

## ON NON-ARCHIMEDEAN GURARIĬ SPACES

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ABSTRACT. A Banach space  $E$  is *non-archimedean* if its norm satisfies the *strong triangle inequality*, so  $E$  is then a complete ultrametric space. Let  $\mathcal{U}_{FNA}$  be the class of all non-archimedean finite-dimensional Banach spaces. A non-archimedean GurariĬ Banach space  $\mathbb{G}$  over a non-archimedean valued field  $\mathbb{K}$  is constructed, i.e. a non-archimedean Banach space  $\mathbb{G}$  of countable type which is of *almost universal disposition* for the class  $\mathcal{U}_{FNA}$ . This means: for every isometry  $g : X \rightarrow Y$ , where  $Y \in \mathcal{U}_{FNA}$  and  $X$  is a subspace of  $\mathbb{G}$ , and every  $\varepsilon \in (0, 1)$  there exists an  $\varepsilon$ -isometry  $f : Y \rightarrow \mathbb{G}$  such that  $f(g(x)) = x$  for all  $x \in X$ . We show that all non-archimedean Banach spaces of almost universal disposition for the class  $\mathcal{U}_{FNA}$  are  $\varepsilon$ -isometric. Furthermore, all non-archimedean Banach spaces of almost universal disposition for the class  $\mathcal{U}_{FNA}$  are isometrically isomorphic if and only if  $\mathbb{K}$  is spherically complete and  $\{|\lambda| : \lambda \in \mathbb{K} \setminus \{0\}\} = (0, \infty)$ .

## 1. INTRODUCTION

In 1966 GurariĬ constructed a separable (real) Banach space  $\mathbb{G}$  of *almost universal disposition* for finite-dimensional spaces (called later the *GurariĬ space*), see [3], which means the following condition:

- (G) *For every isometry  $g : X \rightarrow Y$ , where  $Y$  is a finite-dimensional Banach space and  $X$  is a subspace of  $\mathbb{G}$ , and every  $\varepsilon \in (0, 1)$  there exists an  $\varepsilon$ -isometry  $f : Y \rightarrow \mathbb{G}$  such that  $f(g(x)) = x$  for all  $x \in X$ .*

A linear operator  $f : E \rightarrow F$  between Banach spaces  $E$  and  $F$  is an  $\varepsilon$ -isometry if for  $x \in E$  with  $\|x\| = 1$  one has  $(1 + \varepsilon)^{-1} < \|f(x)\| < 1 + \varepsilon$ . By an *isometry* we mean a linear operator  $f : E \rightarrow F$  that is an  $\varepsilon$ -isometry for every  $\varepsilon > 0$ , that is,  $\|f(x)\| = \|x\|$  for each  $x \in E$ .

One can prove easily that the GurariĬ space  $\mathbb{G}$  is unique up to isomorphism of norm arbitrarily close to one. Nevertheless, the question whether the GurariĬ space is unique up to isometry remained open for a longer time. It was answered affirmatively by Lusky in 1976, see [10], who used quite technical and difficult methods involving techniques developed by Lazar and Lindenstrauss [9]. Much simpler proof has been provided by Kubiś and Solecki in 2013, see [5].

In [5] the authors proved the following

**Theorem 1.1.** *Let  $E, F$  be separable GurariĬ spaces and  $\varepsilon > 0$ . Assume  $X \subset E$  is a finite-dimensional space and  $f : X \rightarrow F$  is an  $\varepsilon$ -isometry. Then there exists a bijective isometry  $h : E \rightarrow F$  such that  $\|h|_X - f\| < \varepsilon$ .*

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Applying this result for  $X$  being the trivial space one gets the result of Lusky [10] stating that the Gurarii space is unique up to isometry.

A Banach space  $E$  is said to be of *universal disposition* for the class  $\mathcal{U}$  of finite-dimensional spaces if it satisfies the following condition:

- (G1) *For every isometry  $j : X \rightarrow Y$ , where  $Y \in \mathcal{U}$  and  $X \subset E$ , there is an isometry  $f : Y \rightarrow E$  such that  $f(j(x)) = x$  for all  $x \in X$ .*

We refer the reader to [1] and [2], where recent developments in the study of Gurarii spaces, spaces of universal disposition, and related topics are surveyed.

In the present paper we study non-archimedean counterparts of the above concepts. The property of being of (almost) universal disposition for finite-dimensional non-archimedean normed spaces is defined precisely in the same way as for the real case mentioned above.

From now on, by  $\mathbb{K}$  we will denote a non-archimedean complete non-trivially valued field, i.e. the valuation satisfies *the strong triangle inequality*:

$$|\lambda + \mu| \leq \max\{|\lambda|, |\mu|\}$$

for all  $\lambda, \mu \in \mathbb{K}$ .

All linear spaces considered in this paper are over  $\mathbb{K}$ . Recall that

$$|\mathbb{K}^*| = \{|\lambda| : \lambda \in \mathbb{K} \setminus \{0\}\}$$

is the *value group* of  $\mathbb{K}$ .

$\mathbb{K}$  is said to be *discretely valued* if 0 is the only accumulation point of  $|\mathbb{K}^*|$ ; then, there exists a *uniformizing element*  $\rho \in \mathbb{K}$  with  $0 < |\rho| < 1$  such that  $|\mathbb{K}^*| = \{|\rho|^n : n \in \mathbb{Z}\}$ . Otherwise, we say that  $\mathbb{K}$  is *densely valued* (then,  $|\mathbb{K}^*|$  is a dense subset of  $[0, \infty)$ ).

By a *non-archimedean Banach space* we mean a Banach space  $E$  equipped with a non-archimedean norm  $\|\cdot\|$ , i.e. a norm for which the triangle inequality is replaced by a stronger condition  $\|x + y\| \leq \max\{\|x\|, \|y\|\}$  for all  $x, y \in E$ .

An infinite-dimensional normed space  $E$  over  $\mathbb{K}$  is of *countable type* if it contains a countable set whose linear hull is dense in  $E$ . If  $\mathbb{K}$  is separable, then a normed space is of countable type if and only if it is separable.

We say that  $E$  (in particular,  $E$  may be equal to  $\mathbb{K}$ ) is *spherically complete* if every shrinking sequence of balls in  $E$  has a non-empty intersection; otherwise,  $E$  is *non-spherically complete*. Every finite-dimensional Banach space over  $\mathbb{K}$  has an equivalent non-archimedean norm. We refer the reader to the monographs [11] and [12] for non-archimedean concepts mentioned above.

We say that a spherically complete Banach space  $\widehat{E}$  is the *spherical completion* of a non-archimedean Banach space  $E$ , if there exists an isometric embedding  $i : E \rightarrow \widehat{E}$  and  $\widehat{E}$  has no proper spherically complete linear subspace containing  $i(E)$ . Applying the natural identification, we will usually identify  $E$  with  $i(E)$ . Every Banach space (in particular  $\mathbb{K}$ ) has the spherical completion and any two spherical completions of  $E$  are isometrically isomorphic ([12, Theorem 4.43]).

Let  $\mathcal{U}_{FNA}$  be the class of all non-archimedean finite-dimensional normed spaces. As it can be expected, properties of spaces  $E$  of (almost) universal disposition for the class  $\mathcal{U}_{FNA}$  strictly depend on the valued field  $\mathbb{K}$ , in particular, on whether it is spherically complete or not. In Section 3 we show that all non-archimedean Banach spaces of almost universal disposition for the class  $\mathcal{U}_{FNA}$  are  $\varepsilon$ -isometric with arbitrarily small  $\varepsilon > 0$  (Corollary 3.3). Moreover, all non-archimedean Banach spaces of almost universal disposition for

the class  $\mathcal{U}_{FNA}$  are isometrically isomorphic if and only if  $\mathbb{K}$  is spherically complete and  $|\mathbb{K}^*| = (0, \infty)$  (Proposition 3.4). The main result of this section is the following

**Theorem 1.2.** *Let  $\mathbb{K}$  be a non-archimedean valued field. The following conditions are equivalent:*

- (a) *Every non-archimedean Banach space of countable type over  $\mathbb{K}$  is of almost universal disposition for the class  $\mathcal{U}_{FNA}$ .*
- (b)  *$\mathbb{K}$  is densely valued.*

Section 4 focuses on the study Banach spaces of universal disposition for the class  $\mathcal{U}_{FNA}$ . It turns out that a real Banach space  $G$  of almost universal disposition for the class  $\mathcal{U}$  can be characterized (see [4]) by the following condition:

- (H) for every  $\varepsilon > 0$ , for every finite-dimensional normed spaces  $X \subset Y$  and for every isometric embedding  $j : X \rightarrow G$ , there is an isometric embedding  $f : Y \rightarrow G$  such that  $\|j - f|_X\| < \varepsilon$ .

In contrast to the real case, in a non-archimedean setting the condition (H) characterizes Banach spaces of universal disposition for the class  $\mathcal{U}_{FNA}$ . We show the following

**Theorem 1.3.** *Let  $G$  be a non-archimedean Banach space which satisfies the following property: For every finite-dimensional non-archimedean normed space  $Y$  and every isometric embedding  $j : X \rightarrow G$ , where  $X \subset Y$  is a linear subspace, there is an isometry  $f : Y \rightarrow G$  such that  $\|j - f|_X\| < 1$ . Then  $G$  is of universal disposition for the class  $\mathcal{U}_{FNA}$ .*

Recall that a Banach space  $X$  is (*isometrically*) *universal* for the class of Banach spaces  $\mathcal{C}$  if  $X \in \mathcal{C}$  and for any  $Y \in \mathcal{C}$ , there is an isometrical embedding  $Y \rightarrow X$ . Note that, as a result of Banach-Mazur theorem, the space  $C[0, 1]$  is isometrically universal for the class of separable real Banach spaces.

If  $\mathbb{K}$  is spherically complete, we can properly select a set  $I$  and a map  $s : I \rightarrow (0, \infty)$  such that every non-archimedean Banach space of countable type can be isometrically embedded into  $E_u = c_0(I : s)$ . However  $E_u$  is isometrically universal for the class of non-archimedean Banach spaces of countable type if and only if  $\mathbb{K}$  is spherically complete and  $(0, \infty)$  is an union of at most countably many cosets of  $|\mathbb{K}^*|$  (Proposition 4.4). On the other hand,  $E_u$  is never separable. If  $\mathbb{K}$  is non-spherically complete, the role of  $c_0(I : s)$  is replaced by  $\ell^\infty$ , which clearly is not of countable type (Remark 4.5).

Applying Theorem 1.3 we prove that the spherical completion  $\widehat{E}_u$  of  $E_u$  is a space of universal disposition for the class  $\mathcal{U}_{FNA}$ , see Theorem 4.6. We show also that the suitably selected proper linear subspace of  $\widehat{E}_u$ , denoted as  $E_h$ , is also of universal disposition for the class  $\mathcal{U}_{FNA}$  (Theorem 4.7). If  $\mathbb{K}$  is spherically complete, then  $E_h = E_u$ ; hence,  $E_h$  has an orthogonal base and is of countable type if and only if  $\mathbb{K}$  is spherically complete and  $(0, \infty)$  is the union of at most countably many cosets of  $|\mathbb{K}^*|$  (Corollary 4.8).

## 2. PRELIMINARIES

Let  $t \in (0, 1]$ . A subset  $\{x_i : i \in I\} \subset E$  is called *t-orthogonal* (*orthogonal* for  $t = 1$ ) if for each finite subset  $J \subset I$  and all  $\{\lambda_i\}_{i \in J} \subset \mathbb{K}$  we have

$$\left\| \sum_{i \in J} \lambda_i x_i \right\| \geq t \cdot \max_{i \in J} \|\lambda_i x_i\|.$$

If additionally  $\overline{\{x_i\}_{i \in I}} = E$ , then  $\{x_i\}_{i \in I}$  is said to be a  $t$ -orthogonal base of  $E$ . Then every  $x \in E$  has an unequivocal expansion

$$x = \sum_{i \in I} \lambda_i x_i \quad (\lambda_i \in \mathbb{K}, i \in I).$$

Every non-archimedean Banach space of countable type has a  $t$ -orthogonal base for each  $t \in (0, 1)$ ; if  $\mathbb{K}$  is spherically complete, then every non-archimedean Banach space of countable type has an orthogonal base ([12, Lemma 5.5]). Every closed linear subspace of a non-archimedean Banach space with an orthogonal base has an orthogonal base ([12, Theorem 5.9]).

Linear subspaces  $D, D_0$  of a non-archimedean Banach space  $E$  are called *orthogonal* if  $\|x + y\| = \max\{\|x\|, \|y\|\}$  for all  $x \in D$  and  $y \in D_0$ ; then we will write  $D \perp D_0$ .

Let  $D$  be a closed linear subspace of  $E$ . Then  $D$  is *orthocomplemented* in  $E$  if there is a linear subspace  $D_0$  of  $E$  such that  $D + D_0 = E$  and  $D \perp D_0$ . Consequently, there exists a surjective projection (called an *orthoprojection*)  $P : E \rightarrow D$  with  $\|P\| \leq 1$ . Observe that  $D_1 \perp D_2$  implies  $D_1 \cap D_2 = \emptyset$ ; hence the sum  $D_1 + D_2$  is direct.

Let  $D$  and  $E_0$  be linear subspaces of a normed space  $E$ . Recall that  $E_0$  is called an *immediate extension* of  $D$  if  $D \subset E_0$  and there is no nonzero element of  $E_0$  that is orthogonal to  $D$ ; in other words, for every  $x \in E_0 \setminus D$  we have  $\text{dist}(x, D) < \|x - d\|$  for all  $d \in D$ . A spherical completion  $\widehat{E}$  of  $E$  is a maximal immediate extension of  $E$ . Let  $I$  be a non-empty set and let  $s : I \rightarrow (0, \infty)$  be a map. By

$$\ell^\infty(I : s) := \{(\lambda^i)_{i \in I} \in \mathbb{K}^I : \sup_{i \in I} |\lambda_i| \cdot s(i) < \infty\}$$

we denote the linear space over  $\mathbb{K}$  equipped with the norm

$$\|(\lambda_i)_{i \in I}\| := \sup_{i \in I} |\lambda_i| \cdot s(i).$$

Then  $\ell^\infty(I : s)$  is a non-archimedean Banach space.

Let  $c_0(I : s)$  be a closed linear subspace of  $\ell^\infty(I : s)$  which consists of all  $(\lambda^i)_{i \in I} \in \ell^\infty(I : s)$  such that for every  $\varepsilon > 0$  there exists a finite  $J \subset I$  for which  $|\lambda^i| \cdot s(i) < \varepsilon$  for every  $i \in I \setminus J$ . If  $s(i) = 1$  for all  $i \in I$  we will write  $\ell^\infty(I)$  and  $c_0(I)$ , respectively. In particular  $\ell^\infty := \ell^\infty(\mathbb{N})$  and  $c_0 := c_0(\mathbb{N})$ .

Every non-archimedean Banach space which has an orthogonal base is isomorphic with  $c_0(I)$  for some set  $I$  (see [12, Ch. 5]).

### 3. NON-ARCHIMEDEAN BANACH SPACE OF ALMOST UNIVERSAL DISPOSITION FOR FINITE-DIMENSIONAL SPACES

First we prove the following technical fact.

**Lemma 3.1.** *Let  $E$  be a non-archimedean Banach space of countable type, let  $F$  be a finite-dimensional linear subspace of  $E$ ,  $t \in (0, 1)$  and  $\{x_1, \dots, x_m\}$  be a  $\sqrt{t}$ -orthogonal base of  $F$ . Then there exist  $x_{m+1}, x_{m+2}, \dots \in E \setminus F$  such that  $(x_n)$  is a  $t$ -orthogonal base of  $E$ .*

*Proof.* By [11, Theorem 2.3.13] there exists a linear subspace  $F_0 \subset E$  such that  $E = F \oplus F_0$  and

$$\|u_1 + u_2\| \geq \sqrt{t} \cdot \max\{\|u_1\|, \|u_2\|\}$$

for all  $u_1 \in F, u_2 \in F_0$ . Applying [11, Theorem 2.3.7] we select a  $\sqrt{t}$ -orthogonal base  $(z_n)$  of  $F_0$ . Denote  $x_{m+n} := z_n$  for every  $n \in \mathbb{N}$ . Then taking any  $k \in \mathbb{N}$  and  $\lambda_1, \dots, \lambda_k \in \mathbb{K}$  one gets

$$\begin{aligned} \left\| \sum_{i=1}^k \lambda_i x_i \right\| &\geq \sqrt{t} \cdot \max \left\{ \left\| \sum_{i=1}^m \lambda_i x_i \right\|, \left\| \sum_{i=m+1}^k \lambda_i x_i \right\| \right\} \\ &\geq \sqrt{t} \cdot \max \left\{ \sqrt{t} \cdot \max_{i=1, \dots, m} \|\lambda_i x_i\|, \sqrt{t} \cdot \max_{i=m+1, \dots, k} \|\lambda_i x_i\| \right\} \geq t \cdot \max_{i=1, \dots, k} \|\lambda_i x_i\|. \end{aligned}$$

Hence,  $(x_n)$  is a required  $t$ -orthogonal base of  $E$ .  $\square$

**Theorem 3.2.** *A non-archimedean Banach space of countable type  $E$  is of almost universal disposition for the class  $\mathcal{U}_{FNA}$  if and only if  $\|E^*\|$  is a dense subset of  $(0, \infty)$ .*

*Proof.* Let  $E$  be a non-archimedean Banach space of countable type. Assume that  $\|E^*\|$  is a dense subset of  $(0, \infty)$ . Let  $X$  be a finite-dimensional subspace of  $E$ ,  $Y$  be a non-archimedean finite-dimensional normed space and  $i : X \rightarrow Y$  be an isometrical embedding. Assume that  $\dim Y = m$  and  $\dim X = m_0$ ; clearly,  $m_0 \leq m$ . Fix  $\varepsilon > 0$  and take  $t \in (\frac{1}{\sqrt[3]{1+\varepsilon}}, 1)$ . Applying Lemma 3.1 and [11, Theorem 2.3.7] we form a  $t$ -orthogonal base  $(x_n)$  of  $E$  such that  $\{x_1, \dots, x_{m_0}\}$  is a  $\sqrt{t}$ -orthogonal base of  $X$ . Now, applying Lemma 3.1 again, we select  $y_{m_0+1}, \dots, y_m \in Y$  such that  $\{y_1, \dots, y_m\}$  is a  $t$ -orthogonal base of  $Y$ , where  $y_k = i(x_k)$  for  $k \in \{1, \dots, m_0\}$ . Since, by assumption  $\|E^*\|$  is a dense subset of  $(0, \infty)$ , we can assume that  $1 \geq \|y_k\| \geq t$  ( $k = 1, \dots, m$ ) and  $1 \geq \|x_n\| \geq t$  ( $n \in \mathbb{N}$ ); thus,  $\|y_k\| \geq t \cdot \|x_k\|$  for each  $k \in \{1, \dots, m\}$ .

Define  $f : Y \rightarrow E$  by setting

$$f \left( \sum_{k=1}^m \lambda_k y_k \right) = \sum_{k=1}^m \lambda_k x_k.$$

Clearly,  $f(i(x)) = x$  for all  $x \in X$ . Let  $y = \sum_{k=1}^m \lambda_k y_k \in Y$ . Then,

$$\|y\| = \left\| \sum_{k=1}^m \lambda_k y_k \right\| \geq t \cdot \max_{k=1, \dots, m} \{\|\lambda_k y_k\|\} \geq t^2 \cdot \max_{k=1, \dots, m} \{\|\lambda_k x_k\|\} \geq t^3 \cdot \left\| \sum_{k=1}^m \lambda_k x_k \right\| = t^3 \cdot \|f(y)\|.$$

On the other hand, we have

$$\|y\| = \left\| \sum_{k=1}^m \lambda_k y_k \right\| \leq \max_{k=1, \dots, m} \{\|\lambda_k y_k\|\} \leq \frac{1}{t} \max_{k=1, \dots, m} \{\|\lambda_k x_k\|\} \leq \frac{1}{t^2} \left\| \sum_{k=1}^m \lambda_k x_k \right\| = \frac{1}{t^2} \cdot \|f(y)\|.$$

Thus  $(1 - \varepsilon) \|y\| \leq \|f(y)\| \leq (1 + \varepsilon) \|y\|$ .

Now assume that  $\|E^*\|$  is not dense in  $(0, \infty)$ . Then there exist  $s_1 \in (0, \infty)$  and  $\varepsilon > 0$  such that

$$\|E^*\| \cap (s_1 - 2\varepsilon \cdot s_1, s_1 + 2\varepsilon \cdot s_1) = \emptyset.$$

Define  $X = \mathbb{K}$  and  $Y = (\mathbb{K}^2, \|\cdot\|_Y)$ , where  $\|(\lambda_1, \lambda_2)\|_Y := \max\{|\lambda_1|, s_1 \cdot |\lambda_2|\}$ ,  $\lambda_1, \lambda_2 \in \mathbb{K}$ .

Assume for a contradiction that there is an  $\varepsilon$ -isometry  $f : Y \rightarrow E$ . But then, taking  $x_0 = (0, 1) \in Y$ , we obtain  $\|x_0\|_Y = s_1$  and  $\|f(x_0)\| \geq s_1 + 2\varepsilon \cdot s_1$  or  $\|f(x_0)\| \leq s_1 - 2\varepsilon \cdot s_1$ . Hence,  $\|f(x_0)\| > (1 + \varepsilon) \|x_0\|$ , or  $\|f(x_0)\| < (1 - \varepsilon) \|x_0\|$ , a contradiction.  $\square$

Next conclusion follows directly from Theorem 3.2.

**Corollary 3.3.** *If  $\mathbb{K}$  is densely valued, every non-archimedean Banach space of countable type is of almost universal disposition for the class  $\mathcal{U}_{FNA}$ . All non-archimedean Banach spaces of almost universal disposition for the class  $\mathcal{U}_{FNA}$  are  $\varepsilon$ -isometric, where  $\varepsilon > 0$  is arbitrarily small.*

However, as the next result shows, they need not be always isometric.

**Proposition 3.4.** *All non-archimedean Banach spaces over  $\mathbb{K}$  of almost universal disposition for the class  $\mathcal{U}_{FNA}$  are isometrically isomorphic if and only if  $\mathbb{K}$  is spherically complete and  $|\mathbb{K}^*| = (0, \infty)$ .*

*Proof.* If  $\mathbb{K}$  is spherically complete and  $|\mathbb{K}^*| = (0, \infty)$  then every non-archimedean Banach space of countable type has an orthonormal base, thus, it is isometrically isomorphic with  $c_0$ . Hence, the conclusion follows.

Now assume that  $\mathbb{K}$  is non-spherically complete. Then, since  $\mathbb{K}$  is densely valued,  $c_0$  and  $\mathbb{K}_v^2 \oplus c_0$  are both of universal disposition for the class  $\mathcal{U}_{FNA}$  by Theorem 3.2 (recall that  $\mathbb{K}_v^2$  is a two-dimensional normed space without two orthogonal elements, see [11, Example 2.3.26]). Clearly,  $\mathbb{K}_v^2 \oplus c_0$  and  $c_0$  are not isometrically isomorphic.

Suppose that  $|\mathbb{K}^*| \neq (0, \infty)$ . Then we can find  $s \in (0, \infty) \setminus |\mathbb{K}^*|$ . Define the norm  $\|x\|_s : c_0 \rightarrow [0, \infty)$  by

$$\|x\|_s := \max\{s \cdot |x_1|, \max_{n>1}\{|x_n|\}\},$$

and  $x = (x_n) \in c_0$ . Then, by Theorem 3.2,  $E = (c_0, \|\cdot\|_s)$  and  $F = (c_0, \|\cdot\|_\infty)$  are of almost universal disposition for the class  $\mathcal{U}_{FNA}$ . Since  $\|E\| \neq \|F\|$ ,  $E$  and  $F$  are not isometrically isomorphic.  $\square$

Now, we are ready to prove Theorem 1.2, which characterizes  $\mathbb{K}$ , formulated in Introduction.

*Proof of Theorem 1.2.* Let  $E$  be a non-archimedean Banach space of countable type. If  $\mathbb{K}$  is densely valued,  $\|E^*\|$  is a dense subset of  $(0, \infty)$  and the conclusion follows from Theorem 3.2. Assume now that  $\mathbb{K}$  is discretely valued and  $\rho$  be a uniformizing element of  $\mathbb{K}$ . Let  $E := c_0$ . Set  $s := \frac{|\rho|+1}{2}$  and take

$$\varepsilon < \frac{1-s}{s} = \frac{s-|\rho|}{s}.$$

Let  $X = [e_1] \subset E$  and  $Y = (\mathbb{K}^2, \|\cdot\|_s)$ , where

$$\|x\|_s := \max\{|x_1|, s \cdot |x_2|\}, (x_1, x_2) \in \mathbb{K}^2.$$

Define an isometry  $i : X \rightarrow Y$  by  $i(\lambda e_1) := (\lambda x_1, 0)$  and assume that there exists an  $\varepsilon$ -isometry  $f : Y \rightarrow E$ . Then, for  $x = (0, 1) \in Y$  we get

$$(1 - \varepsilon) \cdot s \leq \|f(x)\| \leq (1 + \varepsilon) \cdot s.$$

Recall that  $\|E^*\| = \{|\rho|^n : n \in \mathbb{Z}\}$ , hence  $(|\rho|, 1) \cap \|E^*\| = \emptyset$ . But  $(1 - \varepsilon) \cdot s > |\rho|$  and  $(1 + \varepsilon) \cdot s < 1$ , a contradiction.  $\square$

#### 4. NON-ARCHIMEDEAN BANACH SPACE OF UNIVERSAL DISPOSITION FOR FINITE-DIMENSIONAL SPACES

We start with the proof of Theorem 1.3, as promised in Introduction.

*Proof of Theorem 1.3.* We need to prove that there is an isometry  $T : Y \rightarrow G$  which extends  $j$ . Let  $t = \|j - f|_X\|$ ,  $n_X = \dim X$  and  $n_Y = \dim Y$ . Clearly  $n_X \leq n_Y$ . Choose  $\{x_1, \dots, x_{n_Y}\}$ , a  $\sqrt{t}$ -orthogonal base of  $Y$  such that  $[x_1, \dots, x_{n_X}] = X$ . Define  $T : Y \rightarrow G$  by setting

$$T(x_n) := \begin{cases} j(x_n) & \text{if } n \leq n_X \\ f(x_n) & \text{if } n > n_X \end{cases}, n = 1, \dots, n_Y.$$

Clearly  $T$  extends  $j$ . We show that  $T$  is an isometry. Take  $x \in Y$ , written as  $x = \sum_{i=1}^{n_Y} \lambda_i x_i$  ( $\lambda_i \in \mathbb{K}$ ). Then, we obtain

$$\begin{aligned} \|T(x)\| &= \left\| \sum_{i=1}^{n_X} \lambda_i T(x_i) + \sum_{i=n_X+1}^{n_Y} \lambda_i T(x_i) \right\| \\ &= \left\| \sum_{i=1}^{n_X} \lambda_i j(x_i) - \sum_{i=1}^{n_X} \lambda_i f(x_i) + \sum_{i=1}^{n_X} \lambda_i f(x_i) + \sum_{i=n_X+1}^{n_Y} \lambda_i f(x_i) \right\| \\ &= \left\| (j - f)\left(\sum_{i=1}^{n_X} \lambda_i x_i\right) + f\left(\sum_{i=1}^{n_Y} \lambda_i x_i\right) \right\| = \|(j - f)(x_0) + f(x)\|, \end{aligned}$$

where  $x_0 = \sum_{i=1}^{n_X} \lambda_i x_i \in X$ . But,

$$\|x\| \geq \sqrt{t} \cdot \max \left\{ \|x_0\|, \sum_{i=n_X+1}^{n_Y} \lambda_i x_i \right\} \geq \sqrt{t} \cdot \|x_0\|$$

and

$$\|(j - f)(x_0)\| \leq \|j - f|_X\| \cdot \|x_0\| = t \cdot \|x_0\| \leq \sqrt{t} \cdot \|x\| < \|x\| = \|f(x)\|$$

hence, we get

$$\|T(x)\| = \|(j - f)(x_0) + f(x)\| = \|f(x)\| = \|x\|.$$

□

To prove the main result of this section we need a few technical lemmas (see also [12, Exercise 5.B and Lemma 4.42] and [11, Theorem 2.3.16]).

**Lemma 4.1.** *Let  $0 < t \leq 1$  and let  $\{x_1, \dots, x_n\}$  be a  $t$ -orthogonal set in a non-archimedean normed space  $E$ . If  $\{z_1, \dots, z_n\} \subset E$  and  $\|x_i - z_i\| < t \cdot \|x_i\|$  for each  $i \in \{1, \dots, n\}$ , then  $\{z_1, \dots, z_n\}$  is also  $t$ -orthogonal.*

*Proof.* Take any  $\lambda_i \in \mathbb{K}$ ,  $i = 1, \dots, n$ . Since  $\|x_i - z_i\| < t \cdot \|x_i\|$ , we have  $\|z_i\| = \|x_i\|$  for each  $i \in \{1, \dots, n\}$ . Consequently we note

$$\|\lambda_1 x_1 + \dots + \lambda_n x_n\| \geq t \cdot \max_{i=1, \dots, n} \|\lambda_i x_i\|$$

and

$$\|\lambda_1(z_1 - x_1) + \dots + \lambda_n(z_n - x_n)\| \leq \max_{i=1, \dots, n} \|\lambda_i(z_i - x_i)\| < t \cdot \max_{i=1, \dots, n} \|\lambda_i x_i\|.$$

Hence

$$\begin{aligned} \|\lambda_1 z_1 + \dots + \lambda_n z_n\| &= \|\lambda_1 z_1 + \dots + \lambda_n z_n - (\lambda_1 x_1 + \dots + \lambda_n x_n) + (\lambda_1 x_1 + \dots + \lambda_n x_n)\| \\ &= \|\lambda_1(z_1 - x_1) + \dots + \lambda_n(z_n - x_n) + (\lambda_1 x_1 + \dots + \lambda_n x_n)\| \\ &= \|\lambda_1 x_1 + \dots + \lambda_n x_n\| \geq t \cdot \max_{i=1, \dots, n} \|\lambda_i x_i\| = t \cdot \max_{i=1, \dots, n} \|\lambda_i z_i\|, \end{aligned}$$

and we are done.  $\square$

**Lemma 4.2.** *Let  $E, F$  be non-archimedean normed spaces,  $D$  a linear subspace of  $E$  such that  $E$  is an immediate extension of  $D$ , let  $F$  be spherically complete and  $T : D \rightarrow F$  be an isometry. Then  $T$  can be extended to a linear isometry  $T' : E \rightarrow F$ .*

*Proof.* Applying Ingleton's theorem (see [12, Theorem 4.8]) we can extend the isometry  $T : D \rightarrow F$  to the linear operator  $T' : E \rightarrow F$  such that  $\|T'\| \leq 1$ . We prove that  $T'$  is also an isometry.

Set  $x \in E \setminus D$ . Then, since  $E$  is an immediate extension of  $D$ , there is  $x_d \in D$  such that

$$\|x - x_d\| < \|x_d\| = \|x\|.$$

Thus we get

$$\|T'(x) - T'(x_d)\| \leq \|T'\| \cdot \|x - x_d\| < \|x_d\| = \|T'(x_d)\|$$

and

$$\|T'(x)\| = \|T'(x) - T'(x_d) + T'(x_d)\| = \|T'(x_d)\|.$$

Hence, finally  $\|T'(x)\| = \|x\|$ .  $\square$

**Lemma 4.3.** *Let  $Y$  be a finite-dimensional non-archimedean normed space and  $X$  be its linear subspace. Let  $\{u_1, \dots, u_{m_Y}\}$  be a maximal orthogonal set in  $Y$  such that  $\{u_1, \dots, u_{m_X}\}$  is a maximal orthogonal set in  $X$  for some  $m_X \leq m_Y$  and let  $F_Y := [u_{m_X+1}, \dots, u_{m_Y}]$ . Then,  $F_Y \perp X$ .*

*Proof.* Assume that  $m_Y > m_X$  (otherwise nothing is to prove). Take any  $x \in X$  and  $y \in F_Y$ . If  $x \in [u_1, \dots, u_{m_X}]$ , the conclusion is obvious. So, assume that  $x \notin [u_1, \dots, u_{m_X}]$ . But then, since  $X$  is an immediate extension of  $[u_1, \dots, u_{m_X}]$ , there is  $x_0 \in [u_1, \dots, u_{m_X}]$  with

$$\|x - x_0\| < \|x\| = \|x_0\|.$$

Thus, since

$$\|x_0 + y\| = \max\{\|x_0\|, \|y\|\},$$

we have

$$\|x + y\| = \|x - x_0 + x_0 + y\| = \|x_0 + y\| = \max\{\|x_0\|, \|y\|\} = \max\{\|x\|, \|y\|\}.$$

$\square$

Let  $r := |\rho|$  if  $\mathbb{K}$  is discretely valued, where  $\rho \in \mathbb{K}$  is a uniformizing element of  $|\mathbb{K}^*|$  with  $0 < |\rho| < 1$ , and let  $r$  be any number taken from  $(\frac{1}{2}, 1)$  if  $\mathbb{K}$  is densely valued. Note that  $(0, \infty)$  is a multiplicative group. Let

$$(4.2) \quad \pi : (0, \infty) \rightarrow G := (0, \infty) / |\mathbb{K}^*|$$

be the quotient map and let  $S = \{s_g : g \in G\}$  be a set of representatives of elements of  $G$  in  $(r, 1]$ , i.e.  $\pi(s_g) = g$ .



Let  $I_u$  be a set for which  $\text{card}(I_u) = \max\{\aleph_0, \text{card}(G)\}$  and let  $I_u = \bigcup_{g \in G} I_g$  where  $\{I_g : g \in G\}$  is a partition of  $I_u$  such that  $\text{card}(I_g) = \aleph_0$  for each  $g \in G$ . Then, clearly  $c_0(I_u) = \bigoplus_{g \in G} c_0(I_g)$ .

Define the function  $s : I_u \rightarrow (r, 1]$  by  $h(i) := s_g$  if  $i \in I_g$  and the norm on  $c_0(I_u)$  by

$$\|x\|_u := \max_{i \in I_u} \{s(i) \cdot |x_i|\}, \quad x = (x_i)_{i \in I} \in c_0(I_u).$$

Denote  $E_u := (c_0(I_u), \|\cdot\|_u)$ .

**Proposition 4.4.** *Let  $\mathbb{K}$  be spherically complete. Then every non-archimedean Banach space of countable type can be isometrically embedded into  $E_u$ .*

- $E_u$  is of countable type (hence  $E_u$  is isometrically universal for the class of non-archimedean Banach spaces of countable type) if and only if  $(0, \infty)$  is the union of at most countably many cosets of  $|\mathbb{K}^*|$ ;
- $E_u$  is never separable.

*Proof.* Let  $E$  be a non-archimedean Banach space of countable type. Since  $\mathbb{K}$  is spherically complete,  $E$  has an orthogonal base  $(x_n)$  (see [11, Theorem 2.3.25]). Let

$$J_g = \{n : \pi(\|x_n\|) = g\}, \quad g \in G,$$

where  $\pi$  is the map defined in (4.2). Then  $G_0 = \{g \in G : J_g \text{ is nonempty}\}$  is at most countable. So, we can write  $\mathbb{N} = \bigcup_{g \in G_0} J_g$ , where  $J_g$  ( $g \in G_0$ ) are nonempty, finite or infinite, pairwise disjoint subsets of  $\mathbb{N}$ .

Define the map  $l : \mathbb{N} \rightarrow I_u$  (recall that  $I_u = \bigcup_{g \in G} I_g$  and  $I_g$  is countable for every  $g \in G$ ; thus we can write  $I_g = \{m_1^g, m_2^g, \dots\}$ ,  $g \in G$ ) as follows: for every  $n \in \mathbb{N}$  there exist  $g \in G_0$  and  $k \in \mathbb{N}$  such that  $n = n_k^g$  (what means  $n \in J_g$ ). Finally set  $l(n) := m_k^g$ .

Note that for every  $n \in \mathbb{N}$  we can find  $\lambda_n \in \mathbb{K}$  for which

$$\|x_n\| = s_g \cdot |\lambda_n|.$$

Next, define the map  $i_0 : \{x_1, x_2, \dots\} \rightarrow E_u$  by the formula  $x_n \mapsto \lambda_n e_{l(n)}$ , where  $e_n$  are as usual the unit vectors. Since

$$\|\lambda_n e_{l(n)}\|_u = s_g \cdot |\lambda_n| = \|x_n\|,$$

we can extend the map  $i_0$  to an isometric embedding  $E \rightarrow E_u$ .

Now we prove the next claim of the proposition. Suppose that  $(0, \infty)$  is not the union of at most countably many cosets of  $|\mathbb{K}^*|$  and assume that  $E_u$  is of countable type. By [11, Theorem 2.3.25] the space  $E_u$  has an orthogonal base  $(x_n)$ . Since  $(x_n)$  is orthogonal,  $\|E_u\|_u \setminus \{0\}$  consist of at most countably many cosets of  $|\mathbb{K}^*|$ . Hence there exists

$$s \in (0, \infty) \setminus \|E_u\|_u.$$

Define  $E = (\mathbb{K}^2, \|\cdot\|_s)$ , where

$$\|(x, y)\|_s := \max\{|x|, s \cdot |y|\},$$

$(x, y) \in \mathbb{K}^2$ . Then

$$\|(0, 1)\|_s \notin \|E_u\|_u.$$

Hence there is no isometry  $E \rightarrow E_u$ , a contradiction. If  $(0, \infty)$  is the union of at most countably many cosets of  $|\mathbb{K}^*|$ , then  $G$ , and consequently  $I_u$  is countable, and  $E_u$  is of countable type.

Finally, assume that  $\mathbb{K}$  is separable. If  $\mathbb{K}$  is discretely valued, then  $(0, \infty)$  is always the union of more than countably many cosets of  $|\mathbb{K}^*|$ . On the other hand, by [13, Theorem 20.5] there is no separable densely valued spherically complete  $\mathbb{K}$ ; hence  $E_u$  is not separable.  $\square$

**Remark 4.5.** If  $\mathbb{K}$  is non-spherically complete, then  $E_u$  does not contain an isometric image of any non-archimedean Banach spaces of countable type. Indeed, in this case there exists finite-dimensional normed spaces without orthogonal bases, see [11, Example 2.3.26] and [6]. Take  $E = \mathbb{K}_v^2$ , where  $\mathbb{K}_v^2$  is a two-dimensional normed space over  $\mathbb{K}$  without two non-zero orthogonal elements, and assume that there exists an isometric embedding  $i : E \rightarrow E_u$ . Then the image  $i(E)$  has no two non-zero orthogonal elements. But this contradicts the conclusion of Gruson's theorem ([12, Theorem 5.9]) stating that every linear subspace of a non-archimedean Banach spaces with an orthogonal base has an orthogonal base.

If  $\mathbb{K}$  is non-spherically complete the role of  $c_0(I : s)$  takes the space  $\ell^\infty$ . In this case, by [11, Theorem 2.5.13], every non-archimedean Banach space of countable type can be isometrically embedded into  $\ell^\infty$ .

Finally we prove the following

**Theorem 4.6.** *The spherical completion  $\widehat{E}_u$  of  $E_u$  is a non-archimedean Banach space of universal disposition for the class  $\mathcal{U}_{FNA}$ .*

*Proof.* Denote  $F := \widehat{E}_u$ . Let  $X \subset F$  and let  $j : X \rightarrow Y$  be an isometric embedding, where  $Y$  is a finite non-archimedean normed space. We prove, that there exists an isometric embedding  $f : Y \rightarrow F$  such that  $f(j(x)) = x$  for all  $x \in X$ .

Choose a maximal orthogonal set  $\{u_1, \dots, u_{m_X}, \dots, u_{m_Y}\}$  in  $Y$  such that  $\{u_1, \dots, u_{m_X}\}$  is a maximal orthogonal set in  $j(X)$  for some  $m_Y \geq m_X \geq 1$ . Set  $F_Y := [u_{m_X+1}, \dots, u_{m_Y}]$ . By Lemma 4.3 we get  $F_Y \perp j(X)$ .

Set  $v_k = f(u_k) := j^{-1}(u_k)$  for each  $k \in \{1, \dots, m_X\}$ . For every  $k \in \{m_X+1, \dots, m_Y\}$  choose  $i_k \in I$  such that  $\|e_{i_k}\|_u = \|\lambda_k u_k\|$  for some  $\lambda_k \in \mathbb{K}$  and

$$e_{i_k} \perp [v_1, \dots, v_{m_X}, e_{i_{m_X+1}}, \dots, e_{i_{k-1}}].$$

Next set  $f(u_k) := e_{i_k}$  for  $k = m_X+1, \dots, m_Y$ . Define  $f : j(X) + F_Y \rightarrow F$ . Clearly  $f$  is an isometry and  $f(j(X) + F_Y) \subset E_h^0$ .

If  $\mathbb{K}$  is spherically complete, we are done, as  $\{u_1, \dots, u_{m_Y}\}$  is an orthogonal base of  $Y$  by [12, Lemma 5.5 and Theorem 5.15]; hence,  $f$  is a required isometry defined on  $Y$ . If  $\mathbb{K}$  is non-spherically complete and  $j(X) + F_Y \neq Y$ , then, by [7, Proposition 2.1],  $Y$  is an immediate extension of  $j(X) + F_Y$ . Now, using Lemma 4.2, we extend  $f$  to the isometry defined on  $Y$ .  $\square$

The last part of the proof of Proposition 4.6 uses Lemma 4.2 for the spherical completeness of the considered space  $F$ . In fact it is enough to assume that  $F$  contains a spherical completion of its every finite-dimensional linear subspace. This observation suggests another construction.

For each  $g \in G$  set  $I_g = \{i_{g,1}, i_{g,2}, \dots\}$  (note that  $I_g$  is countable). For every  $n \in \mathbb{N}$  set  $F_g^n := [e_{i_{g,1}}, \dots, e_{i_{g,n}}]$ , a finite-dimensional linear subspace of  $c_0(I_u)$  spanned by appropriate unit vectors. By  $\widehat{F}_g^n$  denote a spherical completion of  $F_g^n$  such that for fixed  $g \in G$  we have  $F_g^n \subset \widehat{F}_g^n \subset c_0(I_g)$  and  $\widehat{F}_g^{n-1} \subset \widehat{F}_g^n$  if  $n > 1$ .

Next, for every  $g \in G$  define  $F_g := \bigcup_n \widehat{F}_g^n$ . Let  $E_h := \bigoplus_{g \in G} \overline{F}_g$ .

We are in a position to prove Theorem 4.7.

**Theorem 4.7.** *The space  $E_h$  is of universal disposition for the class  $\mathcal{U}_{FNA}$ .*

*Proof.* Let  $Y$  be a finite-dimensional non-archimedean normed space, let  $X \subset Y$ , and let  $j : X \rightarrow E_h$  be an isometric embedding. We prove that there exists an isometry  $f : Y \rightarrow E_h$  such that  $\|j - f|_X\| < 1$ . Then if we apply Theorem 1.3, the proof will be finished.

Set  $t \in (0, 1)$  and choose a  $t$ -orthogonal base  $\{x_1, \dots, x_{n_X}\}$  of  $X$ . Let  $z_i := j(x_i)$ ,  $i = 1, \dots, n_X$ . Then for each  $i \in \{1, \dots, n_X\}$  there exists a finite  $G_i \subset G$ , say  $G_i = \{g_1, \dots, g_m\}$ , and

$$z'_i \in \bigoplus_{g \in G_i} \overline{F}_g$$

for which

$$\|z'_i - z_i\| < \frac{t}{2} \|z_i\|.$$

Fix  $i \in \{1, \dots, n_X\}$ . Then we can write  $z'_i = z'_{i,1} + \dots + z'_{i,m}$ , where  $z'_{i,k} \in \overline{F}_{g_k}$ ,  $k = 1, \dots, m$ . But, then we can select  $n_i$  and  $w'_{i,1}, \dots, w'_{i,m}$  such that  $w'_{i,k} \in \widehat{F}_{g_k}^{n_i}$ , and  $\|w'_{i,k} - z'_{i,k}\| < \frac{t}{2} \|z'_{i,k}\|$ ,  $k = 1, \dots, m$ . Denote  $w_i := w'_{i,1} + \dots + w'_{i,m}$ . Then

$$\begin{aligned} \|w_i - z_i\| &= \|w_i - z'_i + z'_i - z_i\| \leq \max\{\|w_i - z'_i\|, \|z'_i - z_i\|\} \\ &\leq \max\left\{\max_{k=1, \dots, m} \|w'_{i,k} - z'_{i,k}\|, \|z'_i - z_i\|\right\} < \frac{t}{2} \|z_i\|. \end{aligned}$$

Hence, by Lemma 4.1,  $\{w_i : i \in \{1, \dots, n_X\}\}$  is a  $t$ -orthogonal set in  $\bigoplus_{g \in G_0} \widehat{F}_g^{n_0}$ , where  $G_0 = G_1 \cup \dots \cup G_{n_X}$  and  $n_0 = \max\{n_i : i \in \{1, \dots, n_X\}\}$ .

Define the map  $f : X \rightarrow \bigoplus_{g \in G_0} \widehat{F}_g^{n_0} \subset E_h$  setting  $f(x_i) := w_i$ ,  $i = 1, \dots, n_X$ . Then for all  $\lambda_i \in \mathbb{K}$  ( $i = 1, \dots, n_X$ ) we have

$$f : \sum_{i=1}^{n_X} \lambda_i x_i \mapsto \sum_{i=1}^{n_X} \lambda_i w_i.$$

Consequently

$$\begin{aligned} \left\| f\left(\sum_{i=1}^{n_X} \lambda_i x_i\right) \right\| &= \left\| \sum_{i=1}^{n_X} \lambda_i w_i - \sum_{i=1}^{n_X} \lambda_i z_i + \sum_{i=1}^{n_X} \lambda_i z_i \right\| = \left\| \sum_{i=1}^{n_X} \lambda_i (w_i - z_i) + \sum_{i=1}^{n_X} \lambda_i z_i \right\| \\ &= \left\| \sum_{i=1}^{n_X} \lambda_i z_i \right\| = \left\| \sum_{i=1}^{n_X} \lambda_i j(x_i) \right\| = \left\| \sum_{i=1}^{n_X} \lambda_i x_i \right\| \end{aligned}$$

since

$$\left\| \sum_{i=1}^{n_X} \lambda_i (w_i - z_i) \right\| < \frac{t}{2} \max_{i=1, \dots, n_X} \|\lambda_i z_i\|$$

and, as  $\{z_1, \dots, z_{n_X}\}$  is  $t$ -orthogonal, we have

$$\left\| \sum_{i=1}^{n_X} \lambda_i z_i \right\| \geq t \cdot \max_{i=1, \dots, n_X} \|\lambda_i z_i\|.$$

Thus  $f$  is isometric. Observe that

$$\begin{aligned} \|f - j\| &= \sup_{x \in X} \frac{\|(f - j)(x)\|}{\|x\|} = \sup_{\lambda_i \in \mathbb{K} \ (i=1, \dots, n_X)} \frac{\|\sum_{i=1}^{n_X} \lambda_i (f - j)(x_i)\|}{\|\sum_{i=1}^{n_X} \lambda_i x_i\|} \\ &\leq \sup_{\lambda_i \in \mathbb{K} \ (i=1, \dots, n_X)} \frac{\max_{i=1, \dots, n_X} \|\lambda_i (w_i - z_i)\|}{t \cdot \max_{i=1, \dots, n_X} \|\lambda_i x_i\|} \\ &\leq \sup_{\lambda_i \in \mathbb{K} \ (i=1, \dots, n_X)} \frac{\frac{t}{2} \cdot \max_{i=1, \dots, n_X} \|\lambda_i x_i\|}{t \cdot \max_{i=1, \dots, n_X} \|\lambda_i x_i\|} < 1. \end{aligned}$$

Now, we extend  $f$  on  $Y$ . We argue similarly as in the proof of the previous theorem. Choose  $\{u_1, \dots, u_{m_X}, \dots, u_{m_Y}\}$ , a maximal orthogonal set in  $Y$  such that  $\{u_1, \dots, u_{m_X}\}$  is a maximal orthogonal set in  $X$ . By Lemma 4.3 we get  $F_Y \perp j(X)$ , where  $F_Y = [u_{m_X+1}, \dots, u_{m_Y}]$ .

Denote  $v_k := f(u_k)$  for  $k = 1, \dots, m_X$ , and for each  $k \in \{m_X+1, \dots, m_Y\}$  choose  $g_k \in G$  and  $i_{g_k, n_k} \in I_{g_k}$ , denoting for simplicity  $j_k := i_{g_k, n_k}$ , such that  $\|e_{j_k}\|_u = \|\lambda_k u_k\|$  (for some  $\lambda_k \in \mathbb{K}$ ) and

$$e_{j_k} \perp [v_1, \dots, v_{m_X}, e_{i_{m_X+1}}, \dots, e_{j_{k-1}}].$$

Then, setting  $f(u_k) := e_{j_k}$ , where  $k = m_X+1, \dots, m_Y$ , we extend  $f$  on  $X + F_Y$ . Let  $p_0 = \max\{n_k : k = m_X+1, \dots, m_Y\}$  and  $G_1 = \{g_k : k = m_X+1, \dots, m_Y\}$ . The map  $f$  is an isometry and  $f(X + F_Y) \subset H$ , where

$$H = \bigoplus_{g \in G_0 \cup G_1} \widehat{F_g^{\max\{n_0, p_0\}}}.$$

If  $\mathbb{K}$  is spherically complete, the proof is completed, since  $\{u_1, \dots, u_{n_Y}\}$  is an orthogonal base of  $Y$  by [12, Lemma 5.5 and Theorem 5.15]. This shows that  $f$  is a required isometry defined on  $Y$ . Assume that  $\mathbb{K}$  is non-spherically complete. Since  $Y$  is an immediate extension of  $X + F_Y$  and  $H$  is spherically complete as a finite direct sum of spherically complete spaces (see [12, 4A]), we apply Lemma 4.2 to extend  $f$  to the isometry defined on  $Y$ . Since  $\|j - f|_X\| < 1$ , as we proved above, we apply Theorem 1.3. The proof is finished.  $\square$

This yields the following interesting

**Corollary 4.8.** (1) *If  $\mathbb{K}$  is spherically complete and  $(0, \infty)$  is the union of at most countably many cosets of  $|\mathbb{K}^*|$ , the isometrically universal space  $E_h$  for the class of non-archimedean Banach spaces of countable type is also of universal disposition for the class  $\mathcal{U}_{FNA}$ .*

(2) *There exist non-archimedean Banach spaces of universal disposition for the class  $\mathcal{U}_{FNA}$  which are not isometrically isomorphic.*

*Proof.* Recall that if  $\mathbb{K}$  is spherically complete, then every finite-dimensional normed space over  $\mathbb{K}$  is spherically complete (see [12, Theorem 4.2 and Corollary 4.6]). Thus,  $E_h = (c_0(I_u), \|\cdot\|_u) = E_u$ . The remaining part of the proof of (1) follows from Proposition 4.4. To prove (2) consider the spaces  $E_h$  and  $\widehat{E}_u$  (by Theorems 4.6 and 4.7 both are spaces of universal disposition for the class  $\mathcal{U}_{FNA}$ ) assuming that  $\mathbb{K}$  is spherically complete. Then  $E_h = E_u$ . Since  $\|E_u^*\|_u$  is a dense subset of  $(0, \infty)$ , we can choose  $\{i_1, i_2, \dots\} \subset I_u$  such that  $1 \geq \|e_{i_1}\|_u > \|e_{i_2}\|_u > \dots > \frac{1}{2}$  (where  $e_{i_k}$  are unit vectors of  $E_u$ ). Set

$x_n := \sum_{k=1}^n e_{i_k}$  ( $n \in \mathbb{N}$ ). Then, the balls  $V_n := \{x \in E_u : \|x - x_n\|_u \leq \|e_{i_{n+1}}\|_u\}$  form a shrinking sequence in  $E_u$ . But  $\bigcap_{n \in \mathbb{N}} V_n = \emptyset$ ; hence,  $E_u$  is not spherically complete. Clearly  $\widehat{E}_u$ , as a spherical completion of  $E_u$ , is spherically complete, thus we imply that  $E_h (=E_u)$  and  $\widehat{E}_u$  are not isometrically isomorphic.  $\square$

**Remark 4.9.** Note that (see Remark 4.5) if  $\mathbb{K}$  is non-spherically complete then the space  $\ell^\infty$  is not of universal disposition for the class  $\mathcal{U}_{FNA}$ . Indeed, take  $Y = \mathbb{K}_v^2$ ,  $e_1 = (1, 0, 0, \dots) \in \ell^\infty$  and define the isometric embedding  $i : [e_1] \rightarrow \mathbb{K}_v^2 : e_1 \mapsto (1, 0)$ . On the other hand, by [8, Proposition 3.2], every two-dimensional linear subspace of  $\ell^\infty$  containing  $e_1$  has two non-zero orthogonal elements. Thus, there is no isometric embedding  $f : Y \rightarrow \ell^\infty$  such that  $f(1, 0) = e_1$ .

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