



# On Valdivia strong version of Nikodym boundedness property <sup>☆</sup>



J. Kąkol <sup>a,b</sup>, M. López-Pellicer <sup>c,\*</sup>

<sup>a</sup> Adam Mickiewicz University, Poznań, Poland

<sup>b</sup> Institute of Mathematics, Czech Academy of Sciences, Czech Republic

<sup>c</sup> Department of Applied Mathematics and IUMPA, Universitat Politècnica de València, València, Spain

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## ABSTRACT

Following Schachermer, a subset  $\mathcal{B}$  of an algebra  $\mathcal{A}$  of subsets of  $\Omega$  is said to have the *N-property* if a  $\mathcal{B}$ -pointwise bounded subset  $M$  of  $ba(\mathcal{A})$  is uniformly bounded on  $\mathcal{A}$ , where  $ba(\mathcal{A})$  is the Banach space of the real (or complex) finitely additive measures of bounded variation defined on  $\mathcal{A}$ . Moreover  $\mathcal{B}$  is said to have the *strong N-property* if for each increasing countable covering  $(\mathcal{B}_m)_m$  of  $\mathcal{B}$  there exists  $\mathcal{B}_n$  which has the *N-property*. The classical Nikodym–Grothendieck’s theorem says that each  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  has the *N-property*. The Valdivia’s theorem stating that each  $\sigma$ -algebra  $\mathcal{S}$  has the strong *N-property* motivated the main measure-theoretic result of this paper: We show that if  $(\mathcal{B}_{m_1})_{m_1}$  is an increasing countable covering of a  $\sigma$ -algebra  $\mathcal{S}$  and if  $(\mathcal{B}_{m_1, m_2, \dots, m_p, m_{p+1}})_{m_{p+1}}$  is an increasing countable covering of  $\mathcal{B}_{m_1, m_2, \dots, m_p}$ , for each  $p, m_i \in \mathbb{N}$ ,  $1 \leq i \leq p$ , then there exists a sequence  $(n_i)_i$  such that each  $\mathcal{B}_{n_1, n_2, \dots, n_r}$ ,  $r \in \mathbb{N}$ , has the strong *N-property*. In particular, for each increasing countable covering  $(\mathcal{B}_m)_m$  of a  $\sigma$ -algebra  $\mathcal{S}$  there exists  $\mathcal{B}_n$  which has the strong *N-property*, improving mentioned Valdivia’s theorem. Some applications to localization of bounded additive vector measures are provided.

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## 1. Introduction

Let  $\mathcal{B}$  be a subset of an algebra  $\mathcal{A}$  of subsets of a set  $\Omega$  (in brief, set-algebra  $\mathcal{A}$ ). The normed space  $L(\mathcal{B})$  is the  $span\{\chi_C : C \in \mathcal{B}\}$  of the characteristic functions of each set  $C \in \mathcal{B}$  with the supremum norm  $\|\cdot\|$  and  $ba(\mathcal{A})$  is the Banach space of finitely additive measures on  $\mathcal{A}$  with bounded variation endowed with the variation norm, i.e.,  $|\cdot| := |\cdot|(\Omega)$ . If  $\{C_i : 1 \leq i \leq n\}$  is a measurable partition of  $C \in \mathcal{A}$  and  $\mu \in ba(\mathcal{A})$  then  $|\mu|(C) = \sum_i |\mu|(C_i)$  and, as usual, we represent also by  $\mu$  the linear form in  $L(\mathcal{A})$  determined by

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\* Corresponding author.

E-mail addresses: kakol@amu.edu.pl (J. Kąkol), mlopezpe@mat.upv.es (M. López-Pellicer).

$\mu(\chi_C) := \mu(C)$ , for each  $C \in \mathcal{A}$ . By this identification we get that the dual of  $L(\mathcal{A})$  with the dual norm is isometric to  $ba(\mathcal{A})$  (see e.g., [2, Theorem 1.13]).

Polar sets are considered in the dual pair  $\langle L(\mathcal{A}), ba(\mathcal{A}) \rangle$ ,  $M^\circ$  means the polar of a set  $M$  and if  $\mathcal{B} \subset \mathcal{A}$  the topology in  $ba(\mathcal{A})$  of pointwise convergence in  $\mathcal{B}$  is denoted by  $\tau_s(\mathcal{B})$ .  $(E', \tau_s(E))$  is the vector space of all continuous linear forms defined on a locally convex space  $E$  endowed with the topology  $\tau_s(E)$  of the pointwise convergence in  $E$ . In particular, the topology  $\tau_s(L(\mathcal{A}))$  in  $ba(\mathcal{A})$  is  $\tau_s(\mathcal{A})$ .

The convex (absolutely convex) hull of a subset  $M$  of a topological vector space is denoted by  $co(M)$  ( $absco(M)$ ) and  $absco(M) = co(\cup\{rM : |r| = 1\})$ . An equivalent norm to the supremum norm in  $L(\mathcal{A})$  is the Minkowski functional of  $absco(\{\chi_C : C \in \mathcal{A}\})$  ([14, Propositions 1 and 2]) and its dual norm is the  $\mathcal{A}$ -supremum norm, i.e.,  $\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$ ,  $\mu \in ba(\mathcal{A})$ . The closure of a set is marked by an overline, hence if  $P \subset L(\mathcal{A})$  then  $\overline{span(P)}$  is the closure in  $L(\mathcal{A})$  of the linear hull of  $P$ .  $\mathbb{N}$  is the set  $\{1, 2, \dots\}$  of positive integers.

Recall the classical Nikodym–Dieudonné–Grothendieck theorem (see [1, page 80, named as [Nikodym–Grothendieck boundedness theorem](#)]): *If  $\mathcal{S}$  is a  $\sigma$ -algebra of subsets of a set  $\Omega$  and  $M$  is a  $\mathcal{S}$ -pointwise bounded subset of  $ba(\mathcal{S})$  then  $M$  is a bounded subset of  $ba(\mathcal{S})$  (i.e.,  $\sup\{|\mu(C)| : \mu \in M, C \in \mathcal{S}\} < \infty$ , or, equivalently,  $\sup\{|\mu|(\Omega) : \mu \in M\} < \infty$ ).* This theorem was firstly obtained by Nikodym in [11] for a subset  $M$  of countably additive complex measures defined on  $\mathcal{S}$  and later on by Dieudonné for a subset  $M$  of  $ba(2^\Omega)$ , where  $2^\Omega$  is the  $\sigma$ -algebra of all subsets of  $\Omega$ , see [3].

It is said that a subset  $\mathcal{B}$  of an algebra  $\mathcal{A}$  of subsets of a set  $\Omega$  has the *Nikodym property*,  $N$ -property in brief, if the Nikodym–Dieudonné–Grothendieck theorem holds for  $\mathcal{B}$ , i.e., *if each  $\mathcal{B}$ -pointwise bounded subset  $M$  of  $ba(\mathcal{A})$  is bounded in  $ba(\mathcal{A})$*  (see [12, Definition 2.4] or [15, Definition 1]). Let us note that in this definition we may suppose that  $M$  is  $\tau_s(\mathcal{A})$ -closed and absolutely convex. If  $\mathcal{B}$  has  $N$ -property then the polar set  $\{\chi_C : C \in \mathcal{B}\}^\circ$  is bounded in  $ba(\mathcal{A})$ , hence  $\{\chi_C : C \in \mathcal{B}\}^{\circ\circ} = \overline{absco\{\chi_C : C \in \mathcal{B}\}}$  is a neighborhood of zero in  $L(\mathcal{A})$ , whence  $L(\mathcal{B})$  is dense in  $L(\mathcal{A})$ .

It is well known that *the algebra of finite and co-finite subsets of  $\mathbb{N}$  fails  $N$ -property* [2, Example 5 in page 18] and that Schachermayer proved that *the algebra  $\mathcal{J}(I)$  of Jordan measurable subsets of  $I := [0, 1]$  has  $N$ -property* (see [12, Corollary 3.5] and a generalization in [4, Corollary]). A recent improvement of this result for the algebra  $\mathcal{J}(K)$  of Jordan measurable subsets of a compact  $k$ -dimensional interval  $K := \Pi\{[a_i, b_i] : 1 \leq i \leq k\}$  in  $\mathbb{R}^k$  has been provided in [15, Theorem 2], where Valdivia proved that *if  $\mathcal{J}(K)$  is the increasing countable union  $\cup_m \mathcal{B}_m$  there exists a positive integer  $n$  such that  $\mathcal{B}_n$  has  $N$ -property* (see [8, Theorem 1] for a strong result in  $\mathcal{J}(K)$ ). This fact motivated to say that a subset  $\mathcal{B}$  of a set-algebra  $\mathcal{A}$  has the *strong Nikodym property*,  $sN$ -property in brief, if for each increasing covering  $\cup_m \mathcal{B}_m$  of  $\mathcal{B}$  there exists  $\mathcal{B}_n$  which has  $N$ -property. As far as we know this result suggested the following very interesting Valdivia's open question (2013):

**Problem 1** ([15, Problem 1]). Let  $\mathcal{A}$  be an algebra of subsets of  $\Omega$ . Is it true that  $N$ -property of  $\mathcal{A}$  implies  $sN$ -property?

Note that the Nikodym–Dieudonné–Grothendieck stating that every  $\sigma$ -algebra  $\mathcal{S}$  of subsets of a set  $\Omega$  has property  $N$  is a particular case of the following Valdivia's theorem.

**Theorem 1** ([14, Theorem 2]). *Each  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  has  $sN$ -property.*

Following [7, Chapter 7, 35.1] a family  $\{B_{m_1, m_2, \dots, m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$  of subsets of  $A$  is an *increasing web in  $A$*  if  $(B_{m_1})_{m_1}$  is an increasing covering of  $A$  and  $(B_{m_1, m_2, \dots, m_p, m_{p+1}})_{m_{p+1}}$  is an increasing covering of  $B_{m_1, m_2, \dots, m_p}$ , for each  $p, m_i \in \mathbb{N}$ ,  $1 \leq i \leq p$ . We will say that *a set-algebra  $\mathcal{A}$  of subsets of  $\Omega$  has the web strong  $N$ -property (web- $sN$ -property, in brief) if for each increasing web  $\{B_{m_1, m_2, \dots, m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$  in  $\mathcal{A}$  there exists a sequence  $(n_i)_i$  in  $\mathbb{N}$  such that each  $\mathcal{B}_{n_1, n_2, \dots, n_i}$  has  $sN$ -property, for each  $i \in \mathbb{N}$ .*

The main measure-theoretic result of this paper is the following theorem, motivated by [Theorem 1](#) and covering all mentioned results for  $\sigma$ -algebras.

**Theorem 2.** *Each  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  has web- $sN$ -property.*

In particular, if  $\mathcal{B}_{m_1, m_2, \dots, m_p} = \mathcal{B}_{m_1}$  for each  $p \in N$ , we have the following improvement of [Theorem 1](#): If  $(\mathcal{B}_m)_m$  is an increasing covering of a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  there exists an index  $n$  so that  $\mathcal{B}_n$  has  $sN$ -property.

Next section provides properties concerning  $N$ -property of subsets of a set-algebra  $\mathcal{A}$  and unbounded subsets of  $ba(\mathcal{A})$ . These results will be used in [Section 3](#) to provide necessary facts to complete the proof of our main result ([Theorem 2](#)).

Last section deals with applications of [Theorem 2](#) to localizations of bounded finite additive vector measures.

A characterization of  $sN$ -property of a set-algebra  $\mathcal{A}$  by a locally convex property of  $L(\mathcal{A})$  was obtained in [\[15, Theorem 3\]](#). Analogously a characterization of web- $sN$ -property of a set-algebra  $\mathcal{A}$  by a locally convex property of  $L(\mathcal{A})$  may be found easily following [\[5\]](#) and [\[10\]](#).

## 2. Nikodym property and deep unbounded sets

To keep the paper self-contained we provided a short proof of the next (well known) proposition.

**Proposition 3.** *Let  $\mathcal{A}$  be an algebra of subsets of  $\Omega$  and let  $M$  be an absolutely convex  $\tau_s(\mathcal{A})$ -closed subset of  $ba(\mathcal{A})$ . The following properties are equivalent:*

1. *For each finite subset  $\mathcal{Q}$  of  $\{\chi_A : A \in \mathcal{A}\}$  the set  $M \cap \mathcal{Q}^\circ$  is an unbounded subset of  $ba(\mathcal{A})$ .*
2. *For each finite subset  $\mathcal{Q}$  of  $\{\chi_A : A \in \mathcal{A}\}$  such that  $\text{span}\{M^\circ\} \cap \text{span}\{\mathcal{Q}\} = \{0\}$  the set  $M \cap \mathcal{Q}^\circ$  is unbounded in  $ba(\mathcal{A})$ .*
3.  *$M^\circ$  is not a neighborhood of zero in  $\text{span}\{M^\circ\}$  or the codimension of  $\text{span}\{M^\circ\}$  in  $L(\mathcal{A})$  is infinite.*

*If  $M$  is unbounded and  $\overline{\text{span}\{M^\circ\}} = L(\mathcal{A})$  then  $M$  verifies the previous properties.*

**Proof.** To prove these equivalences recall that if  $M$  is a  $\tau_s(\mathcal{A})$ -closed and absolutely convex subset of  $ba(\mathcal{A})$  then  $M^{\circ\circ} = M$  [\[7, Chapter 4 20.8.5\]](#).

(1)  $\iff$  (2). Let  $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leq i \leq r\}$ . First we prove that if there exists  $m_1 \in M^\circ$  such that  $\chi_{Q_1} = h_1 m_1 + \sum_{2 \leq i \leq r} h_i \chi_{Q_i}$  and if  $h := 2 + \sum_{1 \leq i \leq r} |h_i|$  then

$$\text{absco}(M^\circ \cup \mathcal{Q}) \subset h \text{absco}(M^\circ \cup \{\mathcal{Q} \setminus \{\chi_{Q_1}\}\}). \tag{1}$$

In fact, if  $x \in \text{absco}(M^\circ \cup \mathcal{Q})$  then  $x = \lambda_0 m_0 + \sum_{1 \leq i \leq r} \lambda_i \chi_{Q_i}$ , with  $m_0 \in M^\circ$  and  $\sum_{0 \leq i \leq r} |\lambda_i| \leq 1$ , whence  $x = \lambda_0 m_0 + \lambda_1 h_1 m_1 + \sum_{2 \leq i \leq r} (\lambda_1 h_i + \lambda_i) \chi_{Q_i}$ . From  $m_2 := (1 + |\lambda_0| + |\lambda_1 h_1|)^{-1} (\lambda_0 m_0 + \lambda_1 h_1 m_1) \in M^\circ$  we get the representation  $x = (1 + |\lambda_0| + |\lambda_1 h_1|) m_2 + \sum_{2 \leq i \leq r} (\lambda_1 h_i + \lambda_i) \chi_{Q_i}$  which verifies the inequality  $1 + |\lambda_0| + |\lambda_1 h_1| + \sum_{2 \leq i \leq r} |\lambda_1 h_i + \lambda_i| \leq h$ , whence  $x \in h \text{absco}(M^\circ \cup \{\mathcal{Q} \setminus \{\chi_{Q_1}\}\})$ . Taking polar sets in [\(1\)](#) we obtain that

$$M \cap \{\mathcal{Q} \setminus \{\chi_{Q_1}\}\}^\circ \subset h(M \cap \mathcal{Q}^\circ),$$

hence if  $M \cap \{\mathcal{Q} \setminus \{\chi_{Q_1}\}\}^\circ$  is unbounded one gets that  $M \cap \mathcal{Q}^\circ$  is also unbounded. The rest of this equivalence is obvious.

(2)  $\iff$  (3). If  $M^\circ$  is a neighborhood of zero in  $\text{span}\{M^\circ\}$  and if  $\mathcal{Q} = \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leq i \leq r\}$  is a cobase of  $\text{span}\{M^\circ\}$  in  $L(\mathcal{A})$  then  $\text{absco}(M^\circ \cup \mathcal{Q})$  is a neighborhood of zero in  $L(\mathcal{A})$ , hence

$$(\text{absco}(M^\circ \cup \mathcal{Q}))^\circ = M \cap \mathcal{Q}^\circ$$

is a bounded subset of  $\text{ba}(\mathcal{A})$ .

If  $M^\circ$  is not a neighborhood of zero in  $\text{span}\{M^\circ\}$  or if the codimension of  $\text{span}\{M^\circ\}$  in  $L(\mathcal{A})$  is infinite, then for each finite set  $\mathcal{Q} := \{\chi_{Q_i} : Q_i \in \mathcal{A}, 1 \leq i \leq r\}$  such that  $\text{span}\{M^\circ\} \cap \text{span}\{\mathcal{Q}\} = \{0\}$  the set  $\text{absco}(M^\circ \cup \mathcal{Q})$  is not a neighborhood of zero in  $L(\mathcal{A})$ , whence the set  $(\text{absco}(M^\circ \cup \mathcal{Q}))^\circ = M \cap \mathcal{Q}^\circ$  is unbounded in  $\text{ba}(\mathcal{A})$ .

If  $M$  is an unbounded subset of  $\text{ba}(\mathcal{A})$  then  $M^\circ$  is not a neighborhood of zero in  $L(\mathcal{A})$ . If, additionally,  $\overline{\text{span}\{M^\circ\}} = L(\mathcal{A})$  we have, by denseness, that  $M^\circ$  is not a neighborhood of zero in  $\text{span}\{M^\circ\}$  and we obtain that  $M$  verifies (3).  $\square$

The fact that if a subset  $M$  of  $\text{ba}(\mathcal{A})$  verifies (1) in Proposition 3 then its subsets  $M \cap \mathcal{Q}^\circ$  are unbounded, for each finite subset  $\mathcal{Q}$  of  $\{\chi_A : A \in \mathcal{A}\}$ , motivates the following definition.

**Definition 1.** Let  $B$  be an element of the algebra  $\mathcal{A}$  of subsets of  $\Omega$ . A subset  $M$  of  $\text{ba}(\mathcal{A})$  is deep  $B$ -unbounded if each finite subset  $\mathcal{Q}$  of  $\{\chi_A : A \in \mathcal{A}\}$  verifies that

$$\sup\{|\mu(C)| : \mu \in M \cap \mathcal{Q}^\circ, C \in \mathcal{A}, C \subset B\} = \infty, \tag{2}$$

or, equivalently,  $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} = \infty$ .

In particular, a subset  $M$  of  $\text{ba}(\mathcal{A})$  is deep  $\Omega$ -unbounded if  $M \cap \mathcal{Q}^\circ$  is an unbounded subset of  $\text{ba}(\mathcal{A})$ , for each finite subset  $\mathcal{Q}$  of  $\{\chi_A : A \in \mathcal{A}\}$ . Therefore an absolutely convex  $\tau_s(\mathcal{A})$ -closed subset  $M$  of  $\text{ba}(\mathcal{A})$  is deep  $\Omega$ -unbounded if and only if  $M$  verifies condition (2) or (3) in Proposition 3. If, additionally,  $\overline{\text{span}\{M^\circ\}} = L(\mathcal{A})$  then  $M$  is deep  $\Omega$ -unbounded if and only if it is unbounded.

Next proposition furnishes sequences of deep  $\Omega$ -unbounded subsets of  $\text{ba}(\mathcal{A})$ . The particular case  $\cup_m \mathcal{B}_m = \mathcal{A}$  is Theorem 1 in [15].

**Proposition 4.** Let  $\mathcal{A}$  be an algebra of subsets of  $\Omega$  and let  $(\mathcal{B}_m)_m$  be an increasing sequence of subsets of  $\mathcal{A}$  such that each  $\mathcal{B}_m$  does not have  $N$ -property and  $\overline{\text{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$ . There exists  $n_0 \in \mathbb{N}$  such that for each  $m \geq n_0$  there exists a deep  $\Omega$ -unbounded  $\tau_s(\mathcal{A})$ -closed absolutely convex subset  $M_m$  of  $\text{ba}(\mathcal{A})$  which is pointwise bounded in  $\mathcal{B}_m$ , i.e.,  $\sup\{|\mu(C)| : \mu \in M_m\} < \infty$  for each  $C \in \mathcal{B}_m$ . In particular this proposition holds if  $\cup_m \mathcal{B}_m = \mathcal{A}$  or if  $\cup_m \mathcal{B}_m$  has  $N$ -property.

**Proof.** If for each  $m \in \mathbb{N}$  the subspace  $H_m := \overline{\text{span}\{\chi_C : C \in \mathcal{B}_m\}}$  has infinite codimension in  $L(\mathcal{A})$  then, by (3) in Proposition 3, the polar set of  $P_m := \text{absco}\{\chi_C : C \in \mathcal{B}_m\}$  is the deep  $\Omega$ -unbounded set  $M_m := P_m^\circ$ . The definition of polar set implies that  $\sup\{|\mu(C)| : \mu \in M_m\} \leq 1$ , for each  $C \in \mathcal{B}_m$ . Whence we get the proposition with  $n_0 = 1$ .

If there exists  $p$  such that the codimension of  $F := \overline{\text{span}\{\chi_C : C \in \mathcal{B}_p\}}$  in  $L(\mathcal{A}) = \text{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\}$  is the finite positive number  $q$  then  $\{\chi_C : C \in \cup_m \mathcal{B}_m\} \not\subset F$ , whence there exists  $m_1 \in \mathbb{N}$  and  $D \in \mathcal{B}_{p+m_1}$  such that  $\chi_D \notin F$  and then the codimension of  $\overline{\text{span}\{\chi_C : C \in \mathcal{B}_{p+m_1}\}}$  in  $L(\mathcal{A})$  is less or equal than  $q - 1$ . Therefore there exists  $n_0$  such that  $\overline{\text{span}\{\chi_C : C \in \mathcal{B}_m\}} = L(\mathcal{A})$ , for each  $m \geq n_0$ . As for each  $m \geq n_0$  the set  $\mathcal{B}_m$  does not have  $N$ -property there exists an absolutely convex  $\tau_s(\mathcal{A})$ -closed unbounded subset  $M_m$  of  $\text{ba}(\mathcal{A})$  such that  $\sup\{|\mu(C)| : \mu \in M_m\} < k_C < \infty$ , for each  $C \in \mathcal{B}_m$ , and then it follows that  $\{k_C^{-1}\chi_C : C \in \mathcal{B}_m\} \subset M_m^\circ$ . This inclusion implies that  $\text{span}\{\chi_C : C \in \mathcal{B}_m\} \subset \text{span}\{M_m^\circ\}$ , whence  $\overline{\text{span}\{M_m^\circ\}} = L(\mathcal{A})$ , because  $\overline{\text{span}\{\chi_C : C \in \mathcal{B}_m\}} = L(\mathcal{A})$ . Then, by Proposition 3, the unbounded set  $M_m$  is deep  $\Omega$ -unbounded for each  $m \geq n_0$ .

If  $\cup_m \mathcal{B}_m = \mathcal{A}$  or if  $\cup_m \mathcal{B}_m$  has  $N$ -property then  $\overline{\text{span}\{\chi_C : C \in \cup_m \mathcal{B}_m\}} = L(\mathcal{A})$  and this proposition holds.  $\square$

Next Proposition 5 follows from [15, Proposition 1]. We give a simplified proof according to our current notation.

**Proposition 5.** *Let  $B$  be an element of an algebra  $\mathcal{A}$  and  $\{C_1, C_2, \dots, C_q\}$  a finite partition of  $B$  by elements of  $\mathcal{A}$ . If  $M$  is a deep  $B$ -unbounded subset of  $\text{ba}(\mathcal{A})$  there exists  $C_i$ ,  $1 \leq i \leq q$ , such that  $M$  is deep  $C_i$ -unbounded.*

**Proof.** If for each  $i$ ,  $1 \leq i \leq q$ , there exists a finite set  $\mathcal{Q}^i$  of characteristic functions of elements of  $\mathcal{A}$  such that  $\sup\{|\mu|(C_i) : \mu \in M \cap (\mathcal{Q}^i)^\circ\} < H_i$ ,  $i \in \{1, 2, \dots, q\}$ , then we get the contradiction that the set  $\mathcal{Q} = \cup_{1 \leq i \leq q} \mathcal{Q}^i$  verifies that  $\sup\{|\mu|(B) : \mu \in M \cap \mathcal{Q}^\circ\} < \sum_{1 \leq i \leq q} H_i$ .  $\square$

If  $t = (t_1, t_2, \dots, t_p)$ ,  $s = (s_1, s_2, \dots, s_q)$ ,  $T$  and  $U$  are two elements and two subsets of  $\cup_s \mathbb{N}^s$  we define  $t(i) := (t_1, t_2, \dots, t_i)$  if  $1 \leq i \leq p$ ,  $t(i) := \emptyset$  if  $i > p$ ,  $T(m) := \{t(m) : t \in T\}$ , for each  $m \in \mathbb{N}$ ,  $t \times s := (t_1, t_2, \dots, t_p, t_{p+1}, t_{p+2}, \dots, t_{p+q})$ , with  $t_{p+j} := s_j$ , for  $1 \leq j \leq q$ , and  $T \times U := \{t \times u : t \in T, u \in U\}$ . We simplify  $(t_1)$ ,  $(n)$  and  $T \times \{(n)\}$  by  $t_1$ ,  $n$  and  $T \times n$ . The length of  $t = (t_1, t_2, \dots, t_p)$  is  $p$  and the cardinal of a set  $C$  is denoted by  $|C|$ .

If  $v \in \cup_s \mathbb{N}^s$  and  $t \times v \in U$  then  $t \times v$  is an extension of  $t$  in  $U$ . A sequence  $(t^n)_n$  of elements  $t^n = (t_1^n, t_2^n, \dots, t_n^n, \dots) \in \cup_s \mathbb{N}^s$  is an infinite chain if for each  $n \in \mathbb{N}$  the element  $t^{n+1}$  is an extension of the section  $t^n(n)$ , i.e.,  $\emptyset \neq t^n(n) = t^{n+1}(n)$ .

A subset  $U$  of  $\cup_n \mathbb{N}^n$  is increasing at  $t = (t_1, t_2, \dots, t_p) \in \cup_s \mathbb{N}^s$  if  $U$  contains  $p$  elements  $t^1 = (t_1^1, t_2^1, \dots)$  and  $t^i = (t_1, t_2, \dots, t_{i-1}, t_i^i, t_{i+1}^i, \dots)$ ,  $1 < i \leq p$ , such that  $t_i < t_i^i$ , for each  $1 \leq i \leq p$ . A non-void subset  $U$  of  $\cup_s \mathbb{N}^s$  is increasing (increasing respect to a subset  $V$  of  $\cup_s \mathbb{N}^s$ ) if  $U$  is increasing at each  $t \in U$  (at each  $t \in V$ ), hence  $U$  is increasing if  $|U(1)| = \infty$  and  $|\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty$ , for each  $t = (t_1, t_2, \dots, t_p) \in U$  and  $1 \leq i < p$ .

If  $\{B_u : u \in \cup_s \mathbb{N}^s\}$  is an increasing web in  $A$  and  $U$  is an increasing subset of  $\cup_{s \in \mathbb{N}} \mathbb{N}^s$  then  $\mathcal{B} := \{B_{u(i)} : u \in U, 1 \leq i \leq \text{length } u\}$  verifies that  $(B_{u(1)})_{u \in U}$  is an increasing covering of  $A$  and for each  $u = (u_1, u_2, \dots, u_p) \in U$  and each  $i < p$  the sequence  $(B_{u(i) \times n})_{u(i) \times n \in U(i+1)}$  is an increasing covering of  $B_{u(i)}$ . If, additionally, each element  $u \in U$  has an extension in  $U$  then renumbering the indexes in the elements of  $\mathcal{B}$  we get an increasing web.

The Definition 2 deals with increasing subsets of  $\cup_{s \in \mathbb{N}} \mathbb{N}^s$  and it is motivated by the technical Example 1 which will be used onwards to complete the proof of Theorem 2. A particular class of increasing trees, named NV-trees – surely reminding Nikodym and Valdivia –, is considered in [9, Definition 1].

**Definition 2.** An increasing tree  $T$  is an increasing subset of  $\cup_{s \in \mathbb{N}} \mathbb{N}^s$  without infinite chains.

An increasing tree  $T$  is trivial if  $T = T(1)$ ; then  $T$  is an infinite subset of  $\mathbb{N}$ . The sets  $\mathbb{N}^i$ ,  $i \in \mathbb{N} \setminus \{1\}$ , and the set  $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$  are nontrivial increasing trees.

An increasing subset  $S$  of an increasing tree  $T$  is an increasing tree. From this observation it follows the Claim 6.

**Claim 6.** *If  $(S_n)_n$  is a sequence of non-void subsets of an increasing tree  $T$  such that for each  $n \in \mathbb{N}$  the set  $S_{n+1}$  is increasing respect to  $S_n$ , then  $S := \cup_n S_n$  is an increasing tree.*

**Proof.** It is enough to notice that  $S$  is an increasing subset of  $T$ .  $\square$

**Example 1.** Let  $\mathcal{B} := \{\mathcal{B}_{m_1, m_2, \dots, m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$  be an increasing web in an algebra  $\mathcal{A}$  of subsets of  $\Omega$  with the property that for each sequence  $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$  there exists  $q \in \mathbb{N}$  such that  $\mathcal{B}_{m_1, m_2, \dots, m_q}$  does not have  $sN$ -property. Then there exists an increasing web  $\mathcal{C} := \{\mathcal{C}_{m_1, m_2, \dots, m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$  in  $\mathcal{A}$

and an increasing tree  $T$  such that for each  $(t_1, t_2, \dots, t_p) \in T$  there exists a deep  $\Omega$ -unbounded  $\tau_s(\mathcal{A})$ -closed absolutely convex subset  $M_{t_1, t_2, \dots, t_p}$  of  $\text{ba}(\mathcal{A})$  which is pointwise bounded in  $\mathcal{C}_{t_1, t_2, \dots, t_p}$ , i.e.,

$$\sup\{|\mu(C)| : \mu \in M_{t_1, t_2, \dots, t_p}\} < \infty, \quad (3)$$

for each  $C \in \mathcal{C}_{t_1, t_2, \dots, t_p}$ .

**Proof.** If each  $\mathcal{B}_{m_1}$ ,  $m_1 \in \mathbb{N}$ , does not have  $N$ -property then the example is given by  $\mathcal{C} := \mathcal{B}$  and  $T := \mathbb{N} \setminus \{1, 2, \dots, n_0 - 1\}$ , where  $n_0$  is the natural number obtained in Proposition 4 applied to the increasing covering  $(\mathcal{B}_{m_1})_{m_1}$  of  $\mathcal{A}$ . Hence we may suppose that there exists  $m_1 \in \mathbb{N}$  such that  $\mathcal{B}_{t_1}$  has  $N$ -property for each  $t_1 \geq m_1$  and then:

- (i<sub>1</sub>) Either  $\mathcal{B}_{t_1}$  does not have  $sN$ -property for each  $t_1 \in \mathbb{N}$  and the inductive process finish defining  $T_0 := \{t_1 \in \mathbb{N} : t_1 \geq m_1\}$ .
- (ii<sub>1</sub>) Or there exists  $m'_1 \in \mathbb{N}$  such that  $\mathcal{B}_{t_1}$  has  $sN$ -property for each  $t_1 \geq m'_1$ . Then we write  $Q_1 := \emptyset$  and  $Q'_1 := \{t_1 \in \mathbb{N} : t_1 \geq m'_1\}$ .

Let us assume that for each  $j$ , with  $2 \leq j \leq i$ , we have obtained by induction two disjoint subsets  $Q_j$  and  $Q'_j$  of  $\mathbb{N}^j$  such that each  $t = (t_1, t_2, \dots, t_j) \in Q_j \cup Q'_j$  verifies:

1.  $t(j-1) = (t_1, t_2, \dots, t_{j-1}) \in Q'_{j-1}$ .
2. If  $t \in Q_j$  the set  $\mathcal{B}_t$  has  $N$ -property but it does not have  $sN$ -property and  $S_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$  is a cofinite subset of  $\mathbb{N}$  such that  $t(j-1) \times S_{t(j-1)} \subset Q_j$ .
3. If  $t \in Q'_j$  the set  $\mathcal{B}_t$  has  $sN$ -property and  $S'_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$  is a cofinite subset of  $\mathbb{N}$  such that  $t(j-1) \times S'_{t(j-1)} \subset Q'_j$ .

If  $t := (t_1, t_2, \dots, t_i) \in Q'_i$  then  $\mathcal{B}_{t_1, t_2, \dots, t_i}$  has  $sN$ -property and  $(\mathcal{B}_{t_1, t_2, \dots, t_i, n})_n$  is an increasing covering of  $\mathcal{B}_{t_1, t_2, \dots, t_i}$ , hence there exists  $m_{i+1}$  such that  $\mathcal{B}_{t_1, t_2, \dots, t_i, n}$  has  $N$ -property for each  $n \geq m_{i+1}$ . Then we may have two possible cases:

- (i<sub>i+1</sub>) Either  $\mathcal{B}_{t_1, t_2, \dots, t_i, n}$  does not have  $sN$ -property for each  $n \in \mathbb{N}$  and we define  $S_{t_1, t_2, \dots, t_i} := \{n \in \mathbb{N} : m_{i+1} \leq n\}$  and  $S'_{t_1, t_2, \dots, t_i} := \emptyset$ ,
- (ii<sub>i+1</sub>) or there exists  $m'_{i+1} \in \mathbb{N}$  such that  $\mathcal{B}_{t_1, t_2, \dots, t_i, n}$  has  $sN$ -property for each  $n \geq m'_{i+1}$ . In this case let  $S_{t_1, t_2, \dots, t_i} := \emptyset$  and  $S'_{t_1, t_2, \dots, t_i} := \{n \in \mathbb{N} : m'_{i+1} \leq n\}$ .

We finish this induction procedure by setting  $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$  and  $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$ . By construction  $Q_{i+1}$  and  $Q'_{i+1}$  verify the properties 1., 2. and 3. with  $j = i + 1$ .

The fact that for each sequence  $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$  there exists  $j \in \mathbb{N}$  such that  $\mathcal{B}_{m_1, m_2, \dots, m_j}$  does not have  $sN$ -property imply that  $T_0 := \cup\{Q_i : i \in \mathbb{N}\}$  does not contain infinite chains, because if  $(m_1, m_2, \dots, m_p) \in Q_p$  then  $\mathcal{B}_{m_1, m_2, \dots, m_{p-1}}$  has  $sN$ -property, whence for each  $(t_1, t_2, \dots, t_k) \in Q'_k$  there exists  $q \in \mathbb{N}$  and  $(t_{k+1}, \dots, t_{k+q}) \in \mathbb{N}^q$  such that  $(t_1, t_2, \dots, t_k, t_{k+1}, \dots, t_{k+q}) \in Q_{k+q}$  and then  $T_0(k) = Q_k \cup Q'_k$ , for each  $k \in \mathbb{N}$ . These equalities imply that  $T_0$  is increasing, because  $|T_0(1)| = |Q'_1| = \infty$  and if  $t = (t_1, t_2, \dots, t_p) \in T_0$  the sets  $S'_{t(i-1)}$ ,  $1 < i < p$ , and  $S_{t(p-1)}$  are cofinite subsets of  $\mathbb{N}$ .

This increasing tree  $T_0$  as well as the trivial increasing tree obtained in (i<sub>1</sub>), also named  $T_0$ , verify that for each  $t = (t_1, t_2, \dots, t_p) \in T_0$  the family  $\mathcal{B}_{t_1, t_2, \dots, t_p}$  has  $N$ -property and it does not have  $sN$ -property, whence  $\mathcal{B}_{t_1, t_2, \dots, t_p}$  has an increasing covering  $(\mathcal{B}'_{t_1, t_2, \dots, t_p, n})_n$  such that each  $\mathcal{B}'_{t_1, t_2, \dots, t_p, n}$  does not have  $N$ -property. By Proposition 4 there exists  $n_0 \in \mathbb{N}$  such that for each  $n \geq n_0$  there exists a deep  $\Omega$ -unbounded  $\tau_s(\mathcal{A})$ -closed absolutely convex subset  $M_{t_1, t_2, \dots, t_p, n}$  of  $\text{ba}(\mathcal{A})$  which is  $\mathcal{B}'_{t_1, t_2, \dots, t_p, n}$  pointwise bounded, i.e.,  $\sup\{|\mu(C)| :$



$\mu \in M_{t_1, t_2, \dots, t_p, n} \} < \infty$ , for each  $C \in \mathcal{B}'_{t_1, t_2, \dots, t_p, n}$ . We assume  $n_0 = 1$ , removing  $\mathcal{B}'_{t_1, t_2, \dots, t_p, n}$  when  $n < n_0$  and changing  $n$  by  $n - n_0 + 1$ .

Then we get the example with the increasing tree  $T := T_0 \times \mathbb{N}$  and with the increasing web  $\mathcal{C} := \{C_t : t \in \cup_s \mathbb{N}^s\}$  in the algebra  $\mathcal{A}$  such that for each  $t = (t_1, t_2, \dots, t_p) \in \cup_s \mathbb{N}^s$  either  $C_t := \mathcal{B}'_{t(i)}$  if  $i \leq p$  and  $t(i) \in T$  or  $C_t := \mathcal{B}_t$  if  $\{t(i) : 1 \leq i \leq p\} \cap T = \emptyset$ .  $\square$

Let  $U$  be a subset of  $\cup_s \mathbb{N}^s$ . An element  $t \in \cup_s \mathbb{N}^s$  admits increasing extension in  $U$  if the set of  $\{v \in \cup_s \mathbb{N}^s : t \times v \in U\}$  contains an increasing subset. We need the following obvious properties (a), (b<sub>1</sub>) and (b<sub>2</sub>) to prove Proposition 7, stating that if a subset  $U$  of an increasing tree  $T$  does not contain an increasing tree then  $T \setminus U$  contains an increasing tree.

- (a) If  $U$  is a subset of  $\cup_s \mathbb{N}^s$  and  $U$  does not contain an increasing tree then there exists  $m_1 \in \mathbb{N}$  such that each  $n \in \mathbb{N} \setminus \{1, 2, \dots, m_1\}$  does not admit increasing extension in  $U$ .
- (b) Let  $t \in \cup_s \mathbb{N}^s$  and let  $U$  be a subset of the increasing tree  $T$ . Suppose that  $t$  does not admit increasing extension in  $U$  and that  $T_t := \{v \in \cup_s \mathbb{N}^s : t \times v \in T\} \neq \emptyset$ . Then
  - (b<sub>1</sub>) if the increasing tree  $T_t$  is trivial there exists  $m_{i+1} \in \mathbb{N}$  such that the set

$$(t \times \{\mathbb{N} \setminus \{1, 2, \dots, m_{i+1}\}\}) \cap T$$

is an infinite subset of  $T \setminus U$ ,

- (b<sub>2</sub>) if  $T_t$  is non-trivial there exists  $m'_{i+1} \in \mathbb{N}$  such that each element of

$$(t \times \{\mathbb{N} \setminus \{1, 2, \dots, m'_{i+1}\}\}) \cap T(i + 1)$$

does not admit increasing extension in  $U$ .

**Proposition 7.** *Let  $U$  be a subset of an increasing tree  $T$ . If  $U$  does not contain an increasing tree then  $T \setminus U$  contains an increasing tree.*

**Proof.** It is enough to prove that  $T \setminus U$  contains an increasing subset  $W$ . Now we follow the scheme of the proof in Example 1. In fact, if  $T$  is a trivial increasing tree the proposition is obvious. Hence we may suppose that  $T$  is a non-trivial increasing tree. Then we define  $Q_1 := \emptyset$  and by (a) there exists  $m'_1 \in \mathbb{N}$  such that each element of the set  $Q'_1 := \{n \in T(1) : m'_1 \leq n\}$  does not admit increasing extension in  $U$ . Notice that  $Q'_1 \subset T(1) \setminus T$ .

Let us suppose that we have obtained for each  $j$ , with  $2 \leq j \leq i$ , two disjoint subsets  $Q_j$  and  $Q'_j$  such that  $Q_j \subset T(j) \cap (T \setminus U)$ ,  $Q'_j \subset T(j) \setminus T$  and each  $t \in Q_j \cup Q'_j$  verifies the following properties:

1.  $t(j - 1) \in Q'_{j-1}$ .
2. If  $t \in Q_j$  then the cardinal of  $S_{t(j-1)} := \{n \in \mathbb{N} : t(j - 1) \times n \in Q_j \cup Q'_j\}$  is infinite and  $t(j - 1) \times S_{t(j-1)} \subset Q_j$ .
3. If  $t \in Q'_j$  then  $t$  does not admit increasing extension in  $U$ , the cardinal of  $S'_{t(j-1)} := \{n \in \mathbb{N} : t(j - 1) \times n \in Q_j \cup Q'_j\}$  is infinite and  $t(j - 1) \times S'_{t(j-1)} \subset Q'_j$ .

If  $t \in Q'_i$  then  $t \in T(i) \setminus T$  and it does not admit increasing extension in  $U$ . If  $T_t = \{v \in \cup_s \mathbb{N}^s : t \times v \in T\}$  then, by (b<sub>1</sub>) and (b<sub>2</sub>), it follows that the following two cases may happen:

- i.* If  $T_t$  is trivial then there exists  $m_{i+1} \in \mathbb{N}$  such that the infinite set  $S_t := \{n \in \mathbb{N} : m_{i+1} \leq n, t \times n \in T(i + 1)\}$  verifies that  $t \times S_t \subset T \setminus U$  and we define  $S'_t := \emptyset$ .

ii. If  $T_t$  is non-trivial then there exists  $m'_{i+1} \in \mathbb{N}$  such that the infinite set  $S'_t := \{n \in \mathbb{N} : m'_{i+1} < n, t \times n \in T(i+1)\}$  verifies that  $t \times S'_t \subset T(i+1) \setminus T$  and each element of  $t \times S'_t$  does not admit increasing extension in  $U$ . Now we define  $S_t := \emptyset$ .

We finish this induction procedure by setting  $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$  and  $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$ .

By construction  $Q_{i+1} \subset T(i+1) \cap (T \setminus U)$ ,  $Q'_{i+1} \subset T(i+1) \setminus T$ , and each  $t \in Q_{i+1} \cup Q'_{i+1}$  verifies the properties 1., 2. and 3. changing  $j$  by  $i+1$ .

As  $T$  does not contain infinite chains we deduce from 1. that for each  $(t_1, t_2, \dots, t_i) \in Q'_i$  there exists  $q \in \mathbb{N}$  and  $(t_{i+1}, \dots, t_{i+q}) \in \mathbb{N}^q$  such that  $(t_1, t_2, \dots, t_i, t_{i+1}, \dots, t_{i+q}) \in Q_{i+q}$ . Whence, for each  $i \in \mathbb{N}$ ,  $(\cup_{j>i} Q_j)(i) = Q'_i$  and then  $W := \cup\{Q_j : j \in \mathbb{N}\}$  is a subset of  $T \setminus U$ .

$W$  has the increasing property because from  $W(k) = Q_k \cup Q'_k$ , for each  $k \in \mathbb{N}$ , it follows that  $|W(1)| = |Q'_1| = \infty$  and if  $t = (t_1, t_2, \dots, t_p) \in W$  then  $(t_1, t_2, \dots, t_i) \in Q'_i$ , if  $1 < i < p$ , and  $(t_1, t_2, \dots, t_p) \in Q_p$ , hence the infinite subsets  $S'_{t(i-1)}$  and  $S_{t(p-1)}$  of  $\mathbb{N}$  verify that  $t(i-1) \times S'_{t(i-1)} \subset Q'_i \subset W(i)$  and  $t(p-1) \times S_{t(p-1)} \subset Q_p \subset W$ .  $\square$

Next Proposition 8 follows from [15, Propositions 2 and 3] and we give a simplified proof according to our current notation for the sake of completeness.

**Proposition 8.** *Let  $\{B, Q_1, \dots, Q_r\}$  be a subset of the algebra  $\mathcal{A}$  of subsets of  $\Omega$  and let  $M$  be a deep  $B$ -unbounded absolutely convex subset of  $ba(\mathcal{A})$ . Then given a positive real number  $\alpha$  and a natural number  $q > 1$  there exists a finite partition  $\{C_1, C_2, \dots, C_q\}$  of  $B$  by elements of  $\mathcal{A}$  and a subset  $\{\mu_1, \mu_2, \dots, \mu_q\}$  of  $M$  such that  $|\mu_i(C_i)| > \alpha$  and  $\sum_{1 \leq j \leq r} |\mu_i(Q_j)| \leq 1$ , for  $i = 1, 2, \dots, q$ .*

**Proof.** Let  $\mathcal{Q} = \{\chi_B, \chi_{Q_1}, \chi_{Q_2}, \dots, \chi_{Q_r}\}$ . The deep  $B$ -unboundedness of  $M$  and the inclusion  $M \subset rM$  imply that

$$\sup\{|\mu(D)| : \mu \in rM \cap \mathcal{Q}^\circ, D \subset B, D \in \mathcal{A}\} = \infty.$$

Hence there exists  $P_1 \subset B$ , with  $P_1 \in \mathcal{A}$ , and  $\mu \in rM \cap \mathcal{Q}^\circ$  such that  $|\mu(P_1)| > r(1 + \alpha)$ . Clearly  $\mu_1 = r^{-1}\mu \in M$ ,  $|\mu_1(P_1)| > 1 + \alpha$  and  $|\mu_1(f)| = r^{-1}|\mu(f)| \leq r^{-1}$  for each  $f \in \mathcal{Q}$ , hence  $|\mu_1(B)| \leq r^{-1} \leq 1$  and  $\sum_{1 \leq j \leq r} |\mu_1(Q_j)| \leq r^{-1}r = 1$ . The set  $P_2 := B \setminus P_1$  verifies that

$$|\mu_1(P_2)| \geq |\mu_1(P_1)| - |\mu_1(B)| > 1 + \alpha - 1 = \alpha.$$

From Proposition 5 there exists  $i \in \{1, 2\}$  such that  $M$  is deep  $P_i$ -unbounded. To finish the first step of the proof let  $C_1 := P_1$  if  $M$  is deep  $P_2$ -unbounded and let  $C_1 := P_2$  if  $M$  is deep  $P_1$ -unbounded. Then  $M$  is deep  $B \setminus C_1$ -unbounded.

Apply the same argument in  $B \setminus C_1$  to obtain a measurable set  $C_2 \subset B \setminus C_1$  and a measure  $\mu_2 \in M$  such that  $|\mu_2(C_2)| > \alpha$ ,  $|\mu_2(B \setminus (C_1 \cup C_2))| > \alpha$  and  $\sum\{|\mu_2(Q_j)| : 1 \leq j \leq r\} \leq 1$ , being  $M$  deep  $B \setminus (C_1 \cup C_2)$ -unbounded. Hence the proof is provided by applying  $q - 1$  times this argument. In the last step we define  $\mu_q := \mu_{q-1}$  and  $C_q = B \setminus (C_1 \cup \dots \cup C_{q-1})$ .  $\square$

**Proposition 9.** *Let  $B$  be an element of an algebra  $\mathcal{A}$  and  $\{M_t : t \in T\}$  a family of deep  $B$ -unbounded subsets of  $ba(\mathcal{A})$  indexed by an increasing tree  $T$ . If  $t^j := (t^j_1, t^j_2, \dots, t^j_{p_j}) \in T$ , for each  $1 \leq j \leq k$ , and  $q = 2 + \sum\{p_j : 1 \leq j \leq k\}$  then for each finite partition  $\{C_1, C_2, \dots, C_q\}$  of  $B$  by elements of  $\mathcal{A}$  there exists  $h \in \{1, 2, \dots, q\}$  and an increasing tree  $T_1$  such that  $\{t^1, t^2, \dots, t^k\} \subset T_1 \subset T$  and  $\{M_t : t \in T_1\}$  is a family of deep  $B \setminus C_h$ -unbounded subsets.*

**Proof.** Let  $\{C_1, C_2, \dots, C_q\}$  be a finite partition of  $B$  by elements of  $\mathcal{A}$  with  $q = 2 + \sum\{p_j : 1 \leq j \leq k\}$ . From Proposition 5 it follows that if  $\{M_u : u \in U\}$  is a family of deep  $B$ -unbounded subsets of  $ba(\mathcal{A})$  indexed by



an increasing tree  $U$  and  $V_i := \{u \in U : M_u \text{ is deep } C_i\text{-unbounded}\}$ ,  $1 \leq i \leq q$ , then  $U = \cup_{1 \leq i \leq q} V_i$  and, by [Proposition 7](#), there exists  $l$ , with  $1 \leq l \leq q$ , such that  $V_l$  contains an increasing tree  $U_l$ . Therefore

- (a) If  $\{M_u : u \in U\}$  is a family of deep  $B$ -unbounded subsets indexed by an increasing tree  $U$  there exists  $l \in \{1, 2, \dots, q\}$  and an increasing tree  $U_l$  contained in  $U$  such that  $\{M_u : u \in U_l\}$  is a family of deep  $C_l$ -unbounded subsets.

In particular, for the increasing tree  $T$  and for each element  $t^j \in T$ , with  $1 \leq j \leq k$ , there exist by (a) and [Proposition 5](#):

- (1)  $i_0 \in \{1, 2, \dots, q\}$  and an increasing tree  $T_{i_0}$  contained in  $T$  such that  $\{M_t : t \in T_{i_0}\}$  is a family of deep  $C_{i_0}$ -unbounded subsets,
- (2)  $i^j \in \{1, 2, \dots, q\}$  such that  $M_{t^j}$  is deep  $C_{i^j}$ -unbounded.

Let  $S := \{j : 1 \leq j \leq k, t^j \notin T_{i_0}\}$ . For each  $j \in S$  and each section  $t^j(m-1)$  of  $t^j = (t_1^j, t_2^j, \dots, t_{p_j}^j)$ , with  $2 \leq m \leq p_j$ , the set  $W_m^j := \{v \in \cup_s \mathbb{N}^s : t^j(m-1) \times v \in T\}$  is an increasing tree such that  $\{M_{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times w} : w \in W_m^j\}$  is a family of deep  $B$ -unbounded subsets. By (a) there exists:

- (3)  $i_m^j \in \{1, 2, \dots, q\}$  and an increasing tree  $V_m^j$  contained in  $W_m^j$  such that

$$\{M_{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times v} : v \in V_m^j\}$$

is a family of deep  $C_{i_m^j}$ -unbounded subsets. Clearly  $(t_1^j, t_2^j, \dots, t_{m-1}^j) \times V_m^j \subset T$ .

As the number of sets  $C_{i_0}, C_{i^j}, C_{i_m^j}$ , with  $j \in S$  and  $2 \leq m \leq p_j$ , is less or equal than  $q-1$ , there exists  $h \in \{1, 2, \dots, q\}$  such that

$$D := C_{i_0} \cup (\cup \{C_{i^j} \cup C_{i_m^j} : j \in S, 2 \leq m \leq p_j\}) \subset B \setminus C_h.$$

Let  $T_1$  be the union of the sets  $T_{i_0}, \{t^j : j \in S\}$  and  $\{(t_1^j, t_2^j, \dots, t_{m-1}^j) \times V_m^j\}$ , with  $j \in S$  and  $2 \leq m \leq p_j$ . Clearly for each  $t \in T_1$  the set  $M_t$  is deep  $D$ -unbounded, whence  $M_t$  is also deep  $B \setminus C_h$ -unbounded. By construction  $\{t^1, t^2, \dots, t^k\} \subset T_1$  and  $T_1$  has the increasing property and it is a subset of the increasing tree  $T$ . Whence  $T_1$  is an increasing tree.  $\square$

We finish this section with a combination of [Propositions 8 and 9](#). The obtained [Proposition 10](#) is a fundamental tool for the next section.

**Proposition 10.** *Let  $\{B, Q_1, \dots, Q_r\}$  be a subset of an algebra  $\mathcal{A}$  of subsets of  $\Omega$ , and let  $\{M_t : t \in T\}$  be a family of deep  $B$ -unbounded absolutely convex subsets of  $\text{ba}(\mathcal{A})$ , indexed by an increasing tree  $T$ . Then for each positive real number  $\alpha$  and each finite subset  $\{t^j : 1 \leq j \leq k\}$  of  $T$  there exist  $\{B_j \in \mathcal{A} : 1 \leq j \leq k\}$ , formed by  $k$  pairwise disjoint subsets  $B_j$  of  $B$ ,  $1 \leq j \leq k$ , a set  $\{\mu_j \in M_{t^j}, 1 \leq j \leq k\}$  and an increasing tree  $T^*$  such that:*

- 1.  $|\mu_j(B_j)| > \alpha$  and  $\Sigma\{|\mu_j(Q_i)| : 1 \leq i \leq r\} \leq 1$ , for  $j = 1, 2, \dots, k$ ,
- 2.  $\{t^j : 1 \leq j \leq k\} \subset T^* \subset T$  and  $\{M_t : t \in T^*\}$  is a family of deep  $(B \setminus \cup_{1 \leq j \leq k} B_j)$ -unbounded sets.

**Proof.** Let  $t^j := (t_1^j, t_2^j, \dots, t_{p_j}^j)$ , for  $1 \leq j \leq k$ . By [Proposition 8](#) applied to  $B$ ,  $\alpha$ ,  $q := 2 + \Sigma_{1 \leq j \leq k} p_j$  and  $M_{t^1}$  there exist a partition  $\{C_1^1, C_2^1, \dots, C_q^1\}$  of  $B$  by elements of  $\mathcal{A}$  and  $\{\lambda_1, \lambda_2, \dots, \lambda_q\} \subset M_{t^1}$  such that:

$$|\lambda_k(C_k^1)| > \alpha \quad \text{and} \quad \sum_{1 \leq i \leq r} |\lambda_k(Q_i)| \leq 1, \quad \text{for } k = 1, 2, \dots, q, \tag{4}$$

hence [Proposition 9](#) applied to the sets  $\{C_1^1, C_2^1, \dots, C_q^1\}$ ,  $\{M_t : t \in T\}$  and  $\{t^j : 1 \leq j \leq k\}$  gives  $h \in \{1, 2, \dots, q\}$  and a family  $\{M_t : t \in T_1\}$  of deep  $B \setminus C_h^1$ -unbounded subsets indexed by an increasing tree  $T_1$  such that  $\{t^1, t^2, \dots, t^k\} \subset T_1 \subset T$ . If  $B_1 := C_h^1$  and  $\mu_1 := \lambda_h$  then (4) holds with  $\lambda_k = \mu_1$  and  $C_k^1 = B_1$ . Clearly  $\{M_t : t \in T_1\}$  is a family of deep  $B \setminus B_1$ -unbounded subsets.

If we apply again [Proposition 8](#) to  $B \setminus B_1$ ,  $\alpha$ ,  $q$  and  $M_{t^2}$  we obtain a partition  $\{C_1^2, C_2^2, \dots, C_q^2\}$  of  $B \setminus B_1$  by elements of  $\mathcal{A}$  and  $\{\zeta_1, \zeta_2, \dots, \zeta_q\} \subset M_{t^2}$  such that

$$|\zeta_k(C_k^2)| > \alpha \quad \text{and} \quad \sum_{1 \leq i \leq r} |\zeta_k(Q_i)| \leq 1, \quad \text{for } k = 1, 2, \dots, q,$$

and then by [Proposition 9](#) (applied to  $\{C_1^2, C_2^2, \dots, C_q^2\}$ ,  $\{M_t : t \in T_1\}$  and  $\{t^j : 1 \leq j \leq k\}$ ) there exists  $l \in \{1, 2, \dots, q\}$  and a family  $\{M_t : t \in T_2\}$  of deep  $(B \setminus B_1) \setminus C_l^2$ -unbounded subsets indexed by an increasing tree  $T_2$  such that  $\{t^1, t^2, \dots, t^k\} \subset T_2 \subset T$ . Now if  $B_2 := C_l^2$  and  $\mu_2 := \zeta_l$  then  $|\mu_2(B_2)| > \alpha$ ,  $\sum\{|\mu_2(Q_i)| : 1 \leq i \leq r\} \leq 1$  and  $\{M_t : t \in T_2\}$  is a family of deep  $B \setminus (B_1 \cup B_2)$ -unbounded subsets. With  $k - 2$  new repetitions of this procedure we get the proof with  $T^* := T_k$ .  $\square$

### 3. Proof of [Theorem 2](#)

With a induction procedure based in [Proposition 10](#) we obtain [Proposition 12](#) that together with the next elementary covering property for families indexed by increasing trees enable to prove [Theorem 2](#).

**Proposition 11.** *If  $\mathcal{Y} = \{Y_{m_1, m_2, \dots, m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$  is an increasing web in  $Y$  and  $T$  is an increasing tree then  $Y = \cup\{Y_y : y \in T\}$ .*

**Proof.** Let us suppose that  $y \in Y \setminus (\cup\{Y_t : t \in T\})$ . As  $\mathcal{Y}$  is an increasing web and  $T$  is an increasing tree then  $Y = \cup\{Y_{t(1)} : t \in T\}$ , whence there exists  $u^1 = (u_1^1, u_2^1, \dots) \in T$  such that

$$y \in Y_{u^1} \setminus (\cup\{Y_t : t \in T\}).$$

Assume that there exists  $\{u^2, u^3, \dots, u^n\} \subset T$  such that  $\emptyset \neq u^{j-1}(j-1) = u^j(j-1)$  and  $y \in Y_{u^j(j)} \setminus (\cup\{Y_t : t \in T\})$ , for  $2 \leq j \leq n$ . Then  $y \in Y_{u^n(n)} \setminus (\cup\{Y_t : t \in T\})$ , with  $u^n(n) = (u_1^n, u_2^n, \dots, u_n^n)$ . As  $\mathcal{Y}$  is an increasing web and  $T$  is an increasing tree then  $Y_{u^n(n)} = \cup\{Y_{u^n(n) \times s} : u^n(n) \times s \in T(n+1)\}$ , hence there exists  $u^{n+1} \in T$  such that  $u^n(n) = u^{n+1}(n)$  and

$$y \in Y_{u^{n+1}(n+1)} \setminus (\cup\{Y_t : t \in T\}).$$

This induction procedure gives the contradiction that  $T$  contains the infinite chain  $(u^n)_n$ . Therefore  $Y = \cup\{Y_u : u \in T\}$ .  $\square$

In [Proposition 12](#) we refer to the sequence  $(i_n)_n = (1, 1, 2, 1, 2, 3, \dots)$ , obtained with the first components of  $\mathbb{N}^2$  ordered by the diagonal order, i.e.,  $i_n = n - 2^{-1}h(h+1)$ , if  $n \in ]2^{-1}h(h+1), 2^{-1}(h+1)(h+2)]$  and  $h = 0, 1, 2, \dots$ . Let us note that  $i_n \leq n$ , for each  $n \in \mathbb{N}$ .

**Proposition 12.** *Let  $\{\mathcal{B}_{m_1, m_2, \dots, m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$  be an increasing web in a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  with the property that for each sequence  $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$  there exists  $h \in \mathbb{N}$  such that  $\mathcal{B}_{m_1, m_2, \dots, m_h}$  does not have  $sN$ -property and let  $(i_n)_n = (1, 1, 2, 1, 2, 3, \dots)$ . Then there exist a strictly increasing sequence  $(j_n)_n$  in  $\mathbb{N}$ , a sequence  $(B_{i_n j_n})_n$  of pairwise disjoint elements of  $\mathcal{S}$ , a sequence  $(\mu_{i_n j_n})_n$  in  $ba(\mathcal{S})$  and a covering  $(C_r)_r$  of  $\mathcal{S}$  such that for each  $n \in \mathbb{N}$*

$$\Sigma_s\{|\mu_{i_{n+1}j_{n+1}}(B_{i_sj_s})| : 1 \leq s \leq n\} < 1, \tag{5}$$

$$|\mu_{i_nj_n}(B_{i_nj_n})| > j_n, \tag{6}$$

$$|\mu_{i_nj_n}(\cup_s\{B_{i_sj_s} : n < s\})| < 1, \tag{7}$$

and for each  $r \in \mathbb{N}$  and each strictly increasing sequence  $(n_p)_p$  such that  $i_{n_p} = r$ , for each  $p \in \mathbb{N}$ , the set  $\{\mu_{i_{n_p}j_{n_p}} : p \in \mathbb{N}\}$  is  $\mathcal{C}_r$ -pointwise bounded, i.e., for each  $H \in \mathcal{C}_r$  we have that

$$\sup\{|\mu_{i_{n_p}j_{n_p}}(H)| : p \in \mathbb{N}\} < \infty. \tag{8}$$

**Proof.** Let  $\{\mathcal{C}_t : t \in \cup_s \mathbb{N}^s\}$  and  $T$  be the increasing web in  $\mathcal{S}$  and the increasing tree determined in [Example 1](#) such that for each  $t \in T$  there exists a deep  $\Omega$ -unbounded  $\tau_s(\mathcal{S})$ -closed absolutely convex subset  $M_t$  of  $\text{ba}(\mathcal{S})$  which is  $\mathcal{C}_t$ -pointwise bounded, i.e.,

$$\sup\{|\mu(H)| : \mu \in M_t\} < \infty \tag{9}$$

for each  $H \in \mathcal{C}_t$ .

Then, by induction, we prove that there exist a countable increasing tree  $\{t^i : i \in \mathbb{N}\}$  contained in  $T$ , a strictly increasing sequence of natural numbers  $(k_j)_j$ , a set  $\{B_{ij} : (i, j) \in \mathbb{N}^2, i \leq k_j\}$  of pairwise disjoint elements of  $\mathcal{S}$  and a set  $\{\mu_{ij} \in M_{t^i} : (i, j) \in \mathbb{N}^2, i \leq k_j\}$  such that if  $(i, j) \in \mathbb{N}^2$  and  $i \leq k_j$  then

$$\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leq k_v, 1 \leq v < j\} < 1, \tag{10}$$

$$|\mu_{ij}(B_{ij})| > j, \tag{11}$$

and for each  $i \in \mathbb{N}$  and each  $H \in \mathcal{C}_{t^i}$  we have

$$\sup_j\{|\mu_{ij}(H)| : i \leq j\} < \infty. \tag{12}$$

Fix  $t^1 \in T$ . By [Proposition 10](#) with  $B := \Omega$ ,  $\alpha = 1$ ,  $\{Q_1, \dots, Q_r\} := \emptyset$  and  $\{t^i : 1 \leq i \leq k\} := \{t^1\}$  there exist  $B_{11} \in \mathcal{S}$ ,  $\mu_{11} \in M_{t^1}$  and an increasing tree  $T_1$  such that

1.  $|\mu_{11}(B_{11})| > 1$ ,  $\{M_t : t \in T_1\}$  is a family of deep  $\Omega \setminus B_{11}$ -unbounded subsets and
2.  $t^1 \in T_1 \subset T$ .

We define  $k_1 := 1$ ,  $S^1 := \{t^1\}$  and  $B^1 := B_{11}$ .

Suppose that in the following  $n - 1$  steps of the inductive process we have obtained the finite sequence  $k_2 < k_3 < \dots < k_n$  in  $\mathbb{N} \setminus \{1\}$ , the increasing trees  $T_2 \supset T_3 \supset \dots \supset T_n$  contained in  $T_1$ , the subset  $\{t^1, t^2, \dots, t^{k_n}\}$  of  $T_n$ , the set  $\{B_{ij} : i \leq k_j, j \leq n\}$  formed by pairwise disjoint elements of  $\mathcal{S}$  and the set  $\{\mu_{ij} \in M_{t^i} : i \leq k_j, j \leq n\}$  such that, for each  $1 < j \leq n$  and each  $i \leq k_j$ :

1.  $|\mu_{ij}(B_{ij})| > j$ ,  $\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leq k_v, 1 \leq v < j\} < 1$ , the union  $B^j := \cup\{B_{sv} : s \leq k_v, 1 \leq v \leq j\}$  verifies that  $\{M_t : t \in T_j\}$  is a family of deep  $\Omega \setminus B^j$ -unbounded subsets,
2.  $S^j := \{t^i : i \leq k_j\} \subset T_j$  and  $S^j$  has the increasing property respect to  $S^{j-1}$ .

To finish the induction procedure let  $\{t^{k_n+1}, \dots, t^{k_{n+1}}\}$  be a subset of  $T_n \setminus \{t^i : i \leq k_n\}$  that verifies the increasing property with respect to  $S^n$ . Then applying [Proposition 10](#) to  $\Omega \setminus B^n$ ,  $\{B_{sv} : s \leq k_v, 1 \leq v \leq n\}$ ,  $T_n$ , the finite subset  $S^{n+1} := \{t^i : i \leq k_{n+1}\}$  of  $T_n$  and  $n + 1$  we obtain a family  $\{B_{in+1} : i \leq k_{n+1}\}$  of pairwise disjoint elements of  $\mathcal{S}$  contained in  $\Omega \setminus B^n$ , a subset  $\{\mu_{in+1} \in M_{t^i} : i \leq k_{n+1}\}$  of  $\text{ba}(\mathcal{S})$  and an increasing tree  $T_{n+1}$  contained in  $T_n$  such that for each  $i \leq k_{n+1}$ ,

1.  $|\mu_{in+1}(B_{in+1})| > n + 1, \Sigma_{s,v}\{|\mu_{in+1}(B_{sv})| : s \leq k_v, 1 \leq v \leq n\} < 1$ , the union  $B^{n+1} := \cup\{B_{sv} : s \leq k_s, 1 \leq v \leq n + 1\}$  has the property that  $\{M_t : t \in T_{n+1}\}$  is a family of deep  $\Omega \setminus B^{n+1}$ -unbounded subsets,
2.  $S^{n+1} \subset T_{n+1}$  and  $S^{n+1}$  has the increasing property respect to  $S^n$ .

By Claim 6,  $\cup_n S_n = \{t^i : i \in \mathbb{N}\}$  is an increasing tree, whence, by Proposition 11, the sequence  $(C_{t^i})_i$  is a countable covering of the  $\sigma$ -algebra  $\mathcal{S}$ . As  $(k_j)_j$  is increasing then  $(i, j) \in \mathbb{N}^2$  and  $i \leq j$  imply that  $i \leq k_j$ , whence  $\{\mu_{ij} : j \in \mathbb{N} \setminus \{1, 2, \dots, i - 1\}\} \subset M_{t^i}$  and from this inclusion and (9) with  $t = t^i$  it follows (12), i.e.,  $\sup_j\{|\mu_{ij}(H)| : i \leq j\} < \infty$ , for each  $i \in \mathbb{N}$  and each  $H \in C_{t^i}$ .

With a new induction procedure we determine the increasing sequence  $(j_n)_n$  such that together with the sequence  $(i_n)_n = (1, 1, 2, 1, 2, 3, \dots)$  verifies (5), (6), (7) and (8).

Let  $j_1 := 1$  and suppose that  $|\mu_{i_1 j_1}|(\Omega) < s_1$ , with  $s_1 \in \mathbb{N}$ . Let  $\{N_u^1, 1 \leq u \leq s_1\}$  be a partition of  $\{m \in \mathbb{N} : m > j_1\}$  in  $s_1$  infinite subsets and define  $B_u^1 := \cup\{B_{st} : (s, t) \in \mathbb{N} \times N_u^1, s \leq k_t\}, 1 \leq u \leq s_1$ . From  $\Sigma\{|\mu_{i_1 j_1}|(B_u^1) : 1 \leq u \leq s_1\} < s_1$  it follows that there exists  $u'$ , with  $1 \leq u' \leq s_1$ , such that  $|\mu_{i_1 j_1}|(B_{u'}^1) < 1$ , whence the sets  $N^{(1)} := N_{u'}^1$  and  $B^1 := B_{u'}^1$  verify that  $N^{(1)} \subset \{m \in \mathbb{N} : m > j_1\}$  and

$$|\mu_{i_1 j_1}|(B^1) < 1.$$

Assume that in the first  $l$  steps of this induction we have obtained a finite sequence  $j_1 < j_2 < \dots < j_l$  in  $\mathbb{N}$  and a decreasing finite sequence  $N^{(1)} \supset N^{(2)} \supset \dots \supset N^{(l)}$  of infinite subsets of  $\mathbb{N}$  such that for each  $w \in \mathbb{N}, 1 \leq w \leq l, N^{(w)} \subset \{n \in \mathbb{N} : n > j_w\}$  and the variation of the measure  $\mu_{i_w j_w}$  in the set  $B^w := \cup\{B_{st} : (s, t) \in \mathbb{N} \times N^{(w)}, s \leq k_t\}$  verifies the inequality

$$|\mu_{i_w j_w}|(B^w) < 1.$$

Let  $j_{l+1}$  be the first element in  $N^{(l)}$  and suppose that  $|\mu_{i_{l+1} j_{l+1}}|(\Omega) < s_{l+1}$ , with  $s_{l+1} \in \mathbb{N}$ . Then  $j_l < j_{l+1}$  and if  $\{N_r^{l+1}, 1 \leq r \leq s_{l+1}\}$  is a partition of  $\{m \in \mathbb{N}^{(l)} : m > j_{l+1}\}$  in  $s_{l+1}$  infinite disjoint subfamilies then the subsets  $B_r^{l+1} := \cup\{B_{st} : (s, t) \in \mathbb{N} \times N_r^{l+1}, s \leq k_t\}, 1 \leq r \leq s_{l+1}$ , verify that  $\Sigma\{|\mu_{i_{l+1} j_{l+1}}|(B_r^{l+1}) : 1 \leq r \leq s_{l+1}\} < s_{l+1}$ , whence it follows that there exists  $r'$ , with  $1 \leq r' \leq s_{l+1}$ , such that the set  $B^{l+1} := \cup\{B_{st} : (s, t) \in \mathbb{N} \times N_{r'}^{l+1}, s \leq k_t\}$  verifies that

$$|\mu_{i_{l+1} j_{l+1}}|(B^{l+1}) < 1.$$

Set  $N^{(l+1)} := N_{r'}^{l+1}$ . Then, by induction, we get a strictly increasing sequence  $(j_n)_n$  in  $\mathbb{N}$  and a decreasing sequence  $(N^{(n)})_n$  of infinite subsets of  $\mathbb{N}$ , with  $j_2 \in N^{(1)} \subset \{m \in \mathbb{N} : m > j_1\}$  and  $j_{n+1} \in N^{(n)} \subset \{m \in N^{(n-1)} : m > j_n\}$ , for each  $n > 1$ , such that the measurable sets  $B^n := \cup\{B_{st} : (s, t) \in \mathbb{N} \times N^{(n)}, s \leq k_t\}, n \in \mathbb{N}$ , verify that

$$|\mu_{i_n j_n}|(B^n) < 1. \tag{13}$$

The inclusion  $j_s \in N^{(s-1)} \subset N^{(n)}$  when  $n < s$  and the trivial inequalities  $i_s \leq s \leq k_s \leq k_{j_s}$  imply that  $\cup\{B_{i_s j_s} : s \in \mathbb{N}, n < s\} \subset B^n$ , hence from (13) it follows that

$$|\mu_{i_n j_n}|(\cup_s\{B_{i_s j_s} : n < s\}) < 1,$$

for each  $n \in \mathbb{N}$ , and this inequality imply (7) because the variation  $|\mu|(B)$  of  $\mu$  in a set  $B \in \mathcal{S}$  verifies that  $|\mu(B)| \leq |\mu|(B)$ .

From the proved relation  $i_s \leq k_{j_s}$  and the trivial fact that  $s \leq n$  implies that  $j_s \leq j_n < j_{n+1}$  it follows that (10) implies (5). The inequality (6) is a particular case of (11). Finally from (12) with  $i = r$  we get (8) because each  $(i_{n_p}, j_{n_p})$  verifies that  $r = i_{n_p} \leq n_p \leq j_{n_p}$ .

To finish the proposition define  $\mathcal{C}_r := \mathcal{C}_{tr}$ , for each  $r \in \mathbb{N}$ .  $\square$

We are at the position to present the proof of Theorem 2. Recall again that  $(i_n)_n = (1, 1, 2, 1, 2, 3, \dots)$ .

**Proof of Theorem 2.** Assume Theorem 2 fails. Then by Proposition 12 there exist a strictly increasing sequence  $(j_n)_n$  in  $\mathbb{N}$ , a sequence  $(B_{i_n j_n})_n$  of pairwise disjoint elements of the  $\sigma$ -algebra  $\mathcal{S}$ , a sequence  $(\mu_{i_n j_n})_n$  in  $\text{ba}(\mathcal{S})$  and a covering  $(\mathcal{C}_r)_r$  of  $\mathcal{S}$  such that for each  $n \in \mathbb{N}$

$$\Sigma_s \{ |\mu_{i_n j_n}(B_{i_s j_s})| : s < n \} < 1, \tag{14}$$

$$|\mu_{i_n j_n}(B_{i_n j_n})| > j_n, \tag{15}$$

$$|\mu_{i_n j_n}(\cup_s \{B_{i_s j_s} : n < s\})| < 1, \tag{16}$$

and for each strictly increasing sequence  $(n_p)_p$  such that  $i_{n_p} = r$  for each  $p \in \mathbb{N}$  we have that the sequence  $(\mu_{i_{n_p} j_{n_p}})_p = (\mu_{r j_{n_p}})_p$  is pointwise bounded in  $\mathcal{C}_r$ , i.e., for each  $H \in \mathcal{C}_r$  we have that

$$\sup \{ |\mu_{i_{n_p} j_{n_p}}(H)| : p \in \mathbb{N} \} < \infty. \tag{17}$$

As  $H_0 := \cup \{B_{i_s j_s} : s = 1, 2, \dots\} \in \mathcal{S}$  and  $(\mathcal{C}_r)_r$  is a covering of the  $\sigma$ -algebra  $\mathcal{S}$  there exists  $r' \in \mathbb{N}$  such that  $H_0 \in \mathcal{C}_{r'}$ . Fix a strictly increasing sequence  $(n_q)_q$  in  $\mathbb{N} \setminus \{1\}$  such that  $i_{n_q} = r'$ , for each  $q \in \mathbb{N}$ . Then, by (17),

$$\sup \{ |\mu_{i_{n_q} j_{n_q}}(H_0)| : q \in \mathbb{N} \} < \infty. \tag{18}$$

The sets  $C_q := \cup_s \{B_{i_s j_s} : s < n_q\}$ ,  $B_{i_{n_q} j_{n_q}}$  and  $D_q := \cup_s \{B_{i_s j_s} : n_q < s\}$  are a partition of the set  $H_0$ . By (14), (15) and (16),  $|\mu_{i_{n_q} j_{n_q}}(C)| < 1$ ,  $\mu_{i_{n_q} j_{n_q}}(B_{i_{n_q} j_{n_q}}) > j_{n_q} > n_q$  and  $|\mu_{i_{n_q} j_{n_q}}(D)| < 1$ , for each  $q \in \mathbb{N} \setminus \{1\}$ . Therefore the inequality

$$|\mu_{i_{n_q} j_{n_q}}(H_0)| > -|\mu_{i_{n_q} j_{n_q}}(C)| + \mu_{i_{n_q} j_{n_q}}(B_{i_{n_q} j_{n_q}}) - |\mu_{i_{n_q} j_{n_q}}(D)| > n_q - 2,$$

implies that

$$\lim_p |\mu_{i_{n_p} j_{n_p}}(H_0)| = \infty,$$

contradicting (18).  $\square$

The following corollary extends Theorems 2 and 3 in [14]. Again following [7, 7 Chapter 7, 35.1] a family  $\{B_{m_1 m_2 \dots m_i} : i, m_j \in \mathbb{N}, 1 \leq j \leq i \leq p\}$  of subsets of  $A$  is an *increasing  $p$ -web in  $A$*  if  $(B_{m_1})_{m_1}$  is an increasing covering of  $A$  and  $(B_{m_1 m_2 \dots m_{i+1}})_{m_{i+1}}$  is an increasing covering of  $B_{m_1 m_2 \dots m_i}$ , for each  $m_j \in \mathbb{N}$ ,  $1 \leq j \leq i < p$ .

**Corollary 13.** *Let  $\mathcal{S}$  be a  $\sigma$ -algebra of subsets of  $\Omega$  and let  $\{B_{m_1 m_2 \dots m_i} : i, m_j \in \mathbb{N}, 1 \leq j \leq i \leq p\}$  be an increasing  $p$ -web in  $\mathcal{S}$ . Then there exists  $\mathcal{B}_{n_1, n_2, \dots, n_p}$  such that if  $(\mathcal{B}_{n_1, n_2, \dots, n_p s_{p+1}})_{s_{p+1}}$  is an increasing covering of  $\mathcal{B}_{n_1, n_2, \dots, n_p}$  there exists  $n_{p+1} \in \mathbb{N}$  such that each  $\tau_s(\mathcal{B}_{n_1, n_2, \dots, n_p n_{p+1}})$ -Cauchy sequence  $(\mu_n)_n$  in  $\text{ba}(\mathcal{S})$  is  $\tau_s(\mathcal{S})$ -convergent.*

**Proof.** By [Theorem 2](#) there exists  $\mathcal{B}_{n_1 n_2 \dots n_p}$  which has  $sN$ -property. Hence there exists  $\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}}$  which has  $N$ -property. Then a  $\tau_s(\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}})$ -Cauchy sequence  $(\mu_n)_n$  is  $\tau_s(\mathcal{A})$ -relatively compact. As  $\overline{L(\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}})} = L(\mathcal{S})$  the sequence  $(\mu_n)_n$  has no more than one  $\tau_s(\mathcal{A})$ -adherent point, whence  $(\mu_n)_n$  is  $\tau_s(\mathcal{A})$ -convergent.  $\square$

#### 4. Applications

We present some applications of [Theorem 2](#) concerning localizations of bounded finitely additive vector measures.

A *finitely additive vector measure*, or simply a *vector measure*,  $\mu$  defined in an algebra  $\mathcal{A}$  of subsets of  $\Omega$  with values in a topological vector space  $E$  is a map  $\mu: \mathcal{A} \rightarrow E$  such that  $\mu(B \cup C) = \mu(B) + \mu(C)$ , for each pairwise disjoint subsets  $B, C \in \mathcal{A}$ . The vector measure  $\mu$  is *bounded* if  $\mu(\mathcal{A})$  is a bounded subset of  $E$ , or, equivalently, if the  $E$ -valued linear map  $\mu: L(\mathcal{A}) \rightarrow E$  defined by  $\mu(\chi_B) := \mu(B)$ , for each  $B \in \mathcal{A}$ , is continuous.

A locally convex space  $E(\tau)$  is an  $(LF)$ - or  $(LB)$ -space if it is, respectively, the inductive limit of an increasing sequence  $(E_m(\tau_m))_m$  of Fréchet or Banach spaces where the relative topology  $\tau_{m+1}|_{E_m}$  induced on  $E_m$  is coarser than  $\tau_m$ , for each  $m \in \mathbb{N}$ .  $(E_m(\tau_m))_m$  is a *defining* sequence for  $E(\tau)$  with *steps*  $E_m(\tau_m)$ ,  $m \in \mathbb{N}$ , and we write  $E(\tau) = \Sigma_m E_m(\tau_m)$ . If  $\tau_{m+1}|_{E_m} = \tau_m$ , for each  $m \in \mathbb{N}$ , then  $E(\tau)$  is a *strict*  $(LF)$ -, or  $(LB)$ -space. From [\[7, 19.4\(4\)\]](#) it follows that if  $\mu: \mathcal{A} \rightarrow E(\tau)$  is a vector bounded measure with values in a strict  $(LF)$ -space  $E(\tau) = \Sigma_m E_m(\tau_m)$  then there exists  $n \in \mathbb{N}$  such that  $\mu(\mathcal{A})$  is a bounded subset of the step  $E_n(\tau_n)$ . For  $\sigma$ -algebras the following extension of this result is contained in [\[14, Theorem 4\]](#).

**Theorem 14.** *Let  $\mu$  be a bounded vector measure defined in a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  with values in an  $(LF)$ -space  $E(\tau) = \Sigma_m E_m(\tau_m)$ . Then there exists  $n \in \mathbb{N}$  such that  $\mu(\mathcal{S})$  is a bounded subset of  $E_n(\tau_n)$ .*

[Theorem 2](#) provides the following proposition that contains [Theorem 14](#) as a particular case.

**Proposition 15.** *Let  $\mu$  be a bounded vector measure defined in a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  with values in a topological vector space  $E(\tau)$ . Suppose that  $\{E_{m_1, m_2, \dots, m_i} : m_j \in \mathbb{N}, 1 \leq j \leq i \leq p\}$  is an increasing  $p$ -web in  $E$ . Then there exists  $E_{n_1, n_2, \dots, n_p}$  such that if  $E_{n_1, n_2, \dots, n_p}(\tau_{n_1, n_2, \dots, n_p})$  is an  $(LF)$ -space, the topology  $\tau_{n_1, n_2, \dots, n_p}$  is finer than the relative topology  $\tau|_{E_{n_1, n_2, \dots, n_p}}$  and if  $(E_{n_1, n_2, \dots, n_p, s_{p+1}}(\tau_{n_1, n_2, \dots, n_p, s_{p+1}}))_{s_{p+1}}$  is a defining sequence for  $E_{n_1, n_2, \dots, n_p}(\tau_{n_1, n_2, \dots, n_p})$  there exists  $n_{p+1} \in \mathbb{N}$  such that  $\mu(\mathcal{S})$  is a bounded subset of  $E_{n_1, n_2, \dots, n_p, n_{p+1}}(\tau_{n_1, n_2, \dots, n_p, n_{p+1}})$ .*

**Proof.** Let  $\mathcal{B}_{m_1, m_2, \dots, m_i} := \mu^{-1}(E_{m_1, m_2, \dots, m_i})$  for each  $m_j \in \mathbb{N}$ ,  $1 \leq j \leq i \leq p$ . By [Theorem 2](#) there exists  $(n_1, n_2, \dots, n_p) \in \mathbb{N}^p$  such that  $\mathcal{B}_{n_1, n_2, \dots, n_p}$  has  $sN$ -property. Let  $(E_{n_1, n_2, \dots, n_p, s_{p+1}}(\tau_{n_1, n_2, \dots, n_p, s_{p+1}}))_{s_{p+1}}$  be a defining sequence for  $E_{n_1, n_2, \dots, n_p}(\tau_{n_1, n_2, \dots, n_p})$  and let  $\mathcal{B}_{n_1, n_2, \dots, n_p, s_{p+1}} := \mu^{-1}(E_{n_1, n_2, \dots, n_p, s_{p+1}})$ .

As  $(\mathcal{B}_{n_1, n_2, \dots, n_p, s_{p+1}})_{s_{p+1}}$  is an increasing covering of  $\mathcal{B}_{n_1, n_2, \dots, n_p}$  there exists  $n_{p+1}$  such that  $\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}}$  has  $N$ -property, whence  $L(\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}})$  is a dense subspace of  $L(\mathcal{S})$  and then the map with closed graph

$$\mu|_{L(\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}})}: L(\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}}) \rightarrow E_{n_1, n_2, \dots, n_p, n_{p+1}}(\tau_{n_1, n_2, \dots, n_p, n_{p+1}})$$

has a continuous extension  $v: L(\mathcal{S}) \rightarrow E_{n_1, n_2, \dots, n_p, n_{p+1}}(\tau_{n_1, n_2, \dots, n_p, n_{p+1}})$  (by [\[12, 2.4 Definition and \(N<sub>2</sub>\)\]](#) and [\[13, Theorems 1 and 14\]](#)). The continuity of  $\mu: L(\mathcal{S}) \rightarrow E(\tau)$  implies that  $v(A) = \mu(A)$ , for each  $A \in \mathcal{S}$ . Whence  $\mu(\mathcal{S})$  is a bounded subset of  $E_{n_1, n_2, \dots, n_p, n_{p+1}}(\tau_{n_1, n_2, \dots, n_p, n_{p+1}})$ .  $\square$

[Proposition 15](#) also holds if we replace  $(LF)$ -space by an inductive limit of  $\Gamma_r$ -spaces (see [\[13, Definition 1\]](#) and, taking into account [\[12, Property \(N<sub>2</sub>\) after 2.4 Definition\]](#), apply again [\[13, Theorems 1 and 14\]](#)).



A particular case of this proposition is the next corollary, which it is also a concrete generalization of [Theorem 14](#).

**Corollary 16.** *Let  $\mu$  be a bounded vector measure defined in a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  with values in an inductive limit  $E(\tau) = \Sigma_m E_m(\tau_m)$  of an increasing sequence  $(E_m(\tau_m))_m$  of (LF)-spaces. There exists  $n_1 \in \mathbb{N}$  such that for each defining sequence  $(E_{n_1, m_2}(\tau_{n_1, m_2}))_{m_2}$  of  $E_{n_1}(\tau_{n_1})$  there exists  $n_2 \in \mathbb{N}$  which verifies that  $\mu(\mathcal{S})$  is a bounded subset of  $E_{n_1, n_2}(\tau_{n_1, n_2})$ .*

A sequence  $(x_k)_k$  in a locally convex space  $E$  is *subseries convergent* if for every infinite subset  $J$  of  $\mathbb{N}$  the series  $\Sigma\{x_k : k \in J\}$  converges and  $(x_k)_k$  is *bounded multiplier* if for every bounded sequence of scalars  $(\lambda_k)_k$  the series  $\Sigma_k \lambda_k x_k$  converges.

A Fréchet space  $E$  is Fréchet Montel if each bounded subset of  $E$  is relatively compact. Important classes of Montel and Fréchet Montel spaces are considered and studied while Schwartz Theory of Distributions is described, for instance, in [\[6, Chapter 3, Examples 3, 4, 5 and 6.\]](#)

The following corollary is a generalization of [\[14, Corollary 1.4\]](#) and it follows partially from [Corollary 16](#).

**Corollary 17.** *Let  $(x_k)_k$  be a subseries convergent sequence in an inductive limit  $E(\tau) = \Sigma_m E_m(\tau_m)$  of an increasing sequence  $(E_m(\tau_m))_m$  of (LF)-spaces. Then there exists  $n_1 \in \mathbb{N}$  such that for each defining sequence  $(E_{n_1, m_2}(\tau_{n_1, m_2}))_{m_2}$  for  $E_{n_1}(\tau_{n_1})$  there exists  $n_2 \in \mathbb{N}$  such that  $\{x_k : k \in \mathbb{N}\}$  is a bounded subset of  $E_{n_1, n_2}(\tau_{n_1, n_2})$ . If, additionally,  $E_{n_1, n_2}(\tau_{n_1, n_2})$  is a Fréchet Montel space then the sequence  $(x_k)_k$  is bounded multiplier in  $E_{n_1, n_2}(\tau_{n_1, n_2})$ .*

**Proof.** As the sequence  $(x_k)_k$  is subseries convergent then the additive vector measure  $\mu: 2^{\mathbb{N}} \rightarrow E(\tau)$  defined by  $\mu(J) := \Sigma_{k \in J} x_k$ , for each  $J \in 2^{\mathbb{N}}$ , is bounded, because as  $(f(x_k))_k$  is subseries convergent for each  $f \in E'$  we get that  $\Sigma_{k=1}^{\infty} |f(x_k)| < \infty$ .

By [Corollary 16](#) there exists  $n_1 \in \mathbb{N}$  such that for each defining sequence  $(E_{n_1, m_2}(\tau_{n_1, m_2}))_{m_2}$  for  $E_{n_1}(\tau_{n_1})$  there exists  $n_2 \in \mathbb{N}$  with the property that  $\mu(2^{\mathbb{N}}) = \{\Sigma_{k \in J} x_k : J \in 2^{\mathbb{N}}\}$  is a bounded subset of  $E_{n_1, n_2}(\tau_{n_1, n_2})$ . Then  $\Sigma_k |\lambda_k f(x_k)| < \infty$  for each continuous linear form  $f$  defined on  $E_{n_1, n_2}(\tau_{n_1, n_2})$  and each bounded sequence  $(\lambda_k)_k$  of scalars, whence  $(\Sigma_{j=1}^k \lambda_j x_j)_k$  is a bounded sequence in  $E_{n_1, n_2}(\tau_{n_1, n_2})$  which has at most one adherent point, because  $\Sigma_k \lambda_k f(x_k)$  converges for each  $f \in (E_{n_1, n_2}(\tau_{n_1, n_2}))'$ . If  $E_{n_1, n_2}(\tau_{n_1, n_2})$  is a Montel space then the bounded subset  $\{\Sigma_{j=1}^k \lambda_j x_j : k \in \mathbb{N}\}$  is relatively compact and then the series  $\Sigma_k \lambda_k x_k$  converges in  $E_{n_1, n_2}(\tau_{n_1, n_2})$ .  $\square$

Recall that a vector measure  $\mu$  defined in an algebra  $\mathcal{A}$  of subsets of  $\Omega$  with values in a Banach space  $E$  is *strongly additive* whenever given a sequence  $(B_n)_n$  of pairwise disjoint elements of  $\mathcal{A}$  the series  $\Sigma_n \mu(B_n)$  converges in norm [\[2, I.1. Definition 14\]](#). Each strongly additive vector measure  $\mu$  is bounded [\[2, I.1. Corollary 19\]](#).

**Corollary 18.** *Let  $\mu$  be a bounded vector measure defined in a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  with values in an inductive limit  $E(\tau) = \Sigma_m E_m(\tau_m)$  of an increasing sequence  $(E_m(\tau_m))_m$  of (LB)-spaces such that each  $E_m(\tau_m)$  admit a defining sequence  $(E_{m, m_2}(\tau_{m, m_2}))_{m_2}$  of Banach spaces which does not contain a copy of  $l^\infty$ . If  $H$  is a dense subset of  $E'(\tau_s(E))$  such that  $f\mu$  is countably additive for each  $f \in H$ , then there exists  $(n_1, n_2) \in \mathbb{N}^2$  such that  $\mu$  is a  $E_{n_1, n_2}(\tau_{n_1, n_2})$ -valued countably additive vector measure.*

**Proof.** By [Corollary 16](#) there exists  $(n_1, n_2) \in \mathbb{N}^2$  such that  $\mu(\mathcal{S})$  is a bounded subset of  $E_{n_1, n_2}(\tau_{n_1, n_2})$ . As  $E_{n_1, n_2}(\tau_{n_1, n_2})$  does not contain a copy of  $l^\infty$  then, by [\(\[2, I.4. Theorem 2\]\)](#), the measure  $\mu$  is strongly additive, hence if  $(B_n : n \in \mathbb{N})$  is a sequence of pairwise disjoint subsets of  $\mathcal{S}$  then  $\Sigma_n \mu(B_n)$  converges to the vector  $x$  in  $E_{n_1, n_2}(\tau_{n_1, n_2})$ . Therefore  $f(x) = \Sigma_n f\mu(B_n)$  for each  $f \in E'$  and, by countably additivity of

$f\mu$  when  $f \in H$ , we have that  $f(x) = \sum_n f\mu(B_n) = f\mu(\cup_n B_n)$  for each  $f \in H$ . By density  $x = \mu(\cup_n B_n)$ , whence  $\sum_n \mu(B_n) = \mu(\cup_n B_n)$  in  $E_{n_1, n_2}(\tau_{n_1, n_2})$ .  $\square$

**Proposition 19.** *Let  $\mu$  be a bounded vector measure defined in a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  with values in a topological vector space  $E(\tau)$ . Suppose that  $\{E_{m_1, m_2, \dots, m_i} : m_j \in \mathbb{N}, 1 \leq j \leq i \leq p\}$  is an increasing  $p$ -web in  $E$ . There exists  $E_{n_1, n_2, \dots, n_p}$  such that if  $(E_{n_1, n_2, \dots, n_p, m_{p+1}})_{m_{p+1}}$  is an increasing covering of  $E_{n_1, n_2, \dots, n_p}$  with the property that each relative topology  $\tau|_{E_{n_1, n_2, \dots, n_p, m_{p+1}}}$ ,  $m_{p+1} \in \mathbb{N}$ , is sequentially complete then there exists  $n_{p+1} \in \mathbb{N}^p$  such that  $\mu(\mathcal{S}) \subset E_{n_1, n_2, \dots, n_p, n_{p+1}}$ .*

**Proof.** Let  $\mathcal{B}_{m_1, m_2, \dots, m_i} := \mu^{-1}(E_{m_1, m_2, \dots, m_i})$  for each  $m_j \in \mathbb{N}, 1 \leq j \leq i \leq p+1$ . By Theorem 2 there exists  $(n_1, n_2, \dots, n_p) \in \mathbb{N}^p$  such that  $\mathcal{B}_{n_1, n_2, \dots, n_p}$  has  $sN$ -property, whence there exists  $n_{p+1} \in \mathbb{N}^p$  such that  $\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}}$  has  $N$ -property, therefore  $E_{n_1, n_2, \dots, n_p, n_{p+1}}(\tau|_{E_{n_1, n_2, \dots, n_p, n_{p+1}}})$  is a dense subspace of  $E(\tau)$ , hence density and sequential completeness imply that the continuous restriction of  $\mu$  to  $L(\mathcal{B}_{n_1, n_2, \dots, n_p, n_{p+1}})$  has a continuous extension  $v$  to  $L(\mathcal{S})$  with values in the space  $E_{n_1, n_2, \dots, n_p, n_{p+1}}(\tau|_{E_{n_1, n_2, \dots, n_p, n_{p+1}}})$ . As  $\mu : L(\mathcal{S}) \rightarrow E(\tau)$  is continuous then  $v = \mu$  and we get that  $\mu(\mathcal{S}) \subset E_{n_1, n_2, \dots, n_p, n_{p+1}}$ .  $\square$

**Corollary 20.** *Let  $\mu$  be a bounded additive vector measure defined in a  $\sigma$ -algebra  $\mathcal{S}$  of subsets of  $\Omega$  with values in an inductive limit  $E(\tau) = \sum_{m_1} E_{m_1}(\tau_{m_1})$  of an increasing sequence  $(E_{m_1}(\tau_{m_1}))_{m_1}$  of countable dimensional topological vector spaces. Then there exists  $n_1$  such that  $\text{span}\{\mu(\mathcal{S})\}$  is a finite dimensional subspace of  $E_{n_1}(\tau_{n_1})$ .*

**Proof.** For each  $m_1 \in \mathbb{N}$  let  $(E_{m_1, m_2})_{m_2}$  be an increasing covering of  $E_{m_1}$  by finite dimensional vector subspaces.  $\{E_{m_1, m_2} : m_j \in \mathbb{N}, 1 \leq j \leq i \leq 2\}$  is an increasing 2-web in  $E$ . As the relative topology  $\tau|_{E_{m_1, m_2}}$  induced on  $E_{m_1, m_2}$  is complete then, by Proposition 19, there exists  $(n_1, n_2) \in \mathbb{N}^2$  such that  $\mu(\mathcal{S}) \subset E_{n_1, n_2}$ .  $\square$

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