an Ascoli space. Recall that X is called an *Ascoli space* if any compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous (or, equivalently, if the natural evaluation map $X \hookrightarrow C_k(C_k(X))$ is an embedding, see [3]).

Every linear order $<$ on a set X generates a natural topology on X whose subbase consists of sets of the form $\{z\colon z < x\}$ and $\{z\colon z > x\}$, where $x \in X$. Spaces X whose topology is generated by some linear order are called *linearly ordered topological spaces*. Ordinals with the topology generated by their natural wellorder form is an interesting class of linearly ordered topological spaces, and function spaces over them give a good source of (counter)examples in the corresponding theory. For instance, the space $C_p(\omega_1)$ is Lindelöf, see [13]. On the other hand, Arhangel'skii showed in [2] that the space $C_p(\omega_1 + 1)$ is not normal. In [8] Gul'ko proved that there are no two distinct natural number n and m for which the powers $C_p(\omega_1)^n$ and $C_p(\omega_1)^m$ are homeomorphic. In [11] Morris and Wulbert observed that $C_k(\omega_1)$ is not barrelled.

In this short note we provide complete characterizations of those ordinals κ for which $C_p(\kappa)$ and $C_k(\kappa)$ are $k_{\mathbb{R}}$-spaces or Ascoli spaces. The following theorems are the main results of the paper.

Theorem 1.1 *For every ordinal κ the space $C_p(\kappa)$ is Ascoli.*

Denote by $\mathrm{cf}(\kappa)$ the cofinality of an ordinal κ.

Theorem 1.2 *For an ordinal κ, the space $C_p(\kappa)$ is a $k_{\mathbb{R}}$-space if and only if* $\mathrm{cf}(\kappa) \le \omega$ *if and only if $C_p(\kappa)$ is Fréchet–Urysohn.*

Theorems 1.1 and 1.2 show that the space $C_p(\omega_1)$ is Ascoli but not a $k_{\mathbb{R}}$-space. This answers Question 6.8 in [6] for spaces $C_p(X)$ in the affirmative, and complements [4, 3.3.E] asserting that for uncountable discrete X the space $C_p(X) = \mathbb{R}^X$ is a $k_{\mathbb{R}}$-space but not a k-space.

Theorem 1.3 *For an ordinal κ, the space $C_k(\kappa)$ is an Ascoli space if and only if* $\mathrm{cf}(\kappa) \le \omega$, *so $C_k(\kappa)$ is complete and metrizable.*

2 Proofs

Below we recall some topological concepts used in Theorems 1.1 and 1.3, for other notions we refer the reader to the book [4]. A *k-cover* $\mathcal{U}$ of a topological space X is a family of subsets of X such that every compact subset of X is contained in some member of $\mathcal{U}$.

Definition 2.1 A topological space X is

- *hemicompact* if there exists a countable k-cover of X consisting of compacts;
- *realcompact* if it can be embedded as a closed subset to $\mathbb{R}^\lambda$ for some cardinal λ;
- a *$k_{\mathbb{R}}$-space* if a real-valued function f on X is continuous if and only if its restriction $f|_K$ to any compact subset K of X is continuous;
- *scattered* if every non-empty subspace A of X has an isolated point in A.

Recall that an ordinal κ is *limit* if there is no α such that $\kappa = \alpha + 1$, otherwise κ is called a *successor* ordinal. The *cofinality* $\mathrm{cf}(\kappa)$ of a limit ordinal number κ is the smallest ordinal α which is the order type of a cofinal subset of κ. If κ is a successor ordinal we set $\mathrm{cf}(\kappa) = 1$.

The following simple facts should be well-known (for (i) see [7, Sect. 5.11], the other ones are straightforward).

Lemma 2.2 (i) *κ is compact if and only if it is a successor;*
(ii) *κ is hemicompact non-countably compact if and only if* $\mathrm{cf}(\kappa) = \omega$;
(iii) *κ is countably compact non-compact if and only if* $\mathrm{cf}(\kappa) > \omega$.

For the convenience of the reader we recall also the following two results.

Proposition 2.3 [6] *Assume X admits a family $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of X, a subset $A = \{a_i : i \in I\} \subset X$ and a point $z \in X$ such that:* (i) *$a_i \in U_i$ for every $i \in I$,* (ii) $\big|\{i \in I : C \cap U_i \neq \emptyset\}\big| < \infty$ *for each compact subset C of X, and* (iii) *z is a cluster point of A. Then X is not an Ascoli space.*

A family $\{A_i\}_{i\in I}$ of subsets of a set X is said to be *point-finite* if the set $\{i \in I : x \in A_i\}$ is finite for every $x \in X$. A family $\{A_i\}_{i\in I}$ of subsets of a topological space X is called *strongly point-finite* if for every $i \in I$, there exists an open set U_i of X such that $A_i \subseteq U_i$ and $\{U_i\}_{i\in I}$ is point-finite. Following Sakai [12], a topological space X is said to have the *property* (κ) if every pairwise disjoint sequence of finite subsets of X has a strongly point-finite subsequence. The following result is proved in [5].

Theorem 2.4 *If $C_p(X)$ is Ascoli, then X has the property* (κ).

It is well-known that ordinals are locally compact and scattered (for the last property we note that the smallest element of a subset A of X is isolated in A). The following proposition is of independent interest, it generalizes Corollary 1.5 of [5] and immediately implies Theorem 1.1.

Proposition 2.5 *Let X be a locally compact space. Then $C_p(X)$ is Asoli if and only if X is scattered.*

Proof The "only if" part follows from Theorem 2.4 combined with the fact that the property (κ) is preserved by subspaces, along with the fact that every compact space with the property (κ) is scattered, see [12, Theorem 3.2]. For the "if" direction consider the one-point compactification $X^* = X \cup \{x_\infty\}$ of X and note that it is scattered. Therefore $C_p(X^*)$ is Frechet-Urysohn by [1, II.7.16], and hence so is its subspace Z consisting of those continuous $f : X^* \to \mathbb{R}$ such that $f(x_\infty) = 0$. Now it is easy to see that $Z \restriction X = \{f \restriction X : f \in Z\} \subset C_p(X)$ is homeomorphic to Z and is dense in $C_p(X)$. Therefore $C_p(X)$ has a dense Frechet–Urysohn subspace, and hence every function f belongs to a dense Ascoli subspace $f + Z$ of $C_p(X)$. Thus $C_p(X)$ is Ascoli by Proposition 5.10 of [3]. □

Below we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2 Let $C_p(\kappa)$ be a $k_{\mathbb{R}}$-space and suppose towards a contradiction that $\mathrm{cf}(\kappa) > \omega$. Then κ is countably compact by Lemma 2.2. Hence the space $C(\kappa)$ endowed with the sup-norm is a Banach space. Therefore the space $C_p(\kappa)$ admits a stronger normed topology and is angelic by [9, Proposition 9.6]. Since every compact subset of $C_p(\kappa)$ is Fréchet–Urysohn, every sequentially continuous function on $C_p(\kappa)$ is continuous. In particular, every sequentially continuous functional on $C_p(\kappa)$ is continuous. So κ is a realcompact space by Theorem 1.1 of [14]. Being realcompact and pseudocompact the space κ is compact by [4, 3.11.1]. Hence κ is a successor ordinal, a contradiction. Thus $\mathrm{cf}(\kappa) \le \omega$.

Conversely, let $\mathrm{cf}(\kappa) \le \omega$. Then κ is hemicompact by Lemma 2.2. Thus $C_p(\kappa)$ is Fréchet–Urysohn by [1, II.7.16]. □

Since each ordinal α is the set of all smaller ordinals, in the following proof we adopt the following perhaps standard notation: For a function f whose domain is an ordinal κ and $\alpha \in \kappa$ we denote by $f(\alpha)$ the value of f at α, and by $f[\alpha]$ the set $\{f(\beta) : \beta < \alpha\}$.

Proof of Theorem 1.3 Suppose for a contradiction that $\mathrm{cf}(\kappa) > \omega$. We shall use Proposition 2.3 and show that $C_k(\kappa)$ is not Ascoli. For every $\alpha < \kappa$ we define $f_\alpha : \kappa \to [0, 1]$ by $f_\alpha\big[\alpha + 1\big] = \{0\}$ and $f_\alpha\big[\kappa\backslash(\alpha + 1)\big] = \{1\}$, and set

$$U_\alpha := \{f \in C_k(\kappa) \colon\ f(\alpha) < 1/4,\ f(\alpha + 1) > 3/4\}.$$

To prove that $C_k(\kappa)$ is not Ascoli it is enough to verify the assumptions of Proposition 2.3 for $\{f_\alpha\}_{\alpha<\kappa}$, $\{U_\alpha\}_{\alpha<\kappa}$ and $0 \in C_k(\kappa)$. Clearly, (i) and (iii) hold true. Let us check (ii). Take any compact $C \subseteq C_k(\kappa)$ and assume, contrary to our claim, that there are infinitely many $\alpha < \kappa$ such that $C \cap U_\alpha \ne \emptyset$. Then there exists a strictly increasing sequence $\{\alpha_n\}_{n<\omega}$ such that $C \cap U_{\alpha_n} \ne \emptyset$. Let $\alpha = \lim \alpha_n$. As $\mathrm{cf}(\kappa) > \omega$ we have $\alpha < \kappa$. By the Ascoli theorem used for $\alpha + 1 \subset \kappa$ and $1/2$ we can find a basic neighborhood O_α of α such that $|h(x) - h(y)| < 1/4$ for all $x, y \in O_\alpha$ and $h \in C$. Take n such that $\alpha_n \in O_\alpha$ and fix $h \in C \cap U_{\alpha_n}$. Then

$$\begin{aligned}\frac{1}{4} &> |h(\alpha_n + 1) - h(\alpha_n)|\\ &\ge |f_{\alpha_n}(\alpha_n + 1) - f_{\alpha_n}(\alpha_n)| - |f_{\alpha_n}(\alpha_n + 1) - h(\alpha_n + 1)| - |h(\alpha_n) - f_{\alpha_n}(\alpha_n)|\\ &> 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2},\end{aligned}$$

which is a contradiction. Thus $\mathrm{cf}(\kappa) \le \omega$.

Conversely, if $\mathrm{cf}(\kappa) \le \omega$, then κ is a hemicompact locally compact space by Lemma 2.2. Hence $C_k(\kappa)$ is complete metrizable by Corollary 5.2.2 of [10]. □

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