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The Ascoli property for function spaces

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ABSTRACT

The paper deals with Ascoli spaces $C_p(X)$ and $C_k(X)$ over Tychonoff spaces X. The class of Ascoli spaces X, i.e. spaces X for which any compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous, essentially includes the class of $k_{\mathbb{R}}$-spaces. First we prove that if $C_p(X)$ is Ascoli, then it is κ-Fréchet–Urysohn. If X is cosmic, then $C_p(X)$ is Ascoli iff it is κ-Fréchet–Urysohn. This leads to the following extension of a result of Morishita: If for a Čech-complete space X the space $C_p(X)$ is Ascoli, then X is scattered. If X is scattered and stratifiable, then $C_p(X)$ is an Ascoli space. Consequently: (a) If X is a complete metrizable space, then $C_p(X)$ is Ascoli iff X is scattered. (b) If X is a Čech-complete Lindelöf space, then $C_p(X)$ is Ascoli iff X is scattered iff $C_p(X)$ is Fréchet–Urysohn. Moreover, we prove that for a paracompact space X of point-countable type the following conditions are equivalent: (i) X is locally compact. (ii) $C_k(X)$ is a $k_{\mathbb{R}}$-space. (iii) $C_k(X)$ is an Ascoli space. The Ascoli spaces $C_k(X, \mathbb{I})$ are also studied.

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1. Introduction

Various topological properties generalizing metrizability have been intensively studied both by topologists and analysts for a long time, and the following diagram gathers some of the most important concepts:

$$\begin{array}{ccccccccccc} & & \kappa\text{-Fréchet–Urysohn} & & & & & & & & \\ & & \Uparrow & & & & & & & & \\ \text{metric} & \Longrightarrow & \text{Fréchet–Urysohn} & \Longrightarrow & \text{sequential} & \Longrightarrow & k\text{-space} & \Longrightarrow & k_{\mathbb{R}}\text{-space} & \Longrightarrow & \text{Ascoli space}. \end{array}$$

Note that none of these implications is reversible. The study of the above concepts for the function spaces with various topologies has a rich history and is also nowadays an active area of research, see [2,15,18,30] and references therein.

For Tychonoff topological spaces X and Y, we denote by $C_k(X,Y)$ and $C_p(X,Y)$ the space $C(X,Y)$ of all continuous functions from X into Y endowed with the compact-open topology or the pointwise topology, respectively. If $Y=\mathbb{R}$, we shall write $C_k(X)$ and $C_p(X)$, respectively.

It is well-known that $C_p(X)$ is metrizable if and only if X is countable. Pytkeev, Gerlitz and Nagy (see §3 of [2]) characterized spaces X for which $C_p(X)$ is Fréchet–Urysohn, sequential or a k-space (these properties coincide for the spaces $C_p(X)$). Sakai in [26] described all spaces X for which $C_p(X)$ is κ-Fréchet–Urysohn, see Theorem 2.3 below. However, very little is known about spaces X for which $C_p(X)$ is an Ascoli space or a $k_{\mathbb{R}}$-space.

Following [3], a space X is called an *Ascoli space* if each compact subset $\mathcal{K}$ of $C_k(X)$ is evenly continuous, that is, the map $X\times\mathcal{K}\ni(x,f)\mapsto f(x)\in\mathbb{R}$ is continuous. Equivalently, X is Ascoli if the natural evaluation map $X\hookrightarrow C_k(C_k(X))$ is an embedding, see [3]. Recall that a space X is called a *$k_{\mathbb{R}}$-space* if a real-valued function f on X is continuous if and only if its restriction $f|_K$ to any compact subset K of X is continuous. It is known that every $k_{\mathbb{R}}$-space is Ascoli, but the converse is in general not true, see [3].

The class of Ascoli spaces was introduced in [3]. The question for which spaces X the space $C_p(X)$ is Ascoli or a $k_{\mathbb{R}}$-space is posed in [10]. It turned out that for spaces of the form $C_p(X)$, the Ascoli property is formally stronger than the κ-Fréchet–Urysohn one. This follows from the following

Theorem 1.1.

(i) *If $C_p(X)$ is Ascoli, then it is κ-Fréchet–Urysohn.*
(ii) *If $C_p(X)$ is κ-Fréchet–Urysohn and every compact $K\subset C_k(C_p(X))$ is first-countable, then $C_p(X)$ is Ascoli.*

Recall that a regular space X is *cosmic* if it is a continuous image of a separable metrizable space, see [19]. Michael proved in [19] that every compact subset of a cosmic space is metrizable, and if X is a cosmic space then $C_p(X)$ and hence $C_p(C_p(X))$ are cosmic. So all compact subsets of $C_p(C_p(X))$ and hence $C_k(C_p(X))$ are metrizable. This remark and Theorem 1.1 imply

Corollary 1.2. *If X is a cosmic space, then $C_p(X)$ is Ascoli if and only if it is κ-Fréchet–Urysohn.*

The second main result of Section 2 is the following theorem, which extends an unpublished result of Morishita [14, Theorem 10.7] and [6, Corollary 4.2], see also Corollary 2.12 below.

Theorem 1.3.

(i) *If X is Čech-complete and $C_p(X)$ is Ascoli, then X is scattered.*
(ii) *If X is scattered and stratifiable, then $C_p(X)$ is an Ascoli space.*

Since a metrizable space X is Čech-complete if and only if it is completely metrizable, and since every metrizable space is stratifiable, Theorem 1.3 implies

Corollary 1.4. *If X is a completely metrizable (and separable) space, then $C_p(X)$ is Ascoli if and only if X is scattered (and countable).*

The following corollary strengthens also Proposition 6.6 of [10].

Corollary 1.5. *Let X be a compact space. Then $C_p(X)$ is Ascoli if and only if $C_p(X)$ is Fréchet–Urysohn if and only if X is scattered.*

The second part of our paper deals with the Ascoli spaces $C_k(X)$. In [23] Pol gave a complete characterization of those first-countable paracompact spaces X for which the space $C_k(X, \mathbb{I})$ is a k-space, where $\mathbb{I} = [0, 1]$.

In [9] the first named author described all zero-dimensional metric spaces X for which the space $C_k(X, 2)$ is Ascoli, where $2 = \{0, 1\}$ is the doubleton.

On the other hand, it is proved in [10] that if X is a first-countable paracompact σ-space, then $C_k(X, \mathbb{I})$ is Ascoli if and only if $C_k(X)$ is Ascoli if and only if X is a locally compact metrizable space. However this result does not cover the case for X being a non-metrizable compact space X for which clearly the Banach space $C_k(X)$ is Ascoli. The next theorem, which is the main result of Section 3, extends all results mentioned above. We prove the following

Theorem 1.6. *For a paracompact space X of point-countable type the following conditions are equivalent:*

(i) *X is locally compact;*
(ii) *$X = \bigoplus_{i\in\kappa} X_i$, where all X_i are Lindelöf locally compact spaces;*
(iii) *$C_k(X)$ is a $k_{\mathbb{R}}$-space;*
(iv) *$C_k(X)$ is an Ascoli space;*
(v) *$C_k(X, \mathbb{I})$ is a $k_{\mathbb{R}}$-space;*
(vi) *$C_k(X, \mathbb{I})$ is an Ascoli space.*

In cases (i)–(vi), the spaces $C_k(X)$ and $C_k(X, \mathbb{I})$ are homeomorphic to products of families of complete metrizable spaces.

In our forthcoming paper [11] we show that the paracompactness assumption on X cannot be omitted in Theorem 1.6 and we provide the first C_p-example of an Ascoli space not being a $k_{\mathbb{R}}$-space.

2. The Ascoli property for $C_p(X)$

Let X be a Tychonoff space and $h \in C(X)$. Then the sets of the form

$$[h, F, \varepsilon] := \{f \in C(X) : |f(x) - h(x)| < \varepsilon \text{ for all } x \in F\}, \text{ where } F \in [X]^{<\omega} \text{ and } \varepsilon > 0,$$

form a base at h for the topology τ_p of pointwise convergence on $C(X)$. The space $C(X)$ equipped with τ_p is usually denoted by $C_p(X)$.

Lemma 2.1. *Let $\{U_n : n \in \omega\}$ be a sequence of open subsets of $C_p(X)$ such that $0 \in \overline{U_n}$ for all n. Then for every sequence $\{\mathcal{W}_n : n \in \omega\}$ such that $\mathcal{W}_n$ is an open cover of U_n, for every n there exists $W_n \in \mathcal{W}_n$ such that $0 \in \overline{\bigcup\{W_n : n \in \omega\}}$.*

Proof. By induction on n we can construct an increasing sequence $\{A_n : n \in \omega\}$ of finite subsets of X, a decreasing null-sequence $\{\varepsilon_n : n \in \omega\}$ of positive reals, a sequence $\{W_n \in \mathcal{W}_n : n \in \omega\}$ of open subsets of $C_p(X)$ and a sequence $\{h_n : n \in \omega\}$ in $C_p(X)$ such that

$$[h_n, A_n, \varepsilon_n] \subseteq W_n \text{ and } [h_{n+1}, A_{n+1}, \varepsilon_{n+1}] \subset [0, A_n, 1/(n+1)].$$

We claim that $\{W_n : n \in \omega\}$ is as required. Indeed, fix a finite $F \subset X$ and $\varepsilon > 0$, and find n_0 such that $F \cap (\bigcup_{n\in\omega} A_n) \subset A_{n_0}$ and $\frac{1}{n_0} + \varepsilon_{n_0+1} < \varepsilon$. Then any $h \in [h_{n_0+1}, A_{n_0+1}, \varepsilon_{n_0+1}]$ such that $h|_{F\setminus A_{n_0+1}} = 0$ belongs to $[0, F, \varepsilon]$. □

The following statement is similar to [10, Proposition 2.1].

Lemma 2.2. *Assume that $C_p(X)$ is an Ascoli space and $\{U_n : n \in \omega\}$ is a sequence of open subsets of $C_p(X)$ such that $0 \in \overline{\bigcup\{U_n : n \in \omega\}}$ but $0 \notin \overline{U_n}$ for all n. Then there exists a compact subspace K of $C_p(X)$ such that the set $\{n : K \cap U_n \neq \emptyset\}$ is infinite.*

Proof. Suppose for a contradiction that for every compact $K \subset C_p(X)$, $K \cap U_n \neq \emptyset$ only for finitely many n. For every $n \in \omega$, set $\widetilde{U}_n := \bigcup_{l\geq n} U_l$ and

$$\mathcal{W}_n := \left\{W \subset C_p(X) : \exists l \geq n\, \exists \varphi \in C\big(C_p(X)\big)\, (W \in \mathcal{P}(U_l) \cap \tau_p) \wedge (\varphi|_W > 1) \wedge (\varphi|_{C_p(X)\setminus U_l} = 0)\right\}.$$

Then $0 \in \overline{\widetilde{U}_n}$ and $\mathcal{W}_n$ is an open cover of $\widetilde{U}_n$. Applying Lemma 2.1 for the sequence $\{\widetilde{U}_n : n \in \omega\}$, for every n there exists $W_n \in \mathcal{W}_n$ such that $0 \in \overline{\bigcup\{W_n : n \in \omega\}}$. Let $l_n \geq n$ and φ_n be witnesses for $W_n \in \mathcal{W}_n$. It follows from the above that φ_n converges to 0 in $C_k(C_p(X))$: given any compact $K \subset C_p(X)$, $\varphi_n|_K$ is constant 0 for all but finitely many n (namely for all n such that $K \cap U_{l_n} = \emptyset$). On the other hand, given any open $V \subset C_p(X)$ containing 0 and $m \in \omega$, the inclusion $0 \in \overline{\bigcup\{W_n : n \in \omega\}}$ implies that there exists $n \geq m$ and $f \in V \cap W_n$, which yields $\varphi_n(f) > 1$. This proves that the convergent sequence

$$\{\varphi_n : n \in \omega\} \cup \{0\} \subset C_k(C_p(X))$$

is not evenly continuous, a contradiction. □

Following Arhangel'skii, a topological space X is said to be *κ-Fréchet–Urysohn* if for every open subset U of X and every $x \in \overline{U}$, there exists a sequence $\{x_n\}_{n\in\omega} \subseteq U$ converging to x. Note that the class of κ-Fréchet–Urysohn spaces is much wider than the class of Fréchet–Urysohn spaces [16].

A family $\{A_i\}_{i\in I}$ of subsets of a set X is said to be *point-finite* if the set $\{i \in I : x \in A_i\}$ is finite for every $x \in X$. A family $\{A_i\}_{i\in I}$ of subsets of a topological space X is called *strongly point-finite* if for every $i \in I$, there exists an open set U_i of X such that $A_i \subseteq U_i$ and $\{U_i\}_{i\in I}$ is point-finite. Following Sakai [26], a topological space X is said to have the *property* (κ) if every pairwise disjoint sequence of finite subsets of X has a strongly point-finite subsequence. We shall need the following result of Sakai, see [26, Theorem 2.1].

Theorem 2.3. *The space $C_p(X)$ is κ-Fréchet–Urysohn if and only if X has the property (κ).*

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. (i) By Theorem 2.3 we have to show that X has the property (κ). Consider a sequence $\{F_n : n \in \omega\}$ of finite subsets of X such that $F_n \cap F_m = \emptyset$ for all $n \neq m$. We need to find an infinite $J \subset \omega$ and open sets $U_j \supset F_j$ for all $j \in J$, such that $\{U_j : j \in J\}$ is point-finite.

Let g_k be the constant k function and denote by O_k the set $[g_k, F_k, 1/2]$. It is easy to see that $0 \in \overline{\bigcup\{O_k : k > 0\}}$. By Lemma 2.2 there exists a compact $K \subset C_p(X)$ intersecting infinitely many of the O_k's. Thus there exists an infinite $J \subset \omega$ and for every $j \in J$ a function $h_j \in K \cap O_j$. Set

$$U_j := \{x \in X : h_j(x) > j - 1/2\} \supset F_j,$$

and note that $\{U_j\}_{j\in J}$ is point-finite. Indeed, if x belongs to U_j for all $j \in J'$, where $J' \subseteq J$ is infinite, then $\{h_j(x) : j \in J'\}$ is unbounded, which is impossible because $\{h_j : j \in J'\} \subset K$.

(ii) Suppose that $C_p(X)$ is not Ascoli and find a compact $\mathcal{K} \subset C_k(C_p(X))$ and $\varphi \in \mathcal{K}$ such that the valuation map is discontinuous at $(0, \varphi) \in C_p(X) \times \mathcal{K}$. Without loss of generality we may assume that $\varphi(0) = 0$ whereas the set

$$\{(h, \psi) \in \big(C_p(X) \setminus \{0\}\big) \times \mathcal{K} : \psi(h) > 1\}$$

contains $(0, \varphi)$ in the closure. Let $\{\mathcal{O}_n : n \in \omega\}$ be a base of the topology of $\mathcal{K}$ at φ. For every $n \in \omega$, denote by H_n the set of all nonzero functions h for which there is $\psi_{n,h} \in \mathcal{O}_n$ such that $\psi_{n,h}(h) > 1$, and note that $0 \in \overline{H_n}$. Let $W_{n,h} \subset C_p(X)$ be an open neighborhood of h such that

$$\overline{W_{n,h}} \subset C_p(X) \setminus \{0\}$$

and $\psi_{n,h}(h') > 1$ for all $h' \in W_{n,h}$. Set $\mathcal{W}_n = \{W_{n,h} : h \in H_n\}$ and note that $0 \in \overline{\bigcup \mathcal{W}_n}$ as $\bigcup \mathcal{W}_n \supset H_n$. Applying Lemma 2.1 we can find $h_n \in H_n$ such that

$$0 \in \overline{\bigcup\{W_{n,h_n} : n \in \omega\}}.$$

Since $C_p(X)$ is κ-Fréchet–Urysohn, there exists a convergent to 0 sequence $\{g_n : n \in \omega\}$ such that $g_n \in W_{k_n,h_{k_n}}$ for some $k_n \in \omega$.

Let n_0 be such that $\varphi(g_n) < 1/2$ for all $n \geq n_0$. Such an n_0 exists since φ is continuous and $\varphi(0) = 0$. Since $\{\psi_{k_n,h_{k_n}} : n \in \omega\}$ converges to φ in $C_k(C_p(X))$ and $\{g_n : n \in \omega\} \cup \{0\}$ is a compact subspace of $C_p(X)$, there exists $n_1 \in \omega$ such that

$$\psi_{k_n,h_{k_n}}|_{\{g_m : m \geq n_0\}\cup\{0\}} < 1/2 \text{ for all } n \geq n_1.$$

But this is impossible since $\psi_{k_n,h_{k_n}}(g_n) > 1$ for all n, because $g_n \in W_{k_n,h_{k_n}}$ and $\psi_{k_n,h_{k_n}}(h') > 1$ for all $h' \in W_{k_n,h_{k_n}}$. □

By [26, Theorem 3.2] every separable metrizable space X with the property (κ) is always of the first category (i.e., every dense in itself subset A of X is of the first category in itself). So, if X is a non-meager separable metrizable space without isolated points then $C_p(X)$ is not an Ascoli space.

Having in mind Theorem 1.1 it is natural to ask the following

Question 2.4. *Suppose that $C_p(X)$ is κ-Fréchet–Urysohn. Is it then Ascoli?*

Next proposition complements Theorem 3.4 and Corollary 3.5 of [26].

Proposition 2.5. *Let X be a Čech-complete space. If X is has the property (κ), then X is scattered.*

Proof. By Fact 1 on page 308 in [30] it is sufficient to prove that any compact $K \subset X$ is scattered. Suppose for the contradiction that X contains a non-scattered compact subset K. Since the property (κ) is hereditary by [26, Proposition 3.7], to get a contradiction it is sufficient to show that K does not have the property (κ). As K is not scattered there exists a continuous surjective map $f : K \to [0,1]$, see [27, Theorem 8.5.4]. By [8, Exercise 3.1.C(a)], passing to the restriction of f to some compact subspace of K if necessary (this is possible because the property (κ) is hereditary), we may additionally assume that f is irreducible, i.e., $f[K'] \neq [0,1]$ for any closed $K' \subsetneq K$. It follows that for any $A \subset K$, if $f[A]$ is dense in $[0,1]$, then A is dense in K because $\overline{f[A]} = [0,1]$.

Let $\{B_n : n \in \omega\}$ be a base of the topology of $[0,1]$. Since every B_n is infinite we can choose a disjoint sequence $\{F_n : n \in \omega\}$ of finite subsets of $[0,1]$ such that $F_n \cap B_k \neq \emptyset$ for all $k \leq n$. Note that $\bigcup_{n\in I} F_n$ is dense in $[0,1]$ for every infinite subset I of ω. For every $n \in \omega$ take a finite subset A_n of K such that $f[A_n] = F_n$. It follows from the above that $\bigcup_{n\in I} A_n$ is dense in K for every infinite $I \subseteq \omega$. We show that the sequence $\{F_n : n \in \omega\}$ does not have a strongly point-finite subsequence. Let $I \subseteq \omega$ be infinite and let a sequence $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of K be such that $A_i \subseteq U_i$ for any $i \in I$. Then

$$\bigcap_{m\in\omega} \bigcup_{i\in I, i\geq m} U_i \neq \emptyset$$

by the Baire theorem because $\bigcup_{i\in I, i\geq m} U_i$ is open and dense in K for all m. So $\mathcal{U}$ is not point-finite. Thus K does not have the property (κ). □

Let $X = \prod_{t\in T} X_t$ be the product of an infinite family of topological spaces. For $x = (x_t)$ and $y = (y_t)$ in X, we set $\delta(x, y) := \{t : x_t \neq y_t\}$ and

$$\Sigma(x) := \{y \in X : \delta(x, y) \text{ is countable}\} \text{ and } \sigma(x) := \{y \in X : \delta(x, y) \text{ is finite}\}. \tag{2.1}$$

If each X_t is considered with a structure of a linear topological space, then we standardly mean by $\sigma_{t\in T} X_t := \sigma(0)$ the σ-product with respect to the identity $0 = 0_X := (0_t) \in X$. If $x \in \Sigma(z)$ we set $\mathrm{supp}(x) := \{t \in T : x_t \neq z_t\}$, so $\mathrm{supp}(x)$ is a countable subset of T. Subspaces of $\prod_{t\in T} X_t$ of the form $\Sigma(x)$, where $x \in \prod_{t\in T} X_t$, are called Σ-*subspaces.*

The following (probably folklore) statement generalizes a result of Noble [22]. It is a trivial consequence of Statements 1.5.25 and 1.5.26 of [1].

Proposition 2.6. *Let $\{X_i : i \in I\}$ be a family of topological spaces such that $X = \prod_{i\in I'} X_i$ is Fréchet–Urysohn for any countable subset I' of I. Then $\Sigma(z)$ and hence also $\sigma(z)$ are Fréchet–Urysohn for every $z \in \prod_{i\in I} X_i$. In particular, each Σ-subspace of a product of first countable spaces is a Fréchet–Urysohn space.*

In what follows we need the following consequence of [3, Proposition 5.10].

Lemma 2.7. *Let Y be a dense subset of a homogeneous space (in particular, a topological group) X. If Y is an Ascoli space, then X is also an Ascoli space.*

Proof. Fix arbitrarily $y_0 \in Y$. Let $x \in X$. Take a homeomorphism h of X such that $h(y_0) = x$. Then $x \in h(Y)$ and $h(Y)$ is an Ascoli space. So each element of X is contained in a dense Ascoli subspace of X. Thus X is an Ascoli space by Proposition 5.10 of [3]. □

Let us recall several definitions. For a scattered space X one of the most efficient methods to analyze its structure is the Cantor–Bendixson procedure described below. Set $X^{(0)} := X$,

$$X^{(\gamma+1)} := X^{(\gamma)} \setminus Iso(X^{(\gamma)})$$

(where by $Iso(Z)$ we denote the set of all isolated points of a space Z), and

$$X^{(\gamma)} := \bigcap_{\alpha<\gamma} X^{(\alpha)}$$

for limit ordinals γ. It is easy to see that X is scattered if and only if $X^{(\gamma)} = \emptyset$ for some ordinal γ. If X is scattered, for $x \in X$ we denote by $d(x)$ the (unique) α such that $x \in X^{(\alpha)} \setminus X^{(\alpha+1)}$.

A space X is called *ultraparacompact* [25] if any open cover of X has a clopen disjoint refinement. It has been shown by Telgarsky in [28] that a scattered paracompact space is zero-dimensional and ultraparacompact, see also [25] for generalizations.

Proposition 2.8. *Assume that a paracompact scattered space X has the following property:*

($\star$) *Each $x \in X$ has a clopen neighborhood $O(x)$ such that for any clopen U, $x \in U \subset O(x)$, there exists a compact C, $x \in C \subset U$, for which the difference $U \setminus C$ is paracompact, and there exists a continuous linear operator $\psi : C_p(C) \to C_p(U)$ such that $\psi(f)|_C = f$ for all $f \in C_p(C)$.*

Then $C_p(X)$ is Ascoli.

Proof. Note that if x and its clopen neighborhood $O(x)$ satisfy ($\star$), then for every clopen neighborhood V of x with $V \subseteq O(x)$ the pair x, V satisfies ($\star$). The following claim is the central part of the proof.

Claim 2.9. *Let X be a paracompact scattered space with the property ($\star$). Then for every $x \in X$ there exists a clopen neighborhood $O(x)$ of x with the following property:*

(†) *For any clopen $U \subset O(x)$ there exists a family $\mathcal{K} = \mathcal{K}_{U,x}$ of scattered compact subsets of U such that $C_p(U)$ is linearly homeomorphic to a linear subspace of $\prod_{K\in\mathcal{K}} C_p(K)$ containing $\sigma_{K\in\mathcal{K}} C_p(K)$.*

Proof. The proof will be by transfinite induction on $d(x)$. If $d(x) = 0$, then $x \in Iso(X)$. Set $O(x) := \{x\}$ and $\mathcal{K} := \big\{\{x\}\big\}$. Clearly, $O(x)$ and $\mathcal{K}$ are as required. Assuming that the claim is true for all $x \in X$ with $d(x) < \alpha$, let us fix $x \in X$ with $d(x) = \alpha$ and find a clopen neighborhood $O(x) \subseteq X$ of x such that

$$O(x) \subseteq \{y \in X : d(y) < \alpha\} \cup \{x\}.$$

We claim that $O(x)$ is as required. Indeed, let us fix a clopen $U \subseteq O(x)$. Two cases are possible.

Case 1. Assume that $x \in U$. Since X has the property ($\star$), there exists a compact $C \ni x$ such that $C \subset U$ and $U \setminus C$ is paracompact and there exists a continuous linear operator $\psi : C_p(C) \to C_p(U)$ such that $\psi(f)|_C = f$ for all $f \in C_p(C)$, so $\psi(0) = 0$. For every $y \in U \setminus C$, set

$$V_0(y) = O(y) \cap (U \setminus C).$$

Then $\mathcal{V}_0 = \{V_0(y) : y \in U \setminus C\}$ is an open cover of a paracompact scattered space $U \setminus C$. Thus there exists [28] a clopen cover $\mathcal{V}$ of $U \setminus C$ whose elements are mutually disjoint, and such that $\mathcal{V} \prec \mathcal{V}_0$, i.e., for every $V \in \mathcal{V}$ there exists $V' \in \mathcal{V}_0$ with the property $V \subset V'$. It follows from the above that each $V \in \mathcal{V}$ has the property (†), and hence there exists a family $\mathcal{K}_V$ of scattered compact subsets of V such that $C_p(V)$ can be topologically embedded into $\prod_{K\in\mathcal{K}_V} C_p(K)$ via a linear continuous map

$$\varphi_V : C_p(V) \to \prod_{K\in\mathcal{K}_V} C_p(K)$$

such that

$$\sigma_{K\in\mathcal{K}_V} C_p(K) \subset \varphi_V[C_p(V)].$$

Set

$$\mathcal{K} = \bigcup\{\mathcal{K}_V : V \in \mathcal{V}\} \cup \{C\},$$

so $\mathcal{K}$ is a family of scattered compact subsets of U. Define a continuous linear operator $\varphi : C_p(U) \to \prod\{C_p(K) : K \in \mathcal{K}\}$ as follows: if $f \in C_p(U)$, then

$$\varphi(f)(C) = f|_C; \tag{2.2}$$

and if $K \in \mathcal{K}_V$ for the unique $V \in \mathcal{V}$ such that $K \in \mathcal{K}_V$, then

$$\varphi(f)(K) = \varphi_V\big((f - \psi(f|_C))|_V\big)(K). \tag{2.3}$$

In (i)–(iii) below we prove that φ and $\mathcal{K}$ satisfy (†).

(i) We show that $\sigma_{K\in\mathcal{K}} C_p(K) \subset \varphi[C_p(U)]$. Fix a finite $\mathcal{K}' \subset \mathcal{K}$ and

$$(f_K)_{K\in\mathcal{K}'} \in \prod_{K\in\mathcal{K}'} C_p(K).$$

There is no loss of generality to assume that $C \in \mathcal{K}'$, because otherwise we may consider $\mathcal{K}'' = \mathcal{K}' \cup \{C\}$ and set $f_C = 0$. For every $K \in \mathcal{K}' \setminus \{C\}$ find (the unique) $V_K \in \mathcal{V}$ such that $K \in \mathcal{K}_{V_K}$. For every $V \in \{V_K : K \in \mathcal{K}'\}$ find $f_V \in C_p(V)$ such that for each $K \in \mathcal{K}'$ with $V_K = V$ it follows that

$$\varphi_V(f_V)(K) = f_K. \tag{2.4}$$

Such an f_V exists by our assumptions on φ_V. Set

$$U' := U \setminus \bigcup\{V_K : K \in \mathcal{K}' \setminus \{C\}\},$$

so U' is a clopen subset of X containing C. Define $f \in C_p(U)$ by

$$f(x) := \begin{cases} \psi(f_C)(x) & \text{, if } x \in U', \\ \psi(f_C)(x) + f_{V_K}(x), & \text{if } x \in V_K \text{ and } K \in \mathcal{K}' \setminus \{C\}. \end{cases} \tag{2.5}$$

We claim that $\varphi(f)(K)$ equals f_K for $K \in \mathcal{K}'$ and 0 otherwise, that proves (i). Indeed, fix $K \in \mathcal{K}'$. If $K = C \subseteq U'$, then

$$\varphi(f)(C) \overset{(2.2)}{=} f|_C \overset{(2.5)}{=} \psi(f_C)|_C = f_C.$$

If $K \in \mathcal{K}' \setminus \{C\}$, then

$$\begin{aligned}\varphi(f)(K) &\overset{(2.3)}{=} \varphi_{V_K}\big((f - \psi(f|_C))|_{V_K}\big)(K) = \varphi_{V_K}\big(f|_{V_K} - \psi(f_C)|_{V_K}\big)(K) \\ &\overset{(2.5)}{=} \varphi_{V_K}\big((\psi(f_C)|_{V_K} + f_{V_K}) - \psi(f_C)|_{V_K}\big)(K) = \varphi_{V_K}\big(f_{V_K}\big)(K) \overset{(2.4)}{=} f_K.\end{aligned}$$

Finally, if $K \in \mathcal{K} \setminus \mathcal{K}'$, then $K \subseteq U'$ and

$$\varphi(f)(K) \overset{(2.3)}{=} \varphi_{V_K}\big((f - \psi(f|_C))|_{V_K}\big)(K) = \varphi_{V_K}\big(f|_{V_K} - \psi(f_C)|_{V_K}\big)(K)$$
$$\overset{(2.5)}{=} \varphi_{V_K}\big(\psi(f_C)|_{V_K} - \psi(f_C)|_{V_K}\big)(K) = \varphi_{V_K}\big(0\big)(K) = 0.$$

(ii) Let us prove that φ is injective. Assume that $\varphi(f) = \varphi(g)$. Set $h := \varphi(f)(C) = f|_C = g|_C$. Given any $V \in \mathcal{V}$ and $K \in \mathcal{K}_V$, the equality $\varphi(f)(K) = \varphi(g)(K)$ and (2.3) imply

$$\varphi_V\big((f - \psi(h))|_V\big)(K) = \varphi_V\big((g - \psi(h))|_V\big)(K),$$

and hence $(f - \psi(h))|_V = (g - \psi(h))|_V$ by the injectivity of φ_V. Consequently, $f|_V = g|_V$, and therefore $f = g$ because $V \in \mathcal{V}$ was chosen arbitrarily.

(iii) We show that $\varphi^{-1} : \varphi[C_p(U)] \to C_p(U)$ is continuous. Fix a finite subset F of U and $\varepsilon > 0$. Passing to a larger F if necessary we may assume that $F = F_C \cup \bigcup\{F_i : i \le n\}$, where $F_C \in [C]^{<\omega}$ and $F_i \in [V_i]^{<\omega}$ for some $V_i \in \mathcal{V}$ such that $V_i \ne V_j$ for $i \ne j$. We need to find an open neighborhood W of

$$(0_K) \in \prod_{K \in \mathcal{K}} C_p(K)$$

such that $f \in [0, F, \varepsilon]$ whenever $\varphi(f) \in W$. Let $A_C \in [C]^{<\omega}$ and $\delta > 0$ be such that $F_C \subset A_C$, $\delta < \varepsilon$, and

$$\psi[0, A_C, \delta] \subset [0, F, \varepsilon/2].$$

(Here of course $[0, A_C, \delta]$ and $[0, F, \varepsilon/2]$ are considered as subsets of $C_p(C)$ and $C_p(U)$, respectively.) Since φ_{V_i} is an embedding, there exists an open neighborhood W_i of

$$(0_K) \in \prod_{K \in \mathcal{K}_{V_i}} C_p(K)$$

such that $h \in C_p(V_i)$ lies in $[0, F_i, \varepsilon/2]$ whenever $\varphi_{V_i}(h) \in W_i$. Consider

$$W = W_C \times \prod_{V \in \mathcal{V}} W_V$$

such that

$$W_C = [0, A_C, \delta] \subset C_p(C), \;\; W_{V_i} = W_i \subset \prod_{K \in \mathcal{K}_{V_i}} C_p(K),$$

and $W_V = \prod_{K \in \mathcal{K}_V} C_p(K)$ for $V \notin \{V_i : i \le n\}$. Assume that $\varphi(f) \in W$ for some $f \in C_p(U)$. Then

$$\varphi(f)(C) = f|_C \in [0, A_C, \delta] \subset [0, F_C, \varepsilon],$$

and hence $\psi(f|_C)|_{V_i} \in [0, F_i, \varepsilon/2]$ for all $i \le n$. Fix $i \le n$ and observe that $\varphi(f) \in W$ implies $\varphi(f) \restriction \mathcal{K}_{V_i} \in W_i$; therefore, see also (2.3), $\varphi_{V_i}(h_i) \in W_i$ for $h_i = (f - \psi(f|_C))|_{V_i}$. It follows from the above that $h_i \in [0, F_i, \varepsilon/2] \subset C_p(V_i)$. Since $\psi(f|_C))|_{V_i} \in [0, F_i, \varepsilon/2]$ and $h_i \in [0, F_i, \varepsilon/2]$, we have that

$$f|_{V_i} = h_i + \psi(f|_C))|_{V_i} \in [0, F_i, \varepsilon]$$

which completes our proof in Case 1.

Case 2. Assume that $x \notin U$. This case is similar but simpler than the previous one. Given any $y \in U$, set $V_0(y) = O(y) \cap U$. Then $\mathcal{V}_0 = \{V_0(y) : y \in U\}$ is an open cover of a paracompact scattered space U. So there exists a clopen cover $\mathcal{V} \prec \mathcal{V}_0$ of U whose elements are mutually disjoint, see [28]. It follows from the above that each $V \in \mathcal{V}$ has the property (†), and hence there exists a family $\mathcal{K}_V$ of scattered compact subsets of V such that $C_p(V)$ can be topologically embedded into $\prod_{K\in\mathcal{K}_V} C_p(K)$ via a linear continuous map

$$\varphi_V : C_p(V) \to \prod_{K\in\mathcal{K}_V} C_p(K)$$

such that

$$\varphi_V[C_p(V)] \supset \sigma_{K\in\mathcal{K}_V} C_p(K).$$

Set $\mathcal{K} := \bigcup\{\mathcal{K}_V : V \in \mathcal{V}\}$ and

$$\varphi = (\varphi_V)_{V\in\mathcal{V}} : C_p(U) = \prod_{V\in\mathcal{V}} C_p(V) \to \prod_{V\in\mathcal{V}} \prod_{K\in\mathcal{K}_V} C_p(K) = \prod\{C_p(K) : K \in \mathcal{K}\}.$$

A direct verification shows that φ is a linear embedding and $\varphi[C_p(U)]$ contains $\sigma_{K\in\mathcal{K}} C_p(K)$. □

Now we complete the proof of the proposition. By Claim 2.9, for every $x \in X$ choose a clopen neighborhood $O(x)$ of x with the property (†). Then $\mathcal{V}_0 = \{O(x) : x \in X\}$ is an open cover of a paracompact scattered space X. By the same argument as in the proof of Case 2 of Claim 2.9 we get that there exists a family $\mathcal{K}$ of scattered compact spaces such that $C_p(X)$ is linearly homeomorphic to a linear subspace of $\prod_{K\in\mathcal{K}} C_p(K)$ containing $\sigma_{K\in\mathcal{K}} C_p(K)$. The latter σ-product is dense in $C_p(X)$ as it is dense in $\prod_{K\in\mathcal{K}} C_p(K)$. For any countable $\mathcal{K}' \subset \mathcal{K}$ the topological sum $\oplus\mathcal{K}'$ is a Lindelöf scattered space, and hence

$$\prod_{K\in\mathcal{K}'} C_p(K) = C_p(\oplus\mathcal{K}')$$

is Fréchet–Urysohn by [2, Theorem II.7.16]. So $\sigma_{K\in\mathcal{K}} C_p(K)$ is Fréchet–Urysohn by Proposition 2.5, and hence $C_p(X)$ can be covered by its dense Fréchet–Urysohn subspaces (namely shifts of $\sigma_{K\in\mathcal{K}} C_p(K)$). Thus $C_p(X)$ is Ascoli by Lemma 2.7. □

Clearly if X has finitely many non-isolated points, then X has the property ($\star$). Therefore we have the following

Corollary 2.10. *If X has finitely many non-isolated points then $C_p(X)$ is Ascoli.*

A regular topological space X is *stratifiable* if there is a function G which assigns to every $n \in \omega$ and each closed set $F \subset X$ an open neighborhood $G(n, F) \subset X$ of F such that $F = \bigcap_{n\in\omega} \overline{G(n, F)}$ and $G(n, F) \subset G(n, F')$ for any $n \in \omega$ and closed sets $F \subset F' \subset X$. Borges proved in [4] that to each stratifiable space X the Dugundji extension theorem is applicable: For every closed subset A of X there is a continuous linear operator $\psi : C_k(A) \to C_k(X)$ such that $\psi(g)|_A = g$ for every $g \in C_k(A)$. Any metrizable space is stratifiable, and each stratifiable space is paracompact, see [13, Theorem 5.7]. Any subspace of a stratifiable space is stratifiable and hence is paracompact.

Proof of Theorem 1.3. (i) follows from Theorems 2.3 and 1.1 and Proposition 2.5.

(ii) By Proposition 2.8 it is sufficient to show that every scattered stratifiable space has the property ($\star$). For every $x \in X$, let $O(x)$ be an arbitrary clopen neighborhood of x and let $C = \{x\}$. Now for every clopen U with $x \in U \subseteq O(x)$, the difference $U \setminus C$ is paracompact and there is a continuous linear operator $\psi : C_k(C) \to C_k(U)$. At the end of page 9 in [4] Borges proved that the operator ψ is also continuous as a map from $C_p(C)$ to $C_p(U)$. Thus X satisfies the property $(\star)$. □

In light of Theorem 1.3 it is natural to ask the following

Question 2.11. *Does every scattered Čech-complete space have the property* $(\star)$*?*

The following corollary complements Theorem II.7.16 of [2] and immediately implies Corollary 1.5.

Corollary 2.12. *For a Čech-complete Lindelöf space X, the following assertions are equivalent:*

(i) *$C_p(X)$ is Ascoli;*
(ii) *$C_p(X)$ is Fréchet–Urysohn;*
(iii) *X is scattered;*
(iv) *X is scattered and σ-compact.*

Proof. (i)⇒(iii) follows from (i) of Theorem 1.3, (iii)⇒(ii) follows from [2, Theorem II.7.16], and (ii)⇒(i) is trivial. (iii) ⇒ (iv): If X is a Čech-complete Lindelöf space, then by Frolik's theorem, see [7], there exists a Polish space Y and a perfect map from X onto Y. As being scattered is inherited by perfect maps, the space Y is scattered, hence countable by [27, 8.5.5]. Consequently X is σ-compact. □

The famous Pytkeev–Gerlitz–Nagy theorem, see [2, Theorem II.3.7], states that $C_p(X)$ is a k-space if and only if $C_p(X)$ is Fréchet–Urysohn if and only if X has the covering property (γ) introduced in [12]. Below we give an example of a separable metrizable space X for which $C_p(X)$ is Ascoli but is not a k-space. So the property to be an Ascoli space is strictly weaker than the property to be a k-space for $C_p(X)$ even in the class of separable metric spaces.

Recall that a separable metric space X is said to be a λ-*space* if every countable subset of X is a G_δ-set of X. Every λ-space has the property (κ) by [26, Theorem 3.2]. So $C_p(X)$ is Ascoli by Corollary 1.2 for such space X.

Example 2.13. Rothberger proved in [24] that there is an unbounded subset X of ω^ω which is a λ-space, see also [20, p. 215]. So X is a separable metrizable space with the property (κ) by Theorem 3.2 of [26]. Therefore $C_p(X)$ is an Ascoli space by Theorem 2.3 and Corollary 1.2. However, it follows from the results of Gerlits and Nagy [12] that no unbounded subset of ω^ω has the property (γ), and hence $C_p(X)$ is not Fréchet–Urysohn. So $C_p(X)$ is not a k-space by the Pytkeev–Gerlitz–Nagy theorem.

Question 2.14. *Let X be an uncountable cosmic space such that $C_p(X)$ is Ascoli (for example, X is a λ-space). Is then $C_p(X)$ a $k_{\mathbb{R}}$-space?*

The negative answer to this question would give an example of an Ascoli space $C_p(X)$ for separable metrizable X which is not a $k_{\mathbb{R}}$-space. Let us note that the example provided in [11] is not metrizable.

The assumption to be Čech-complete is essential for the results of this section as the metrizable space $C_p(\mathbb{Q})$ shows. We end this section with the following question.

Question 2.15. *For which metrizable spaces X the space $C_p(X)$ is Ascoli?*

3. The Ascoli property for $C_k(X)$

Let X be a Tychonoff space and $\mathcal{K}(X)$ be the set of all compact subsets of X. For $h \in C(X)$ the sets of the form

$$[h, K, \varepsilon] := \{f \in C(X) : |f(x) - h(x)| < \varepsilon \text{ for all } x \in K\}, \text{ where } K \in \mathcal{K}(X) \text{ and } \varepsilon > 0,$$

form a base at h for the *compact-open* topology τ_k on $C(X)$. The space $C(X)$ equipped with τ_k is usually denoted by $C_k(X)$.

Theorem 2.5 of [10] states in particular that, for a first-countable paracompact σ-space X, the space $C_k(X)$ is an Ascoli space if and only if $C_k(X)$ is a $k_{\mathbb{R}}$-space if and only if X is a locally compact metrizable space. In this section we prove an analogous result using the following proposition.

Proposition 3.1 *([10]). Assume X admits a family $\mathcal{U} = \{U_i : i \in I\}$ of open subsets of X, a subset $A = \{a_i : i \in I\} \subset X$ and a point $z \in X$ such that*

(i) *$a_i \in U_i$ for every $i \in I$;*
(ii) *$\big|\{i \in I : C \cap U_i \neq \emptyset\}\big| < \infty$ for each compact subset C of X;*
(iii) *z is a cluster point of A.*

Then X is not an Ascoli space.

Recall that X is of *point-countable type* if for every $x \in X$ there exists a compact K containing x such that K has countable basis of neighborhoods, i.e. there is a sequence of open sets $\{U_n\}_{n<\omega}$ such that $K \subseteq U_n$ for all $n < \omega$ and for every open O containing K there is $n < \omega$ such that $U_n \subseteq O$. The following statement is reminiscent of [10, Proposition 2.3], and substantially uses the idea of R. Pol from [23]. We say that a space X is *locally pseudocompact* if for every $x \in X$ there exists an open $U \ni x$ whose closure $\bar{U}$ is pseudocompact.

Lemma 3.2. *Let X be a space of point-countable type. If $C_k(X)$ or $C_k(X, \mathbb{I})$ is an Ascoli space, then X is locally pseudocompact.*

Proof. Assume that X is not locally pseudocompact, so there exists $x_0 \in X$ such that no neighborhood of x_0 is pseudocompact. Because X is of point-countable type there is a compact set $K \subset X$ such that $x_0 \in K$ and there is a base of neighborhoods $\{U_n\}_{n\in\omega}$ of K such that $\overline{U_{n+1}} \subsetneq U_n$ (here we use the fact that K is compact and X is Tychonoff).

We show that there is a strictly increasing sequence $\{n_k\}_{k\in\omega}$ such that $n_{k+1} > n_k + 1$ and for every $k \in \omega$, the difference $\overline{U_{n_k}} \setminus U_{n_k+1}$ is not pseudocompact. Indeed, otherwise there exists n_0 such that $\overline{U_n} \setminus U_{n+1}$ is pseudocompact for all $n \geq n_0$. We claim that $\overline{U_{n_0}}$ is a pseudocompact neighborhood of x_0 which leads to a contradiction. Given any continuous $f : \overline{U_{n_0}} \to \mathbb{R}$, there exists $m \in \mathbb{R}$ such that $f^{-1}[(-m, m)]$ is an open set containing K, and therefore it contains some U_{n_1}, which together with the pseudocompactness of $\overline{U_{n_0}} \setminus U_{n_1}$ implies that f is bounded.

Set $P_k := \overline{U_{n_k}} \setminus U_{n_k+1}$. Since every P_k is not pseudocompact, by [8, Theorem 3.10.22] there exists a locally finite collection $\{U_{i,k}\}_{i<\omega}$ of nonempty open subsets of P_k. We may assume in addition that every $U_{i,k} \subseteq \mathrm{Int}(P_k)$. Pick any $x_{i,k} \in U_{i,k}$, and for $1 \leq k < i$ find continuous functions $f_{i,k} : X \to [0, 1]$ such that

$$f_{i,k}(x_{i,k}) = 1, \ f_{i,k}(x_{i,i}) = 0, \text{ and } \ f_{i,k}(x) = \frac{1}{k} \text{ for } \ x \notin U_{i,k} \cup U_{i,i}.$$

Set $A := \{f_{i,k} : 1 \leq k < i < \omega\}$ and $\mathcal{V} := \{V_{i,k}\}_{1\leq k<i<\omega}$, where $V_{i,k} \subset C_k(X)$ or $V_{i,k} \subset C_k(X, \mathbb{I})$ and $h \in V_{i,k}$ if

$$|h(x_{i,k})-1|<\frac{1}{4^{i+k}},\ |h(x_{i,i})|<\frac{1}{4^{i+k}},\ \text{and}\ \left|h(x)-\frac{1}{k}\right|<\frac{1}{4^{i+k}}\ \text{for all}\ x\in K.$$

We shall complete the proof by showing that A, $\mathcal{V}$ and 0 fulfill the assumption of Proposition 3.1. The first one is by definition. For (iii), assume that $Z \subset X$ is compact and fix $\varepsilon > 0$. Find $k < \omega$ such that $\frac{1}{k} < \varepsilon$ and $i > k$ such that $Z \cap U_{i,k} = \emptyset$ (this is possible because Z is compact and $\{U_{i,k}\}_{i<\omega}$ is a locally finite collection). It follows that

$$f_{i,k}(z) \le \frac{1}{k} < \varepsilon$$

for every $z \in Z$. Thus $0 \in \overline{A}$.

Let us check (ii): any compact subset C of $C_k(X)$ or of $C_k(X, \mathbb{I})$ meets only finitely many elements of $\mathcal{V}$. By the Ascoli theorem [8, Theorem 3.4.20], for every compact $Z \subset X$, $x \in Z$ and $\varepsilon > 0$ there is a neighborhood O_x of x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in O_x \cap Z$ and $f \in C$. Define

$$Z_0 := \{x_{i,k} : 1 \le i \le k < \omega\} \cup K,$$

and note that Z_0 is a compact subset of X.

We claim that for every $k < \omega$ there is $i_0 > k$ such that $C \cap V_{i,k} = \emptyset$ for every $i > i_0$. Indeed, assume the converse. Using the Ascoli theorem for C, Z_0 and $\varepsilon = \frac{1}{3k}$, for every $x \in Z_0$ we find a neighborhood O_x of x such that $|h(y) - h(x)| < \varepsilon$ for every $y \in O_x$ and $h \in C$. Then the collection $\{O_x\}_{x\in Z_0}$ covers $K \subset Z_0$, so there exists i_0 such that $U_{i_0} \subset \bigcup_{x\in Z_0} O_x$. Take any $i > i_0$, $h \in C \cap V_{i,k}$ and $x \in K$ such that $x_{i,i} \in O_x$ (recall that $x_{i,i} \in U_{i,i} \subset P_i \subset U_{n_{i_0}}$, and clearly $n_{i_0} \ge i_0$). By construction,

$$K \cap (U_{i,k} \cup U_{i,i}) = \emptyset,$$

so $f_{i,k}(x) = 1/k$ and $f_{i,k}(x_{i,i}) = 0$. Since $h \in C \cap V_{i,k}$ we obtain

$$\begin{aligned}\frac{1}{3k} &> |h(x_{i,i})-h(x)| \ge |f_{i,k}(x_{i,i})-f_{i,k}(x)| - |f_{i,k}(x_{i,i})-h(x_{i,i})| - |h(x)-f_{i,k}(x)|\\ &> \frac{1}{k}-\frac{1}{4^{i+k}}-\frac{1}{4^{i+k}} > \frac{1}{3k},\end{aligned}$$

a contradiction. This contradiction proves the claim.

To finish the proof it is sufficient to show that there is no sequence $\{(i_n, k_n)\}_{n<\omega}$ such that

$$\ldots < k_n < i_n < k_{n+1} < i_{n+1} < \ldots$$

and $V_{i_n,k_n} \cap C \ne \emptyset$. If not, consider the compact subset $Z_1 := \{x_{i_n,k_n} : n < \omega\} \cup K$ of X. Using the Ascoli theorem for C, Z_1 and $1/3$, for every $x \in Z_1$ we find a neighborhood O_x of x such that $|h(y) - h(x)| < 1/3$ for every $y \in O_x$ and $h \in C$. Again, the collection $\{O_x\}_{x\in K}$ covers $K \subset Z_1$, so there exists $k > 10$ such that $U_k \subset \bigcup_{x\in K} O_x$. Pick n such that $k < k_n$ and note that there is $x \in K$ such that $x_{i_n,k_n} \in O_x$. Then, as above, for any $h \in V_{i_n,k_n} \cap C$ we have

$$\begin{aligned}\frac{1}{3} &> |h(x_{i_n,k_n})-h(x)|\\ &\ge |f_{i_n,k_n}(x_{i_n,k_n})-f_{i_n,k_n}(x)| - |f_{i_n,k_n}(x_{i_n,k_n})-h(x_{i_n,k_n})| - |h(x)-f_{i_n,k_n}(x)|\\ &> (1-1/k_n) - 4^{-(i_n+k_n)} - 4^{-(i_n+k_n)} > \frac{1}{3},\end{aligned}$$

which is the desired contradiction. □

We need the following result.

Lemma 3.3. *Every paracompact locally pseudocompact space X is locally compact.*

Proof. Let $x \in X$ and take a neighborhood U of x with pseudocompact closure $\overline{U}$. Then $\overline{U}$ is compact being pseudocompact and paracompact, see, e.g., [8, 3.10.21, 5.1.5, and 5.1.20]. □

Now we are ready to prove the main result of this section.

Proof of Theorem 1.6. (i)⇒(ii) follows from [8, 5.1.27].

(ii)⇒(iii),(v): If $X = \bigoplus_{i\in\kappa} X_i$, then

$$C_k(X) = \prod_{i\in\kappa} C_k(X_i) \quad \text{and} \quad C_k(X,\mathbb{I}) = \prod_{i\in\kappa} C_k(X_i,\mathbb{I}),$$

where all the spaces $C_k(X_i)$ and $C_k(X_i,\mathbb{I})$ are complete metrizable. So $C_k(X)$ and $C_k(X,\mathbb{I})$ are $k_{\mathbb{R}}$-spaces by [22, Theorem 5.6].

(iii)⇒(iv) and (v)⇒(vi) follow from [21]. The implications (iv)⇒(i) and (vi)⇒(i) follow from Lemmas 3.2 and 3.3. □

Theorem 1.6 also holds for some spaces without point-countable type.

Example 3.4. Let $X = D\cup\{\infty\}$ be the one point Lindelöfication of an uncountable discrete space D. Clearly, X is scattered and Lindelöf. Since any compact subset of X is finite and D is uncountable, the space X is not of point-countable type. Nevertheless, $C_k(X) = C_p(X)$ is Ascoli by Corollary 2.10.

The following statement probably belongs to folklore.

Lemma 3.5. *Let X be a paracompact space which is not Lindelöf. Then ω^{ω_1} can be embedded into $C_k(X)$ as a closed subspace, where ω is considered with the discrete topology.*

Proof. Since X is paracompact and non-Lindelöf, Lemma 2.2 of [5] implies that there is an uncountable $A \subset X$ and open $U_a \ni a$ for every $a \in A$ such that each $x \in X$ has a neighborhood which meets at most one of the U_a's. Set

$$Z := \{f \in C_k(X) : f \restriction (X \setminus \bigcup_{a\in A} U_a) = 0\} \ \text{ and } \ Z_a := \{f \in C_k(X) : f \restriction (X \setminus U_a) = 0\}.$$

Then Z is a closed subspace of $C_k(X)$ and $Z = \prod_{a\in A} Z_a$. It suffices to note that each Z_a contains a closed copy of $\mathbb{R}$ (and hence of ω) being a linear topological space. □

Recall that a *compact resolution* in a topological space X is a family $\{K_\alpha : \alpha \in \omega^\omega\}$ of compact subsets of X which covers X and satisfies the condition: $K_\alpha \subseteq K_\beta$ whenever $\alpha \le \beta$ for all $\alpha, \beta \in \omega^\omega$.

Lemma 3.6. *Let X be a paracompact space with compact resolution. Then X is Lindelöf.*

Proof. Suppose for a contradiction that X is not Lindelöf. Then X contains a closed discrete uncountable subset Y by [5, Lemma 2.2]. Hence the compact resolution restricted to Y is also a compact resolution on Y. So Y is a metric space with a compact resolution. Therefore Y is separable by [15, Corollary 6.2], and hence it is countable being discrete, a contradiction. □

Recall that a space X is *hemicompact* if it has a countable family of compact subspaces which is cofinal with respect to inclusion in the family of all of its compact subspaces. The following theorem extends Corollary 4 of [17].

Theorem 3.7. *Let X be a paracompact space of point-countable type. Then the following conditions are equivalent:*

(i) *X is hemicompact;*
(ii) *$C_k(X)$ is a k-space;*
(iii) *$C_k(X)$ is Ascoli and X has a compact resolution.*

Proof. (i)⇒(ii) is clear. (ii)⇒(iii) Assume that $C_k(X)$ is a k-space. Then $C_k(X)$ is Ascoli. Hence X is locally compact by Theorem 1.6. Moreover X is Lindelöf. Indeed, if not, then $C_k(X)$ contains as a closed subset the product ω^{ω_1} by Lemma 3.5, a contradiction since ω^{ω_1} is not a k-space. Hence X is Lindelöf. Consequently X is hemicompact. Thus X has a compact resolution. (iii)⇒(i) Since $C_k(X)$ is Ascoli, X is locally compact by Theorem 1.6. Now Lemma 3.6 implies that X is Lindelöf, so X is hemicompact. □

We need the following lemma.

Lemma 3.8. *Let X be a non-discrete locally compact space. Then $C_p(X, \mathbb{I})$ contains a closed infinite discrete subspace.*

Proof. Observe that $C_p(X, \mathbb{I})$ is countably compact if and only if X is a P-space by Problem 397 of [29], and a non-discrete locally compact space X is never has the P-property, so $C_p(X, \mathbb{I})$ is not countably compact. □

The next theorem generalizes a result of R. Pol [23].

Theorem 3.9. *Let X be a paracompact space of point-countable type. Then:*

(i) *$C_k(X, \mathbb{I})$ is a k-space if and only if X is the topological sum of a Lindelöf locally compact space L and a discrete space D; so $C_k(X, \mathbb{I}) = C_k(L, \mathbb{I}) \times \mathbb{I}^{|D|}$, where $C_k(L, \mathbb{I})$ is a complete metrizable space;*
(ii) *$C_k(X, \mathbb{I})$ is a sequential space if and only if $C_k(X, \mathbb{I})$ is a complete metrizable space if and only if X is a Lindelöf locally compact space.*

Proof. (i) If $C_k(X, \mathbb{I})$ is a k-space, then X is a locally compact space by Lemmas 3.2 and 3.3. So $X = \bigoplus_{i \in I} X_i$ is the direct sum of a family $\{X_i\}_{i \in I}$ of Lindelöf locally compact spaces by [8, 5.1.27]. Denote by J the set of all $i \in I$ for which X_i is not discrete. To prove (i) we have to show that J is countable. Suppose for a contradiction that J is uncountable. Then $C_p(X_i, \mathbb{I})$ and hence $C_k(X_i, \mathbb{I})$ contains a closed infinite discrete subspace D_i topologically isomorphic to ω by Lemma 3.8. So the space

$$C_k(X, \mathbb{I}) = \prod_{i \in J} C_k(X_i, \mathbb{I}) \times \prod_{i \in I \setminus J} C_k(X_i, \mathbb{I})$$

contains $\omega^{|J|}$ as a closed subspace. As J is uncountable we obtain that $\omega^{|J|}$ is not a k-space. This contradiction shows that J must be countable. Setting $L := \bigcup_{i \in J} X_i$ and $D := \bigcup_{i \in I \setminus J} X_i$ we obtain the desired decomposition. The converse assertion is trivial.

(ii) If $C(X, \mathbb{I})$ is a sequential space, it follows from (i) that D is countable. Indeed, the space $\mathbb{I}^{|D|}$ contains $2^{|D|}$ as a closed subspace and it is well-known that $2^{|D|}$ is sequential (even has countable tightness) if and only if D is countable. So X is a Lindelöf locally compact space. If X is Lindelöf and locally compact space, then $C_k(X)$ and hence its closed subspace $C(X, \mathbb{I})$ are complete metrizable spaces. □

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