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On a theorem of D. P. Baturov

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Abstract A deep result of C_p-theory, due to D. P. Baturov, states that if X is a Lindelöf Σ-space, for every subset H of $C_p(X)$ the extent of H equals the Lindelöf number of H. The most useful consequence of this result asserts that if X is a Lindelöf Σ-space, every countably compact subset H of $C_p(X)$ is a monolithic Fréchet–Urysohn compact. Since every compact set of $C_p(X)$ is Gul'ko compact whenever X is Lindelöf Σ and for such X a well known angelicity theorem due to Orihuela assures that $C_p(X)$ is angelic, both ingredients yield Baturov's utility grade theorem. In this paper we get a simple proof of Orihuela's theorem, which provides in turn a simple proof of Baturov's utility grade result. Our approach is independent of the original techniques used in both proofs.

Keywords Lindelöf Σ-space · Gul'ko compact set · Angelic space · Fréchet–Urysohn space · Locally convex space · Relatively countably compact set

Mathematics Subject Classification 54C35 · 46A03 · 54D30

1 Introduction

Baturov's theorem mentioned in the abstract can be found in [1, III.6.1 Theorem] and Orihuela's angelicity result can be seen in [5] or in [3, Theorem 4.5]. Both results are useful tools in C_p-theory, Descriptive Topology and Functional Analysis.

Let us recall that a completely regular space X is called *Lindelöf* Σ if there is a set $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ and a *usc* (upper semi-continuous) map $T : \Sigma \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ stands for the family of all compact subspaces of X, such that $\bigcup \{T(\alpha) : \alpha \in \Sigma\} = X$. A compact space X is called

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Gul'ko compact if $C_p(X)$ is a Lindelöf Σ-space. A compact space X is *monolithic* if each separable subspace Y of X is metrizable. A topological space X is called *Fréchet–Urysohn* if for each $Y \subseteq X$ and each $y \in \overline{Y}$ there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in Y such that $y_n \to y$ in X. Every Gul'ko compact set is both Fréchet–Urysohn and monolithic. A topological space X is called *web-compact* if there is a map T from a subspace Σ of $\mathbb{N}^{\mathbb{N}}$ into X such that $\overline{\bigcup \{T(\alpha) : \alpha \in \Sigma\}} = X$ and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{x_n\}$ has a cluster point in X (see [5, Definition] or [3, Chapter 4]). Clearly, every Lindelöf Σ-space is web-compact and necessarily each set $T(\alpha)$ of the previous definition is relatively countably compact. The family $\{T(\alpha) : \alpha \in \Sigma\}$ is usually referred to as *a web-compact representation* in X. If X is completely regular, we shall denote by υX the Hewitt *real compactification* of X. If $\mathcal{M} = \{L_\alpha : \alpha \in A\}$ is a family of linear subspaces of a topological vector space E, a linear functional u on E is called $\mathcal{M}$-*continuous* if u is continuous on each L_α.

2 Preliminary background

If X is a completely regular (Hausdorff) space, we denote by $C(X)$ the linear space of real-valued functions defined on X, or by $C_p(X)$ when equipped with the *pointwise* topology τ_p. The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when provided with the weak* topology. The linear space $C(X)$ equipped with the *compact-open* topology τ_c is represented by $C_c(X)$. We will not use Fremlin's angelic lemma.

Theorem 1 *Let X be completely regular. In order that $C_p(X)$ be angelic, the following conditions are equivalent*

1. *X is a Lindelöf Σ-space*
2. *X contains a Lindelöf Σ-subspace that separates the functions of $C(X)$*
3. *X contains a dense Lindelöf Σ-subspace*
4. *X is a realcompact web-space*
5. *X is a web space*

Proof Let us shorten by A the proposition 'the space $C_p(X)$ is angelic' and by H_i the ith-statement, for $1 \le i \le 5$, listed above.

1. Assume that proposition $H_1 \to A$ is true. If X contains a Lindelöf Σ-subspace Y that separates the functions of $C(X)$, then $L_p(Y)$ is a Lindelöf Σ-space by [1, 0.5.14 Corollary]. Moreover, since Y separates the functions of $C(X)$, it turns out that $\langle C(X), L(Y)\rangle$ is a dual pair, so that the weak topology $\sigma(C(X), L(Y))$ on $C(X)$ is Hausdorff. Thanks to $H_1 \to A$ is supposed to be true, $C_p(L_p(Y))$ is angelic. Since $(C(X), \sigma(C(X), L(Y)))$ is embedded in $C_p(L_p(Y))$ and angelicity respects the subspaces, $(C(X), \sigma(C(X), L(Y)))$ is angelic. Due to $\sigma(C(X), L(X))$ is a regular topology stronger than $\sigma(C(X), L(Y))$, one has that $C_p(X) = (C(X), \sigma(C(X), L(X)))$ is angelic too. So $H_2 \to A$ is true.
2. Assume that $H_2 \to A$ is true. If X contains a dense Lindelöf Σ-subspace Y, then Y separates the functions of $C(X)$. So, by hypothesis $C_p(X)$ is angelic and $H_3 \to A$ holds.
3. Assume that $H_3 \to A$ is true. Let X be a realcompact web-space and let $\{T(\alpha) : \alpha \in \Sigma\}$, with $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$, be a web-compact representation in X consisting of relatively countably compact sets. Since X is realcompact, each relatively countably compact set in X is relatively compact. According to [3, Proposition 4.6], these two facts imply that X contains a dense Lindelöf Σ-space. Thus condition 3 holds, which means that $C_p(X)$ must be angelic. Consequently $H_4 \to A$ is true.

4. Assume that $H_4 \to A$ is true. If X is a completely regular web-space, clearly υX is also a web-space. Since υX is realcompact, according to our assumption the space $C_p(\upsilon X)$ is angelic. But in this case, a standard argument using the properties of the Hewitt real compactification υX of X shows that the space $C_p(X)$ is angelic as well. Thus the proposition $H_5 \to A$ is true.
5. Assume finally that proposition $H_5 \to A$ is true. If X is a Lindelöf Σ-space, then obviously X is web-compact. So condition 5 holds and, according to our assumption, $C_p(X)$ ought be angelic. Thus proposition $H_1 \to A$ holds and we are done. □

The following property was proclaimed to be true for web-spaces X in [5], and actually proved in [2, Theorem 2]. We include it here for the sake of completeness. Another closed related result, consequence of Baturov's theorem, can be read in [1, IV.9.10].

Lemma 2 *Let X be completely regular. If X contains a Lindelöf Σ-subspace that separates the functions of $C(X)$, then every compact set of $C_p(X)$ is a Gul'ko compact space, hence a monolithic Fréchet–Urysohn space.*

Proof Let Y be a Lindelöf Σ-space of X that separates the functions of $C(X)$. So, there is a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a map T from Σ into the family $\mathcal{K}(Y)$ of compact sets of Y such that $\{T(\alpha) : \alpha \in \Sigma\}$ covers Y and if $\alpha_n \to \alpha$ in Σ and $y_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{y_n\}_{n=1}^{\infty}$ has a cluster point $y \in T(\alpha)$. Now let H be a compact set of $C_p(X)$ and define

$$S_\alpha = \left\{ \delta_y\big|_H : y \in T(\alpha) \right\}$$

for each $\alpha \in \Sigma$. Clearly S_α is a compact set of $C_p(H)$. Setting $M := \cup\{S_\alpha : \alpha \in \Sigma\} = \left\{\delta_y\big|_H : y \in Y\right\}$, we may define $S : \Sigma \to \mathcal{K}\left(C_p(H)\right)$ by the rule $S(\alpha) = S_\alpha$. Then it is obvious that M is a Lindelöf Σ-subspace of $C_p(H)$. Since Y separates the functions of $C(X)$, if $f, g \in H$ with $f \neq g$ there exists $z \in Y$ such that $f(z) \neq g(z)$, that is, $\left\langle \delta_z|_H, f\right\rangle \neq \left\langle \delta_z|_H, g\right\rangle$. Hence, M is a Lindelöf Σ-subspace of $C_p(H)$ which separates the points of H. So we may apply [7, Theorem 3.4] to guarantee that $C_p(H)$ is a Lindelöf Σ-space. This shows that H is a Gul'ko compact space, as stated. □

If X is a completely regular space, in what follows we shall denote by $\mathcal{N}$ the family of all countable subsets of X and by $\mathcal{M}$ the family of linear subspaces of $L(X)$ defined by

$$\mathcal{M} := \{\overline{L(N)} : N \in \mathcal{N}\},$$

where $L(N)$ is the linear span of $\{\delta_x : x \in N\}$ and the closure is in $L_p(X)$. If $f \in \mathbb{R}^X$, the *linearization* u_f of f on $L(X)$ is the linear functional on $L(X)$ given by

$$\left\langle u_f, \sum_{i=1}^{n} a_i\, \delta_{x_i} \right\rangle = \sum_{i=1}^{n} a_i\, f(x_i)$$

for $(a_i, x_i) \in \mathbb{R} \times X$.

Lemma 3 *If X is completely regular, the linear space $F_p(X)$ consisting of all real-valued functions defined on X whose linearization is $\mathcal{M}$-continuous, equipped with the pointwise topology on X, is realcompact.*

Proof Since $C_p(X) \hookrightarrow F_p(X) \hookrightarrow \mathbb{R}^X$ in a natural way, both spaces $C_p(X)$ and $F_p(X)$ have the same topological dual, which coincides with $L(X)$. Note that $\sigma(L(X), C(X))$

and $\sigma(L(X), F(X))$ coincide on $\overline{L(N)}$, where the closure is in $\sigma(L(X), C(X))$, for each $N \in \mathcal{N}$. Hence

$$\overline{L(N)}^{\sigma(L(X),C(X))} = \overline{L(N)}^{\sigma(L(X),F(X))}$$

and this space is separable when equipped with both relative topologies. Let us write $\mathcal{T}$ instead of $\sigma(L(X), F(X))$.

Let w be a linear functional on $L(X)$ such that $w|_E$ is continuous on every closed and separable linear subspace E of $(L(X), \mathcal{T})$. Then choose $f \in \mathbb{R}^X$ such that $u_f = w$, namely $f := w|_X$, where u_f stands for the linearization of f on $L(X)$. If $N \in \mathcal{N}$, the $\mathcal{T}$-closure $M = \overline{\langle N\rangle_{\mathbb{Q}}}^{\mathcal{T}}$ of the rational span $\langle N\rangle_{\mathbb{Q}}$ of N is a closed and separable linear subspace of $(L(X), \mathcal{T})$, hence $w|_M$ is $\mathcal{T}$-continuous on M. Observe that $\overline{L(N)}^{\sigma(L(X),C(X))} = \overline{L(N)}^{\mathcal{T}} = M$, so that the linear functional w is $\mathcal{M}$-continuous. Since w coincides with the linearization u_f of f, from the definition of the space $F(X)$ it follows that $f \in F(X)$. Therefore w is a continuous linear functional on $(L(X), \mathcal{T})$. According to Corson's criterion of weak realcompactness (see [8, Chapter 1, 8(6) and (7)]), the space $F_p(X)$ is realcompact. □

3 Simple proof of Orihuela's theorem

Let us provide a purely C_p-theory proof of Orihuela's angelicity theorem from we shall derive Baturov's utility grade theorem.

Theorem 4 (Orihuela) *Let X be a completely regular space. If X is web-compact, then $C_p(X)$ is angelic.*

Proof By Theorem 1 we may assume that X is a Lindelöf Σ -space. So let us suppose that there are a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a map T from Σ into the family $\mathcal{K}(X)$ of compact sets of X such that $\{T(\alpha) : \alpha \in \Sigma\}$ covers X and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x \in T(\alpha)$.

Select a relatively countably compact subset H of $C_p(X)$, whose closure $\overline{H}$ in $F_p(X)$ we shall represent by K. Since H is relatively countably compact in $C_p(X)$, clearly H is relatively countably compact in the realcompact (by Lemma 3) space $F_p(X)$. Hence K is a compact subset of $F_p(X)$.

Note that each $\delta_x \in L(X)$ is a $\sigma(C(X), L(X))$-continuous linear form on $C(X)$. Denote by $\overline{\delta}_x$ the (unique) continuous linear extension of δ_x to $F_p(X)$ and define

$$S_\alpha = \{\overline{\delta}_x\big|_K : x \in T(\alpha)\} \subseteq C(K)$$

for each $\alpha \in \Sigma$. We claim that S_α is a compact subset of $C_p(K)$. Let us show in first place that S_α is countably compact. If $\{\overline{\delta}_{x_n}\big|_K : d \in D\}$ is a sequence in S_α there are $x \in T(\alpha)$ and a subnet $\{y_d : d \in D\}$ of $\{x_n\}_{n=1}^{\infty}$ such that $y_d \to x$ in $T(\alpha)$ under the relative topology of X, so that $f(y_d) \to f(x)$ or rather $\left\langle \delta_{y_d}, f\right\rangle \to \langle \delta_x, f\rangle$ for all $f \in C(X)$. Using the fact that $y_d \to x$ happens in $\overline{N}$ with $N = \{x_n : n \in \mathbb{N}\}$ and that on $\overline{N}$ coincide the original topology of X with the initial topology defined by $F(X)$, we get that $f(y_d) \to f(x)$ or, written in another form, that $\langle \overline{\delta}_{y_d}\big|_K, f\rangle \to \langle \overline{\delta}_x\big|_K, f\rangle$ for all $f \in F_p(X)$. In particular $\langle \overline{\delta}_{y_d}\big|_K, h\rangle \to \langle \overline{\delta}_x\big|_K, h\rangle$ for each $h \in K$, so that $\overline{\delta}_{y_d}\big|_K \to \overline{\delta}_x\big|_K$ on S_α. This shows that S_α is a countably compact subspace of $C_p(K)$. But, given that K is compact, $C_p(K)$ is angelic by virtue of the classic Grothendieck theorem [1, III.4.1 Theorem]. So we must conclude that S_α is a compact set, as stated.

Set $M := \cup \{S_\alpha : \alpha \in \Sigma\} \subseteq C(K)$ and define the map $S : \Sigma \to \mathcal{K}(C_p(K))$ by the rule $S(\alpha) = S_\alpha$. If $\alpha_n \to \alpha$ in Σ and $u_n \in S(\alpha_n)$ for each $n \in \mathbb{N}$, then $u_n = \overline{\delta}_{z_n}\big|_K$ for some $z_n \in T(\alpha_n)$ and $n \in \mathbb{N}$. Let $z \in T(\alpha)$ be a cluster point of the sequence $\{z_n\}_{n=1}^{\infty}$ in X, so that δ_z is a cluster point of the sequence $\left\{\delta_{z_n}\right\}_{n=1}^{\infty}$ in $L_p(X)$. Setting $u := \overline{\delta}_z\big|_K \in S(\alpha)$ and taking into account that $N = \{z_n : n \in \mathbb{N}\}$ is a countable set, we conclude as before that u is a cluster point of $\{u_n\}_{n=1}^{\infty}$ in $C_p(K)$.

This shows that M is a Lindelöf Σ-subspace of $C_p(K)$. Let us prove next that M separates the points of K. Indeed, since $\langle L(X), F(X)\rangle$ is a dual pair, choose $g, h \in K$ with $g \neq h$. Given that $g - h \in F(X)$ there exists $\xi \in L(X)$ with $\langle \xi, g - h\rangle \neq 0$. So if $\xi = \sum_{i=1}^{n} b_i\, \delta_{y_i}$ one has

$$\sum_{i=1}^{n} b_i \,\langle \overline{\delta}_{y_i}, g - h\rangle \neq 0.$$

This entails the existence of some $j \in \{1, \ldots, n\}$ such that $\langle \overline{\delta}_{y_j}, g - h\rangle \neq 0$. In other words, we have that $\langle \overline{\delta}_{y_j}\big|_K, g\rangle \neq \langle \overline{\delta}_{y_j}\big|_K, h\rangle$.

Since K is a compact set and M is a Lindelöf Σ-subspace of $C_p(K)$ that separates the points of K, we can use [7, Theorem 3.4] to ensure that $C_p(K)$ is a Lindelöf Σ-space. Consequently K is a Gul'ko compact subset of $F_p(X)$. Now we claim that $K \subseteq C(X)$. Indeed, if $f \in K$ then $f \in \overline{H}$, where the closure is in $F_p(X)$. Hence, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in H such that $f_n \to f$ in $F_p(X)$. But since the set H is relatively countably compact in $C_p(X)$, there exists in $C(X)$ a τ_p-cluster point g of $\{f_n\}_{n=1}^{\infty}$. So, there is a subnet $\left\{f_{n_d} : d \in D\right\}$ such that $f_{n_d} \to g$, which implies that $f = g \in C(X)$.

The latter argument shows that H is a relatively compact subset of $C_p(X)$. So we must conclude that every relatively countably compact set of $C_p(X)$ is relatively compact. In particular, every relatively sequentially compact set of $C_p(X)$ is relatively compact. Finally, if Q is a compact set of $C_p(X)$ and $\{g_n\}_{n=1}^{\infty}$ is a sequence in Q , there is a weak cluster point g of $\{g_n\}_{n=1}^{\infty}$ in $C(X)$. Since $g \in \overline{\{g_n : n \in \mathbb{N}\}}$ and Q is Fréchet–Urysohn, there exists a subsequence $\left\{g_{n_k}\right\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ that converges to g. This shows that Q is sequentially compact. Therefore $C_p(X)$ is angelic. □

Remark 5 If $f \in F(X)$ then f is continuous on N for each $N \in \mathcal{N}$. Since u_f is continuous on $L(N)$, if v_f is a Hahn–Banach extension of $u_f\big|_{L(N)}$ to $L_p(X)$, we see that $f_N := v_f\big|_X \in C(X)$ coincides with f on N. Hence every $f \in F(X)$ is strictly $\aleph_0$-continuous in the sense of Arkhangel'skiĭ [1, Chapter II]. So, it follows from [1, II.4.17 Corollary] and Lemma 3 above that $C_p(X)$ is realcompact if and only if $F_p(X) = C_p(X)$.

Corollary 6 (Baturov) *Let X be a completely regular space. If X contains a Lindelöf Σ-subspace that separates the functions of $C(X)$, every compact set of $C_p(X)$ is a Gul'ko compact space and $C_p(X)$ is angelic. Consequently, every countably compact set of $C_p(X)$ is a monolithic Fréchet–Urysohn compact.*

Proof The first statement follows from Lemma 2. Since, by Theorem 4, $C_p(X)$ is angelic, each countably compact set of $C_p(X)$ is a monolithic Fréchet–Urysohn compact.

4 Approach by locally convex space techniques

In this complementary section we are going to see that usual locally convex machinery is capable to get at least the weak angelicity of $C_c(X)$. We shall only consider the case when X is a Lindelöf Σ-space.

Theorem 7 *Let X be a Lindelöf Σ-space. Usual locally convex techniques suffice to show that $C_c(X)$ is weakly angelic.*

Proof Let us suppose that there are a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a map T from Σ into the family $\mathcal{K}(X)$ of compact sets of X such that $\{T(\alpha) : \alpha \in \Sigma\}$ covers X and if $\alpha_n \to \alpha$ in Σ and $x_n \in T(\alpha_n)$ for all $n \in \mathbb{N}$ then $\{x_n\}_{n=1}^{\infty}$ has a cluster point $x \in T(\alpha)$. Let F denote the completion of $(C(X), \rho(C(X), L(X)))$, where $\rho(C(X), L(X))$ represents the topology on $C(X)$ of uniform convergence on the compact sets of $L_p(X)$. Since X is a μ-space, according to the Nachbin–Shirota theorem $C_c(X)$ is barrelled, so if E stands for the topological dual of $C_c(X)$ then τ_c coincides with $\beta(C(X), E)$. Since $\tau_c \leq \rho(C(X), L(X))$ and $\rho(C(X), L(X)) \leq \beta(C(X), L(X)) \leq \beta(C(X), E)$, it follows that $\rho(C(X), L(X)) = \tau_c$ on $C(X)$. Furthermore, the barrelledness of $C_c(X)$ also implies that $\rho(C(X), L(X)) = \mu(F, E)$ (this follows from [4, 21.4(4)]). Let us point out that, due to the fact that F is the completion of $(C(X), \rho(C(X), L(X)))$, the system $\langle F, L(X)\rangle$ is a dual pair. We follow as close as possible the proof of Theorem 6.

Select a relatively countably compact subset H of $(C(X), \sigma(C(X), E))$, whose closure $\overline{H}$ under the weak topology $\sigma(F, E)$ we shall represent by K. Since H is relatively countably compact in $(C(X), \sigma(C(X), E))$, clearly H is relatively countably compact in $(F, \sigma(F, E))$. So, given that $(F, \mu(F, E))$ is complete, Eberlein's theorem for locally convex spaces (see [6, Chapter 6, Theorem 4]) assures that K is a $\sigma(F, E)$-compact set.

Denote by $\widehat{\delta}_x$ the (linear and) $\sigma(F, L(X))$-continuous extension of δ_x to F and set

$$S_\alpha = \{\widehat{\delta}_x\big|_K : x \in T(\alpha)\} \subseteq C(K)$$

for each $\alpha \in \Sigma$. Note that S_α is a compact subset of $C_p(K)$, for if $\{\widehat{\delta}_{x_d}\big|_K : d \in D\}$ is a net in S_α there is $x \in T(\alpha)$ such that $x_d \to x$ in $T(\alpha)$ as a subspace of X, so that $\delta_{x_d} \to \delta_x$ in $L_p(X)$, i.e. under the weak topology $\sigma(L(X), C(X))$. But since $\rho(F, L(X))$ is the topology of uniform convergence on the compact sets of $L_p(X)$, that is, on the $\sigma(L(X), C(X))$-compact subsets of $L(X)$, both topologies $\sigma(L(X), C(X))$ and $\sigma(L(X), F)$ coincide on S_α (see [6, Chapter VI, Corollary 3]). This guarantees that $\widehat{\delta}_{x_d} \to \widehat{\delta}_x$ in S_α under $\sigma(L(X), F)$, which implies that $\langle \widehat{\delta}_{x_d}\big|_K, h\rangle \to \langle \widehat{\delta}_x\big|_K, h\rangle$ for every $h \in K$. This shows that S_α is a compact set in $C_p(K)$, as stated.

Set $M := \cup\{S_\alpha : \alpha \in \Sigma\}$ and define $S : \Sigma \to \mathcal{K}(C_p(K))$ by the rule $S(\alpha) = S_\alpha$. If $\alpha_n \to \alpha$ in Σ and $\varphi_n \in S(\alpha_n)$ for each $n \in \mathbb{N}$, then $\varphi_n = \widehat{\delta}_{y_n}\big|_K$ for some $y_n \in T(\alpha_n)$ and $n \in \mathbb{N}$. Let $y \in T(\alpha)$ be a cluster point of the sequence $\{y_n\}_{n=1}^{\infty}$ under the topology of X, so that δ_y is a cluster point of the sequence $\left\{\delta_{x_n}\right\}_{n=1}^{\infty}$ in $L_p(X)$. Setting $\varphi := \widehat{\delta}_y\big|_K \in S(\alpha)$, we claim that φ is a cluster point of $\{\varphi_n\}_{n=1}^{\infty}$ in $C_p(K)$. In fact, observe that $\bigcup_{n=1}^{\infty} T(\alpha_n)$ is a countably compact subset of X. So, setting $U_\alpha := \{\delta_x : x \in T(\alpha)\} \subseteq C(X)$ for each $\alpha \in \Sigma$, the set $\bigcup_{n=1}^{\infty} U_{\alpha_n}$ is countably compact in $L_p(X)$. But since X is a Lindelöf Σ-space, it turns out that $L_p(X)$ is a Lindelöf space too, so that $\bigcup_{n=1}^{\infty} U_{\alpha_n}$ is actually a compact subset of $L_p(X)$. Thus both topologies $\sigma(L(X), C(X))$ and $\sigma(L(X), F)$ coincide on $\bigcup_{n=1}^{\infty} U_{\alpha_n}$, or rather on $\bigcup_{n=1}^{\infty} \widehat{U}_{\alpha_n}$, where $\widehat{U}_{\alpha_n} := \{\widehat{\delta}_x : x \in T(\alpha_n)\}$ for $n \in \mathbb{N}$. Therefore $\widehat{\delta}_y$ is a $\sigma(L(X), F)$-cluster point of the sequence $\{\widehat{\delta}_{y_n}\}_{n=1}^{\infty}$, which implies that φ is a cluster point of $\{\varphi_n\}_{n=1}^{\infty}$ in $C_p(K)$. Hence M is a Lindelöf Σ-subspace of $C_p(K)$.

Let us prove next that M separates the points of K. So, choose $u, v \in K$ with $u \neq v$. Since $u - v \in F$ and $\langle L(X), F\rangle$ is a dual pair, there exists $\xi \in L(X)$ with $\langle \xi, u - v\rangle \neq 0$. But $\xi = \sum_{i=1}^{m} a_i\, \delta_{x_i}$ for $a_1, \ldots, a_m \in \mathbb{R}$ and $x_1, \ldots, x_m \in X$. Thus $\sum_{i=1}^{m} a_i \langle \widehat{\delta}_{x_i}, u - v\rangle \neq 0$ and there is $j \in \{1, \ldots, m\}$ with $\langle \widehat{\delta}_{x_j}\big|_K, u\rangle \neq \langle \widehat{\delta}_{x_j}\big|_K, v\rangle$.

Since K is a $\sigma(F, E)$-compact set and M is a Lindelöf Σ-subspace of $C_p(K)$ that separates the points of K, we have that $C_p(K)$ is a Lindelöf Σ-space. Consequently K is a

Fréchet–Urysohn space. Let us see that $K \subseteq C(X)$. Indeed, if $u \in K$ then $u \in \overline{H}$, where the closure is in $\sigma(F, E)$. Hence, there is a sequence $\{f_n\}_{n=1}^{\infty}$ in H such that $f_n \to u$ under $\sigma(F, E)$. But since H is relatively countably compact in $(C(X), \sigma(C(X), E))$, there exists in $C(X)$ a $\sigma(C(X), E)$-cluster point f of $\{f_n\}_{n=1}^{\infty}$. Hence $u = f \in C(X)$.

Thus every weakly relatively countably compact set of $C_c(X)$ is weakly relatively compact. Finally the same argument that was used in the final part of the proof of Theorem 6 shows that every weakly (relatively) sequentially compact set of $C_c(X)$ is (relatively) compact. So $(C(X), \sigma(C(X), E))$ is angelic. □

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