ON NON-ARCHIMEDEAN GURARIĬ SPACES

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Abstract. A Banach space E is *non-archimedean* if its norm satisfies the *strong triangle inequality*, so E is then a complete ultrametric space. Let $\mathcal{U}_{FNA}$ be the class of all non-archimedean finite-dimensional Banach spaces. A non-archimedean Gurariĭ Banach space $\mathbb{G}$ over a non-archimedean valued field $\mathbb{K}$ is constructed, i.e. a non-archimedean Banach space $\mathbb{G}$ of countable type which is of *almost universal disposition* for the class $\mathcal{U}_{FNA}$. This means: for every isometry $g : X \to Y$, where $Y \in \mathcal{U}_{FNA}$ and X is a subspace of $\mathbb{G}$, and every $\varepsilon \in (0, 1)$ there exists an ε-isometry $f : Y \to \mathbb{G}$ such that $f(g(x)) = x$ for all $x \in X$. We show that all non-archimedean Banach spaces of almost universal disposition for the class $\mathcal{U}_{FNA}$ are ε-isometric. Furthermore, all non-archimedean Banach spaces of almost universal disposition for the class $\mathcal{U}_{FNA}$ are isometrically isomorphic if and only if $\mathbb{K}$ is spherically complete and $\{|\lambda| : \lambda \in \mathbb{K}\setminus \{0\}\} = (0, \infty)$.

1. Introduction

In 1966 Gurariĭ constructed a separable (real) Banach space $\mathbb{G}$ of *almost universal disposition* for finite-dimensional spaces (called later the *Gurariĭ space*), see [3], which means the following condition:

(G) *For every isometry $g : X \to Y$, where Y is a finite-dimensional Banach space and X is a subspace of $\mathbb{G}$, and every $\varepsilon \in (0, 1)$ there exists an ε-isometry $f : Y \to \mathbb{G}$ such that $f(g(x)) = x$ for all $x \in X$.*

A linear operator $f : E \to F$ between Banach spaces E and F is an *ε-isometry* if for $x \in E$ with $\|x\| = 1$ one has $(1 + \varepsilon)^{-1} < \|f(x)\| < 1 + \varepsilon$. By an *isometry* we mean a linear operator $f : E \to F$ that is an ε-isometry for every $\varepsilon > 0$, that is, $\|f(x)\| = \|x\|$ for each $x \in E$.

One can prove easily that the Gurariĭ space $\mathbb{G}$ is unique up to isomorphism of norm arbitrarily close to one. Nevertheless, the question whether the Gurariĭ space is unique up to isometry remained open for a longer time. It was answered affirmatively by Lusky in 1976, see [10], who used quite technical and difficult methods involving techniques developed by Lazar and Lindenstrauss [9]. Much simpler proof has been provided by Kubiś and Solecki in 2013, see [5].

In [5] the authors proved the following

Theorem 1.1. *Let E, F be separable Gurariĭ spaces and $\varepsilon > 0$. Assume $X \subset E$ is a finite-dimensional space and $f : X \to F$ is an $\varepsilon-$isometry. Then there exists a bijective isometry $h : E \to F$ such that $\|h|_X - f\| < \varepsilon$.*

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Applying this result for X being the trivial space one gets the result of Lusky [10] stating that the Gurariĭ space is unique up to isometry.

A Banach space E is said to be of *universal disposition* for the class $\mathcal{U}$ of finite-dimensional spaces if it satisfies the following condition:

(G1) *For every isometry $j : X \to Y$, where $Y \in \mathcal{U}$ and $X \subset E$, there is an isometry $f : Y \to E$ such that $f(j(x)) = x$ for all $x \in X$.*

We refer the reader to [1] and [2], where recent developments in the study of Gurariĭ spaces, spaces of universal disposition, and related topics are surveyed.

In the present paper we study non-archimedean counterparts of the above concepts. The property of being of (almost) universal disposition for finite-dimensional non-archimedean normed spaces is defined precisely in the same way as for the real case mentioned above.

From now on, by $\mathbb{K}$ we will denote a non-archimedean complete non-trivially valued field, i.e. the valuation satisfies *the strong triangle inequality*:

$$|\lambda + \mu| \le \max\{|\lambda|, |\mu|\}$$

for all $\lambda, \mu \in \mathbb{K}$.

All linear spaces considered in this paper are over $\mathbb{K}$. Recall that

$$|\mathbb{K}^*| = \{|\lambda| : \lambda \in \mathbb{K} \setminus \{0\}\}$$

is the *value group* of $\mathbb{K}$.

$\mathbb{K}$ is said to be *discretely valued* if 0 is the only accumulation point of $|\mathbb{K}^*|$; then, there exists a *uniformizing element* $\rho \in \mathbb{K}$ with $0 < |\rho| < 1$ such that $|\mathbb{K}^*| = \{|\rho|^n : n \in \mathbb{Z}\}$. Otherwise, we say that $\mathbb{K}$ is *densely valued* (then, $|\mathbb{K}^*|$ is a dense subset of $[0, \infty)$).

By a *non-archimedean Banach space* we mean a Banach space E equipped with a non-archimedean norm $\|.\|$, i.e. a norm for which the triangle inequality is replaced by a stronger condition $\|x + y\| \le \max\{\|x\|, \|y\|\}$ for all $x, y \in E$.

An infinite-dimensional normed space E over $\mathbb{K}$ is of *countable type* if it contains a countable set whose linear hull is dense in E. If $\mathbb{K}$ is separable, then a normed space is of countable type if and only if it is separable.

We say that E (in particular, E may be equal to $\mathbb{K}$) is *spherically complete* if every shrinking sequence of balls in E has a non-empty intersection; otherwise, E is *non-spherically complete*. Every finite-dimensional Banach space over $\mathbb{K}$ has an equivalent non-archimedean norm. We refer the reader to the monographs [11] and [12] for non-archimedean concepts mentioned above.

We say that a spherically complete Banach space $\widehat{E}$ is the *spherical completion* of a non-archimedean Banach space E, if there exists an isometric embedding $i : E \to \widehat{E}$ and $\widehat{E}$ has no proper spherically complete linear subspace containing $i(E)$. Applying the natural identification, we will usually identify E with $i(E)$. Every Banach space (in particular $\mathbb{K}$) has the spherical completion and any two spherical completions of E are isometrically isomorphic ([12, Theorem 4.43]).

Let $\mathcal{U}_{FNA}$ be the class of all non-archimedean finite-dimensional normed spaces. As it can be expected, properties of spaces E of (almost) universal disposition for the class $\mathcal{U}_{FNA}$ strictly depend on the valued field $\mathbb{K}$, in particular, on whether it is spherically complete or not. In Section 3 we show that all non-archimedean Banach spaces of almost universal disposition for the class $\mathcal{U}_{FNA}$ are ε-isometric with arbitrarily small $\varepsilon > 0$ (Corollary 3.3). Moreover, all non-archimedean Banach spaces of almost universal disposition for the class $\mathcal{U}_{FNA}$ are isometrically isomorphic if and only if $\mathbb{K}$ is spherically complete and $|\mathbb{K}^*| = (0, \infty)$ (Proposition 3.4). The main result of this section is the following

Theorem 1.2. *Let $\mathbb{K}$ be a non-archimedean valued field. The following conditions are equivalent:*

(a) *Every non-archimedean Banach space of countable type over $\mathbb{K}$ is of almost universal disposition for the class $\mathcal{U}_{FNA}$.*
(b) *$\mathbb{K}$ is densely valued.*

Section 4 focuses on the study Banach spaces of universal disposition for the class $\mathcal{U}_{FNA}$. It turns out that a real Banach space G of almost universal disposition for the class $\mathcal{U}$ can be characterized (see [4]) by the following condition:

(H) for every $\varepsilon > 0$, for every finite-dimensional normed spaces $X \subset Y$ and for every isometric embedding $j : X \to G$, there is an isometric embedding $f : Y \to G$ such that $||j - f|_X|| < \varepsilon$.

In contrast to the real case, in a non-archimedean setting the condition (H) characterizes Banach spaces of universal disposition for the class $\mathcal{U}_{FNA}$. We show the following

Theorem 1.3. *Let G be a non-archimedean Banach space which satisfies the following property: For every finite-dimensional non-archimedean normed space Y and every isometric embedding $j : X \to G$, where $X \subset Y$ is a linear subspace, there is an isometry $f : Y \to G$ such that $||j - f|_X|| < 1$. Then G is of universal disposition for the class $\mathcal{U}_{FNA}$.*

Recall that a Banach space X is (*isometrically*) *universal* for the class of Banach spaces $\mathcal{C}$ if $X \in \mathcal{C}$ and for any $Y \in \mathcal{C}$, there is an isometrical embedding $Y \to X$. Note that, as a result of Banach-Mazur theorem, the space $C[0, 1]$ is isometrically universal for the class of separable real Banach spaces.

If $\mathbb{K}$ is spherically complete, we can properly select a set I and a map $s : I \to (0, \infty)$ such that every non-archimedean Banach space of countable type can be isometrically embedded into $E_u = c_0 (I : s)$. However E_u is isometrically universal for the class of non-archimedean Banach spaces of countable type if and only if $\mathbb{K}$ is spherically complete and $(0, \infty)$ is an union of at most countably many cosets of $|\mathbb{K}^*|$ (Proposition 4.4). On the other hand, E_u is never separable. If $\mathbb{K}$ is non-spherically complete, the role of $c_0 (I : s)$ is replaced by ℓ^∞, which clearly is not of countable type (Remark 4.5).

Applying Theorem 1.3 we prove that the spherical completion $\widehat{E_u}$ of E_u is a space of universal disposition for the class $\mathcal{U}_{FNA}$, see Theorem 4.6. We show also that the suitably selected proper linear subspace of $\widehat{E_u}$, denoted as E_h, is also of universal disposition for the class $\mathcal{U}_{FNA}$ (Theorem 4.7). If $\mathbb{K}$ is spherically complete, then $E_h = E_u$; hence, E_h has an orthogonal base and is of countable type if and only if $\mathbb{K}$ is spherically complete and $(0, \infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$ (Corollary 4.8).

2. Preliminaries

Let $t \in (0, 1]$. A subset $\{x_i : i \in I\} \subset E$ is called $t-$*orthogonal* (*orthogonal* for $t = 1$) if for each finite subset $J \subset I$ and all $\{\lambda_i\}_{i \in J} \subset \mathbb{K}$ we have

$$\left|\left|\sum_{i \in J} \lambda_i x_i\right|\right| \geq t \cdot \max_{i \in J} ||\lambda_i x_i|| .$$

If additionally $\overline{\left[\{x_i\}_{i\in I}\right]} = E$, then $\{x_i\}_{i\in I}$ is said to be a $t-$*orthogonal base* of E. Then every $x \in E$ has an unequivocal expansion

$$x = \sum_{i\in I} \lambda_i x_i \ (\lambda_i \in \mathbb{K},\ i \in I).$$

Every non-archimedean Banach space of countable type has a $t-$orthogonal base for each $t \in (0,1)$; if $\mathbb{K}$ is spherically complete, then every non-archimedean Banach space of countable type has an orthogonal base ([12, Lemma 5.5]). Every closed linear subspace of a non-archimedean Banach space with an orthogonal base has an orthogonal base ([12, Theorem 5.9]).

Linear subspaces D, D_0 of a non-archimedean Banach space E are called *orthogonal* if $||x+y|| = \max\{||x||, ||y||\}$ for all $x \in D$ and $y \in D_0$; then we will write $D \perp D_0$.

Let D be a closed linear subspace of E. Then D is *orthocomplemented* in E if there is a linear subspace D_0 of E such that $D + D_0 = E$ and $D \perp D_0$. Consequently, there exists a surjective projection (called an *orthoprojection*) $P : E \to D$ with $||P|| \le 1$. Observe that $D_1 \perp D_2$ implies $D_1 \cap D_2 = \varnothing$; hence the sum $D_1 + D_2$ is direct.

Let D and E_0 be linear subspaces of a normed space E. Recall that E_0 is called an *immediate extension* of D if $D \subset E_0$ and there is no nonzero element of E_0 that is orthogonal to D; in other words, for every $x \in E_0 \backslash D$ we have $dist\,(x, D) < ||x - d||$ for all $d \in D$. A spherical completion $\widehat{E}$ of E is a maximal immediate extension of E. Let I be a non-empty set and let $s : I \to (0,\infty)$ be a map. By

$$\ell^\infty\,(I:s) := \{\left(\lambda^i\right)_{i\in I} \in \mathbb{K}^I : \sup_{i\in I} |\lambda_i| \cdot s\,(i) < \infty\}$$

we denote the linear space over $\mathbb{K}$ equipped with the norm

$$\left|\left|(\lambda_i)_{i\in I}\right|\right| := \sup_{i\in I} |\lambda_i| \cdot s\,(i)\,.$$

Then $\ell^\infty\,(I:s)$ is a non-archimedean Banach space.

Let $c_0\,(I:s)$ be a closed linear subspace of $\ell^\infty\,(I:s)$ which consists of all $(\lambda^i)_{i\in I} \in \ell^\infty\,(I:s)$ such that for every $\varepsilon > 0$ there exists a finite $J \subset I$ for which $|\lambda^i| \cdot s\,(i) < \varepsilon$ for every $i \in I\backslash J$. If $s\,(i) = 1$ for all $i \in I$ we will write $\ell^\infty\,(I)$ and $c_0\,(I)$, respectively. In particular $\ell^\infty := \ell^\infty\,(\mathbb{N})$ and $c_0 := c_0\,(\mathbb{N})$.

Every non-archimedean Banach space which has an orthogonal base is isomorphic with $c_0\,(I)$ for some set I (see [12, Ch. 5]).

3. Non-archimedean Banach space of almost universal disposition for finite-dimensional spaces

First we prove the following technical fact.

Lemma 3.1. *Let E be a non-archimedean Banach space of countable type, let F be a finite-dimensional linear subspace of E, $t \in (0,1)$ and $\{x_1, ..., x_m\}$ be a $\sqrt{t}-$orthogonal base of F. Then there exist $x_{m+1}, x_{m+2}, ... \in E\backslash F$ such that (x_n) is a $t-$orthogonal base of E.*

Proof. By [11, Theorem 2.3.13] there exists a linear subspace $F_0 \subset E$ such that $E = F \oplus F_0$ and

$$||u_1 + u_2|| \ge \sqrt{t} \cdot \max\{||u_1||, ||u_2||\}$$

for all $u_1 \in F, u_2 \in F_0$. Applying [11, Theorem 2.3.7] we select a $\sqrt{t}-$orthogonal base (z_n) of F_0. Denote $x_{m+n} := z_n$ for every $n \in \mathbb{N}$. Then taking any $k \in \mathbb{N}$ and $\lambda_1, ..., \lambda_k \in \mathbb{K}$ one gets

$$\left|\left|\sum_{i=1}^{k} \lambda_i x_i\right|\right| \geq \sqrt{t} \cdot \max\left\{\left|\left|\sum_{i=1}^{m} \lambda_i x_i\right|\right|, \left|\left|\sum_{i=m+1}^{k} \lambda_i x_i\right|\right|\right\}$$
$$\geq \sqrt{t} \cdot \max\left\{\sqrt{t} \cdot \max_{i=1,...,m} ||\lambda_i x_i||, \sqrt{t} \cdot \max_{i=m+1,...,k} ||\lambda_i x_i||\right\} \geq t \cdot \max_{i=1,...,k} ||\lambda_i x_i||.$$

Hence, (x_n) is a required $t-$orthogonal base of E. □

Theorem 3.2. *A non-archimedean Banach space of countable type E is of almost universal disposition for the class $\mathcal{U}_{FNA}$ if and only if $||E^*||$ is a dense subset of $(0, \infty)$.*

Proof. Let E be a non-archimedean Banach space of countable type. Assume that $||E^*||$ is a dense subset of $(0, \infty)$. Let X be a finite-dimensional subspace of E, Y be a non-archimedean finite-dimensional normed space and $i : X \to Y$ be an isometrical embedding. Assume that $\dim Y = m$ and $\dim X = m_0$; clearly, $m_0 \leq m$. Fix $\varepsilon > 0$ and take $t \in (\frac{1}{\sqrt[3]{1+\varepsilon}}, 1)$. Applying Lemma 3.1 and [11, Theorem 2.3.7] we form a $t-$orthogonal base (x_n) of E such that $\{x_1, ..., x_{m_0}\}$ is a $\sqrt{t}-$orthogonal base of X. Now, applying Lemma 3.1 again, we select $y_{m_0+1}, ..., y_m \in Y$ such that $\{y_1, ..., y_m\}$ is a $t-$orthogonal base of Y, where $y_k = i(x_k)$ for $k \in \{1, ..., m_0\}$. Since, by assumption $||E^*||$ is a dense subset of $(0, \infty)$, we can assume that $1 \geq ||y_k|| \geq t$ $(k = 1, ..., m)$ and $1 \geq ||x_n|| \geq t$ $(n \in \mathbb{N})$; thus, $||y_k|| \geq t \cdot ||x_k||$ for each $k \in \{1, ..., m\}$.

Define $f : Y \to E$ by setting

$$f\left(\sum_{k=1}^{m} \lambda_k y_k\right) = \sum_{k=1}^{m} \lambda_k x_k.$$

Clearly, $f(i(x)) = x$ for all $x \in X$. Let $y = \sum_{k=1}^{m} \lambda_k y_k \in Y$. Then,

$$||y|| = \left|\left|\sum_{k=1}^{m} \lambda_k y_k\right|\right| \geq t \cdot \max_{k=1,...,m} \{||\lambda_k y_k||\} \geq t^2 \cdot \max_{k=1,...,m} \{||\lambda_k x_k||\} \geq t^3 \cdot \left|\left|\sum_{k=1}^{m} \lambda_k x_k\right|\right| = t^3 \cdot ||f(y)||.$$

On the other hand, we have

$$||y|| = \left|\left|\sum_{k=1}^{m} \lambda_k y_k\right|\right| \leq \max_{k=1,...,m} \{|\,||\lambda_k y_k||\} \leq \frac{1}{t} \max_{k=1,...,m} \{||\lambda_k x_k||\} \leq \frac{1}{t^2} \left|\left|\sum_{k=1}^{m} \lambda_k x_k\right|\right| = \frac{1}{t^2} \cdot ||f(y)||.$$

Thus $(1 - \varepsilon) ||y|| \leq ||f(y)|| \leq (1 + \varepsilon) ||y||$.

Now assume that $||E^*||$ is not dense in $(0, \infty)$. Then there exist $s_1 \in (0, \infty)$ and $\varepsilon > 0$ such that

$$||E^*|| \cap (s_1 - 2\varepsilon \cdot s_1, s_1 + 2\varepsilon \cdot s_1) = \varnothing.$$

Define $X = \mathbb{K}$ and $Y = (\mathbb{K}^2, ||.||_Y)$, where $||(\lambda_1, \lambda_2)||_Y := \max\{|\lambda_1|, s_1 \cdot |\lambda_2|\}$, $\lambda_1, \lambda_2 \in \mathbb{K}$.

Assume for a contradiction that there is an $\varepsilon-$isometry $f : Y \to E$. But then, taking $x_0 = (0, 1) \in Y$, we obtain $||x_0||_Y = s_1$ and $||f(x_0)|| \geq s_1 + 2\varepsilon \cdot s_1$ or $||f(x_0)|| \leq s_1 - 2\varepsilon \cdot s_1$. Hence, $||f(x_0)|| > (1 + \varepsilon) ||x_0||$, or $||f(x_0)|| < (1 - \varepsilon) ||x_0||$, a contradiction. □

Next conclusion follows directly from Theorem 3.2.

Corollary 3.3. *If $\mathbb{K}$ is densely valued, every non-archimedean Banach space of countable type is of almost universal disposition for the class $\mathcal{U}_{FNA}$. All non-archimedean Banach spaces of almost universal disposition for the class $\mathcal{U}_{FNA}$ are ε-isometric, where $\varepsilon > 0$ is arbitrarily small.*

However, as the next result shows, they need not be always isometric.

Proposition 3.4. *All non-archimedean Banach spaces over $\mathbb{K}$ of almost universal disposition for the class $\mathcal{U}_{FNA}$ are isometrically isomorphic if and only if $\mathbb{K}$ is spherically complete and $|\mathbb{K}^*| = (0, \infty)$.*

Proof. If $\mathbb{K}$ is spherically complete and $|\mathbb{K}^*| = (0,\infty)$ then every non-archimedean Banach space of countable type has an orthonormal base, thus, it is isometrically isomorphic with c_0. Hence, the conclusion follows.

Now assume that $\mathbb{K}$ is non-spherically complete. Then, since $\mathbb{K}$ is densely valued, c_0 and $\mathbb{K}_v^2 \oplus c_0$ are both of universal disposition for the class $\mathcal{U}_{FNA}$ by Theorem 3.2 (recall that $\mathbb{K}_v^2$ is a two-dimensional normed space without two orthogonal elements, see [11, Example 2.3.26]). Clearly, $\mathbb{K}_v^2 \oplus c_0$ and c_0 are not isometrically isomorphic.

Suppose that $|\mathbb{K}^*| \neq (0,\infty)$. Then we can find $s \in (0,\infty) \setminus |\mathbb{K}^*|$. Define the norm $||x||_s : c_0 \to [0,\infty)$ by

$$||x||_s := \max\{s \cdot |x_1|, \max_{n>1}\{|x_n|\}\},$$

and $x = (x_n) \in c_0$. Then, by Theorem 3.2, $E = (c_0, ||.||_s)$ and $F = (c_0, ||.||_\infty)$ are of almost universal disposition for the class $\mathcal{U}_{FNA}$. Since $||E|| \neq ||F||$, E and F are not isometrically isomorphic. □

Now, we are ready to prove Theorem 1.2, which characterizes $\mathbb{K}$, formulated in Introduction.

Proof of Theorem 1.2. Let E be a non-archimedean Banach space of countable type. If $\mathbb{K}$ is densely valued, $||E^*||$ is a dense subset of $(0,\infty)$ and the conclusion follows from Theorem 3.2. Assume now that $\mathbb{K}$ is discretely valued and ρ be a uniformizing element of $\mathbb{K}$. Let $E := c_0$. Set $s := \frac{|\rho|+1}{2}$ and take

$$\varepsilon < \frac{1-s}{s} = \frac{s-|\rho|}{s}.$$

Let $X = [e_1] \subset E$ and $Y = (\mathbb{K}^2, ||.||_s)$, where

$$||x||_s := \max\{|x_1|, s \cdot |x_2|\}, (x_1, x_2) \in \mathbb{K}^2.$$

Define an isometry $i : X \to Y$ by $i(\lambda e_1) := (\lambda x_1, 0)$ and assume that there exists an $\varepsilon-$isometry $f : Y \to E$. Then, for $x = (0,1) \in Y$ we get

$$(1-\varepsilon) \cdot s \leq ||f(x)|| \leq (1+\varepsilon) \cdot s.$$

Recall that $||E^*|| = \{|\rho|^n : n \in \mathbb{Z}\}$, hence $(|\rho|, 1) \cap ||E^*|| = \varnothing$. But $(1-\varepsilon) \cdot s > |\rho|$ and $(1+\varepsilon) s < 1$, a contradiction. □

4. Non-archimedean Banach space of universal disposition for finite-dimensional spaces

We start with the proof of Theorem 1.3, as promised in Introduction.

Proof of Theorem 1.3. We need to prove that there is an isometry $T : Y \to G$ which extends j. Let $t = \|j - f|_X\|$, $n_X = \dim X$ and $n_Y = \dim Y$. Clearly $n_X \le n_Y$. Choose $\{x_1, ..., x_{n_Y}\}$, a $\sqrt{t}-$orthogonal base of Y such that $[x_1, ..., x_{n_X}] = X$. Define $T : Y \to G$ by setting

$$T(x_n) := \begin{cases} j(x_n) & \text{if } n \le n_X \\ f(x_n) & \text{if } n > n_X \end{cases}, n = 1, ..., n_Y.$$

Clearly T extends j. We show that T is an isometry. Take $x \in Y$, written as $x = \sum_{i=1}^{n_Y} \lambda_i x_i$ ($\lambda_i \in \mathbb{K}$). Then, we obtain

$$\begin{aligned} \|T(x)\| &= \left\| \sum_{i=1}^{n_X} \lambda_i T(x_i) + \sum_{i=n_X+1}^{n_Y} \lambda_i T(x_i) \right\| \\ &= \left\| \sum_{i=1}^{n_X} \lambda_i j(x_i) - \sum_{i=1}^{n_X} \lambda_i f(x_i) + \sum_{i=1}^{n_X} \lambda_i f(x_i) + \sum_{i=n_X+1}^{n_Y} \lambda_i f(x_i) \right\| \\ &= \left\| (j-f)(\sum_{i=1}^{n_X} \lambda_i x_i) + f(\sum_{i=1}^{n_Y} \lambda_i x_i) \right\| = \|(j-f)(x_0) + f(x)\|, \end{aligned}$$

where $x_0 = \sum_{i=1}^{n_X} \lambda_i x_i \in X$. But,

$$\|x\| \ge \sqrt{t} \cdot \max \left\{ \|x_0\|, \sum_{i=n_X+1}^{n_Y} \lambda_i x_i \right\} \ge \sqrt{t} \cdot \|x_0\|$$

and

$$\|(j-f)(x_0)\| \le \|j - f|_X\| \cdot \|x_0\| = t \cdot \|x_0\| \le \sqrt{t} \cdot \|x\| < \|x\| = \|f(x)\|$$

hence, we get

$$\|T(x)\| = \|(j-f)(x_0) + f(x)\| = \|f(x)\| = \|x\|.$$

□

To prove the main result of this section we need a few technical lemmas (see also [12, Exercise 5.B and Lemma 4.42] and [11, Theorem 2.3.16]).

Lemma 4.1. *Let $0 < t \le 1$ and let $\{x_1, ..., x_n\}$ be a $t-$orthogonal set in a non-archimedean normed space E. If $\{z_1, ..., z_n\} \subset E$ and $\|x_i - z_i\| < t \cdot \|x_i\|$ for each $i \in \{1, ..., n\}$, then $\{z_1, ..., z_n\}$ is also $t-$orthogonal.*

Proof. Take any $\lambda_i \in \mathbb{K}$, $i = 1, ..., n$. Since $\|x_i - z_i\| < t \cdot \|x_i\|$, we have $\|z_i\| = \|x_i\|$ for each $i \in \{1, ..., n\}$. Consequently we note

$$\|\lambda_1 x_1 + ... + \lambda_n x_n\| \ge t \cdot \max_{i=1,...,n} \|\lambda_i x_i\|$$

and

$$\|\lambda_1(z_1 - x_1) + ... + \lambda_n(z_n - x_n)\| \le \max_{i=1,...,n} \|\lambda_i(z_i - x_i)\| < t \cdot \max_{i=1,...,n} \|\lambda_i x_i\|.$$

Hence

$$\begin{aligned}||\lambda_1 z_1 + ... + \lambda_n z_n|| &= ||\lambda_1 z_1 + ... + \lambda_n z_n - (\lambda_1 x_1 + ... + \lambda_n x_n) + (\lambda_1 x_1 + ... + \lambda_n x_n)|| \\ &= ||\lambda_1 (z_1 - x_1) + ... + \lambda_n (z_n - x_n) + (\lambda_1 x_1 + ... + \lambda_n x_n)|| \\ &= ||\lambda_1 x_1 + ... + \lambda_n x_n|| \geq t \cdot \max_{i=1,...,n} ||\lambda_i x_i|| = t \cdot \max_{i=1,...,n} ||\lambda_i z_i||,\end{aligned}$$

and we are done. □

Lemma 4.2. *Let E, F be non-archimedean normed spaces, D a linear subspace of E such that E is an immediate extension of D, let F be spherically complete and $T : D \to F$ be an isometry. Then T can be extended to a linear isometry $T' : E \to F$.*

Proof. Applying Ingleton's theorem (see [12, Theorem 4.8]) we can extend the isometry $T : D \to F$ to the linear operator $T' : E \to F$ such that $||T'|| \leq 1$. We prove that T' is also an isometry.

Set $x \in E \backslash D$. Then, since E is an immediate extension of D, there is $x_d \in D$ such that

$$||x - x_d|| < ||x_d|| = ||x||.$$

Thus we get

$$||T'(x) - T'(x_d)|| \leq ||T'|| \cdot ||x - x_d|| < ||x_d|| = ||T'(x_d)||$$

and

$$||T'(x)|| = ||T'(x) - T'(x_d) + T'(x_d)|| = ||T'(x_d)||.$$

Hence, finally $||T'(x)|| = ||x||$. □

Lemma 4.3. *Let Y be a finite-dimensional non-archimedean normed space and X be its linear subspace. Let $\{u_1, ...u_{m_Y}\}$ be a maximal orthogonal set in Y such that $\{u_1, ..., u_{m_X}\}$ is a maximal orthogonal set in X for some $m_X \leq m_Y$ and let $F_Y := [u_{m_X+1}, ..., u_{m_Y}]$. Then, $F_Y \perp X$.*

Proof. Assume that $m_Y > m_X$ (otherwise nothing is to prove). Take any $x \in X$ and $y \in F_Y$. If $x \in [u_1, ..., u_{m_X}]$, the conclusion is obvious. So, assume that $x \notin [u_1, ..., u_{m_X}]$. But then, since X is an immediate extension of $[u_1, ..., u_{m_X}]$, there is $x_0 \in [u_1, ..., u_{m_X}]$ with

$$||x - x_0|| < ||x|| = ||x_0||.$$

Thus, since

$$||x_0 + y|| = \max\{||x_0||, ||y||\},$$

we have

$$||x + y|| = ||x - x_0 + x_0 + y|| = ||x_0 + y|| = \max\{||x_0||, ||y||\} = \max\{||x||, ||y||\}.$$

□

Let $r := |\rho|$ if $\mathbb{K}$ is discretely valued, where $\rho \in \mathbb{K}$ is a uniformizing element of $|\mathbb{K}^*|$ with $0 < |\rho| < 1$, and let r be any number taken from $\left(\frac{1}{2}, 1\right)$ if $\mathbb{K}$ is densely valued. Note that $(0, \infty)$ is a multiplicative group. Let

$$\pi : (0, \infty) \to G := (0, \infty) / |\mathbb{K}^*| \tag{4.2}$$

be the quotient map and let $S = \{s_g : g \in G\}$ be a set of representatives of elements of G in $(r, 1]$, i.e. $\pi(s_g) = g$.

Let I_u be a set for which $\operatorname{card}(I_u) = \max\{\aleph_0, \operatorname{card}(G)\}$ and let $I_u = \bigcup_{g\in G} I_g$ where $\{I_g : g \in G\}$ is a partition of I_u such that $\operatorname{card}(I_g) = \aleph_0$ for each $g \in G$. Then, clearly $c_0(I_u) = \bigoplus_{g\in G} c_0(I_g)$.

Define the function $s : I_u \to (r, 1]$ by $h(i) := s_g$ if $i \in I_g$ and the norm on $c_0(I_u)$ by

$$||x||_u := \max_{i\in I_u} \{s(i) \cdot |x_i|\}, \ x = (x_i)_{i\in I} \in c_0(I_u).$$

Denote $E_u := (c_0(I_u), ||.||_u)$.

Proposition 4.4. *Let $\mathbb{K}$ be spherically complete. Then every non-archimedean Banach space of countable type can be isometrically embedded into E_u.*

- *E_u is of countable type (hence E_u is isometrically universal for the class of non-archimedean Banach spaces of countable type) if and only if $(0,\infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$;*
- *E_u is never separable.*

Proof. Let E be a non-archimedean Banach space of countable type. Since $\mathbb{K}$ is spherically complete, E has an orthogonal base (x_n) (see [11, Theorem 2.3.25]). Let

$$J_g = \{n : \pi(||x_n||) = g\}, \quad g \in G,$$

where π is the map defined in (4.2). Then $G_0 = \{g \in G : \ J_g \text{ is nonempty}\}$ is at most countable. So, we can write $\mathbb{N} = \bigcup_{g\in G_0} J_g$, where J_g $(g \in G_0)$ are nonempty, finite or infinite, pairwise disjoint subsets of $\mathbb{N}$.

Define the map $l : \mathbb{N} \to I_u$ (recall that $I_u = \bigcup_{g\in G} I_g$ and I_g is countable for every $g \in G$; thus we can write $I_g = \{m_1^g, m_2^g, ...\}$, $g \in G$) as follows: for every $n \in \mathbb{N}$ there exist $g \in G_0$ and $k \in \mathbb{N}$ such that $n = n_k^g$ (what means $n \in J_g$). Finally set $l(n) := m_k^g$.

Note that for every $n \in \mathbb{N}$ we can find $\lambda_n \in \mathbb{K}$ for which

$$||x_n|| = s_g \cdot |\lambda_n|.$$

Next, define the map $i_0 : \{x_1, x_2, ...\} \to E_u$ by the formula $x_n \longmapsto \lambda_n e_{l(n)}$, where e_n are as usual the unit vectors. Since

$$\left|\left|\lambda_n e_{l(n)}\right|\right|_u = s_g \cdot |\lambda_n| = ||x_n||,$$

we can extend the map i_0 to an isometric embedding $E \to E_u$.

Now we prove the next claim of the proposition. Suppose that $(0,\infty)$ is not the union of at most countably many cosets of $|\mathbb{K}^*|$ and assume that E_u is of countable type. By [11, Theorem 2.3.25] the space E_u has an orthogonal base (x_n). Since (x_n) is orthogonal, $||E_u||_u \setminus \{0\}$ consist of at most countably many cosets of $|\mathbb{K}^*|$. Hence there exists

$$s \in (0,\infty) \setminus ||E_u||_u.$$

Define $E = (\mathbb{K}^2, ||.||_s)$, where

$$||(x,y)||_s := \max\{|x|, s \cdot |y|\},$$

$(x, y) \in \mathbb{K}^2$. Then

$$||(0,1)||_s \notin ||E_u||_u.$$

Hence there is no isometry $E \to E_u$, a contradiction. If $(0,\infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$, then G, and consequently I_u is countable, and E_u is of countable type.

Finally, assume that $\mathbb{K}$ is separable. If $\mathbb{K}$ is discretely valued, then $(0,\infty)$ is always the union of more than countably many cosets of $|\mathbb{K}^*|$. On the other hand, by [13, Theorem 20.5] there is no separable densely valued spherically complete $\mathbb{K}$; hence E_u is not separable. ☐

Remark 4.5. If $\mathbb{K}$ is non-spherically complete, then E_u does not contain an isometric image of any non-archimedean Banach spaces of countable type. Indeed, in this case there exists finite-dimensional normed spaces without orthogonal bases, see [11, Example 2.3.26] and [6]. Take $E = \mathbb{K}_v^2$, where $\mathbb{K}_v^2$ is a two-dimensional normed space over $\mathbb{K}$ without two non-zero orthogonal elements, and assume that there exists an isometric embedding $i : E \to E_u$. Then the image $i(E)$ has no two non-zero orthogonal elements. But this contradicts the conclusion of Gruson's theorem ([12, Theorem 5.9]) stating that every linear subspace of a non-archimedean Banach spaces with an orthogonal base has an orthogonal base.

If $\mathbb{K}$ is non-spherically complete the role of $c_0(I:s)$ takes the space ℓ^∞. In this case, by [11, Theorem 2.5.13], every non-archimedean Banach space of countable type can be isometrically embedded into ℓ^∞.

Finally we prove the following

Theorem 4.6. *The spherical completion $\widehat{E_u}$ of E_u is a non-archimedean Banach space of universal disposition for the class $\mathcal{U}_{FNA}$.*

Proof. Denote $F := \widehat{E_u}$. Let $X \subset F$ and let $j : X \to Y$ be an isometric embedding, where Y is a finite non-archimedean normed space. We prove, that there exists an isometric embedding $f : Y \to F$ such that $f(j(x)) = x$ for all $x \in X$.

Choose a maximal orthogonal set $\{u_1, ..., u_{m_X}, ..., u_{m_Y}\}$ in Y such that $\{u_1, ..., u_{m_X}\}$ is a maximal orthogonal set in $j(X)$ for some $m_Y \geq m_X \geq 1$. Set $F_Y := [u_{m_X+1}, ..., u_{m_Y}]$. By Lemma 4.3 we get $F_Y \perp j(X)$.

Set $v_k = f(u_k) := j^{-1}(u_k)$ for each $k \in \{1, ..., m_X\}$. For every $k \in \{m_{X+1}, ..., m_Y\}$ choose $i_k \in I$ such that $||e_{i_k}||_u = ||\lambda_k u_k||$ for some $\lambda_k \in \mathbb{K}$ and

$$e_{i_k} \perp [v_1, ..., v_{m_X}, e_{i_{m_{X+1}}}, ..., e_{i_{k-1}}].$$

Next set $f(u_k) := e_{i_k}$ for $k = m_{X+1}, ..., m_Y$. Define $f : j(X) + F_Y \to F$. Clearly f is an isometry and $f(j(X) + F_Y) \subset E_h^0$.

If $\mathbb{K}$ is spherically complete, we are done, as $\{u_1, ..., u_{m_Y}\}$ is an orthogonal base of Y by [12, Lemma 5.5 and Theorem 5.15]; hence, f is a required isometry defined on Y. If $\mathbb{K}$ is non-spherically complete and $j(X) + F_Y \neq Y$, then, by [7, Proposition 2.1], Y is an immediate extension of $j(X) + F_Y$. Now, using Lemma 4.2, we extend f to the isometry defined on Y. ☐

The last part of the proof of Proposition 4.6 uses Lemma 4.2 for the spherical completeness of the considered space F. In fact it is enough to assume that F contains a spherical completion of its every finite-dimensional linear subspace. This observation suggests another construction.

For each $g \in G$ set $I_g = \{i_{g,1}, i_{g,2}, ...\}$ (note that I_g is countable). For every $n \in \mathbb{N}$ set $F_g^n := \left[e_{i_{g,1}}, ..., e_{i_{g,n}}\right]$, a finite-dimensional linear subspace of $c_0(I_u)$ spanned by appropriate unit vectors. By $\widehat{F_g^n}$ denote a spherical completion of F_g^n such that for fixed $g \in G$ we have $F_g^n \subset \widehat{F_g^n} \subset \widehat{c_0(I_g)}$ and $\widehat{F_g^{n-1}} \subset \widehat{F_g^n}$ if $n > 1$.

Next, for every $g \in G$ define $F_g := \bigcup_n \widehat{F_g^n}$. Let $E_h := \bigoplus_{g\in G} \overline{F_g}$.
We are in a position to prove Theorem 4.7.

Theorem 4.7. *The space E_h is of universal disposition for the class $\mathcal{U}_{FNA}$.*

Proof. Let Y be a finite-dimensional non-archimedean normed space, let $X \subset Y$, and let $j : X \to E_h$ be an isometric embedding. We prove that there exists an isometry $f : Y \to E_h$ such that $||j - f|_X|| < 1$. Then if we apply Theorem 1.3, the proof will be finished.

Set $t \in (0, 1)$ and choose a $t-$orthogonal base $\{x_1, ..., x_{n_X}\}$ of X. Let $z_i := j(x_i)$, $i = 1, ..., n_X$. Then for each $i \in \{1, ..., n_X\}$ there exists a finite $G_i \subset G$, say $G_i = \{g_1, ..., g_m\}$, and

$$z_i' \in \bigoplus_{g\in G_i} \overline{F_g}$$

for which

$$||z_i' - z_i|| < \frac{t}{2} ||z_i|| .$$

Fix $i \in \{1, ..., n_X\}$. Then we can write $z_i' = z_{i,1}' + ... + z_{i,m}'$, where $z_{i,k}' \in \overline{F_{g_k}}$, $k = 1, .., m$. But, then we can select n_i and $w_{i,1}', ..., w_{i,m}'$ such that $w_{i,k}' \in \widehat{F_{g_k}^{n_i}}$, and $||w_{i,k}' - z_{i,k}'|| < \frac{t}{2}||z_{i,k}'||$, $k = 1, .., m$. Denote $w_i := w_{i,1}' + ... + w_{i,m}'$. Then

$$\begin{aligned} ||w_i - z_i|| = ||w_i - z_i' + z_i' - z_i|| &\le \max\{||w_i - z_i'|| , ||z_i' - z_i||\} \\ &\le \max\{\max_{k=1,...,m} \left|\left|w_{i,k}' - z_{i,k}'\right|\right| , ||z_i' - z_i||\} < \frac{t}{2} ||z_i|| . \end{aligned}$$

Hence, by Lemma 4.1, $\{w_i : i \in \{1, ..., n_X\}\}$ is a $t-$orthogonal set in $\bigoplus_{g\in G_0} \widehat{F_g^{n_0}}$, where $G_0 = G_1 \cup ... \cup G_{n_X}$ and $n_0 = \max\{n_i : i \in \{1, ..., n_X\}\}$.

Define the map $f : X \to \bigoplus_{g\in G_0} \widehat{F_g^{n_0}} \subset E_h$ setting $f(x_i) := w_i$, $i = 1, ..., n_X$. Then for all $\lambda_i \in \mathbb{K}$ $(i = 1, ..., n_X)$ we have

$$f : \sum_{i=1}^{n_X} \lambda_i x_i \longmapsto \sum_{i=1}^{n_X} \lambda_i w_i.$$

Consequently

$$\begin{aligned} \left|\left| f(\sum_{i=1}^{n_X} \lambda_i x_i) \right|\right| = \left|\left| \sum_{i=1}^{n_X} \lambda_i w_i - \sum_{i=1}^{n_X} \lambda_i z_i + \sum_{i=1}^{n_X} \lambda_i z_i \right|\right| &= \left|\left| \sum_{i=1}^{n_X} \lambda_i (w_i - z_i) + \sum_{i=1}^{n_X} \lambda_i z_i \right|\right| \\ &= \left|\left| \sum_{i=1}^{n_X} \lambda_i z_i \right|\right| = \left|\left| \sum_{i=1}^{n_X} \lambda_i j(x_i) \right|\right| = \left|\left| \sum_{i=1}^{n_X} \lambda_i x_i \right|\right| \end{aligned}$$

since

$$\left|\left| \sum_{i=1}^{n_X} \lambda_i (w_i - z_i) \right|\right| < \frac{t}{2} \max_{i=1,...,n_X} ||\lambda_i z_i||$$

and, as $\{z_1, ..., z_{n_X}\}$ is $t-$orthogonal, we have

$$\left|\left| \sum_{i=1}^{n_X} \lambda_i z_i \right|\right| \ge t \cdot \max_{i=1,...,n_X} ||\lambda_i z_i|| .$$

Thus f is isometric. Observe that

$$||f-j|| = \sup_{x\in X} \frac{||(f-j)(x)||}{||x||} = \sup_{\lambda_i\in\mathbb{K}\ (i=1,\dots,n_X)} \frac{||\sum_{i=1}^{n_X}\lambda_i (f-j)(x_i)||}{||\sum_{i=1}^{n_X}\lambda_i x_i||}$$
$$\leq \sup_{\lambda_i\in\mathbb{K}\ (i=1,\dots,n_X)} \frac{\max_{i=1,\dots,n_X}||\lambda_i(w_i-z_i)||}{t\cdot\max_{i=1,\dots,n_X}||\lambda_i x_i||}$$
$$\leq \sup_{\lambda_i\in\mathbb{K}\ (i=1,\dots,n_X)} \frac{\frac{t}{2}\cdot\max_{i=1,\dots,n_X}||\lambda_i x_i||}{t\cdot\max_{i=1,\dots,n_X}||\lambda_i x_i||} < 1.$$

Now, we extend f on Y. We argue similarly as in the proof of the previous theorem. Choose $\{u_1,\dots,u_{m_X},\dots,u_{m_Y}\}$, a maximal orthogonal set in Y such that $\{u_1,\dots,u_{m_X}\}$ is a maximal orthogonal set in X. By Lemma 4.3 we get $F_Y\perp j(X)$, where $F_Y = [u_{m_X+1},\dots,u_{m_Y}]$.

Denote $v_k := f(u_k)$ for $k=1,\dots,m_X$, and for each $k\in\{m_{X+1},\dots,m_Y\}$ choose $g_k\in G$ and $i_{g_k,n_k}\in I_{g_k}$, denoting for simplicity $j_k := i_{g_k,n_k}$, such that $||e_{j_k}||_u = ||\lambda_k u_k||$ (for some $\lambda_k\in\mathbb{K}$) and

$$e_{j_k}\perp [v_1,\dots,v_{m_X},e_{i_{m_{X+1}}},\dots,e_{j_{k-1}}].$$

Then, setting $f(u_k) := e_{j_k}$, where $k = n_{X+1},\dots,n_Y$, we extend f on $X+F_Y$. Let $p_0 = \max\{n_k : k = m_{X+1},\dots,m_Y\}$ and $G_1 = \{g_k : k = m_{X+1},\dots,m_Y\}$. The map f is an isometry and $f(X+F_Y)\subset H$, where

$$H = \bigoplus_{g\in G_0\cup G_1} F_g^{\widehat{\max\{n_0,p_0\}}}.$$

If $\mathbb{K}$ is spherically complete, the proof is completed, since $\{u_1,\dots,u_{n_Y}\}$ is an orthogonal base of Y by [12, Lemma 5.5 and Theorem 5.15]. This shows that f is a required isometry defined on Y. Assume that $\mathbb{K}$ is non-spherically complete. Since Y is an immediate extension of $X+F_Y$ and H is spherically complete as a finite direct sum of spherically complete spaces (see [12, 4A]), we apply Lemma 4.2 to extend f to the isometry defined on Y. Since $||j - f|_X|| < 1$, as we proved above, we apply Theorem 1.3. The proof is finished. □

This yields the following interesting

Corollary 4.8. (1) *If $\mathbb{K}$ is spherically complete and $(0,\infty)$ is the union of at most countably many cosets of $|\mathbb{K}^*|$, the isometrically universal space E_h for the class of non-archimedean Banach spaces of countable type is also of universal disposition for the class $\mathcal{U}_{FNA}$.*

(2) *There exist non-archimedean Banach spaces of universal disposition for the class $\mathcal{U}_{FNA}$ which are not isometrically isomorphic.*

Proof. Recall that if $\mathbb{K}$ is spherically complete, then every finite-dimensional normed space over $\mathbb{K}$ is spherically complete (see [12, Theorem 4.2 and Corollary 4.6]). Thus, $E_h = (c_0(I_u), ||.||_u) = E_u$. The remaining part of the proof of (1) follows from Proposition 4.4. To prove (2) consider the spaces E_h and $\widehat{E_u}$ (by Theorems 4.6 and 4.7 both are spaces of universal disposition for the class $\mathcal{U}_{FNA}$) assuming that $\mathbb{K}$ is spherically complete. Then $E_h = E_u$. Since $||E_u^*||_u$ is a dense subset of $(0,\infty)$, we can choose $\{i_1,i_2,\dots\}\subset I_u$ such that $1\geq ||e_{i_1}||_u > ||e_{i_2}||_u > \dots > \frac{1}{2}$ (where e_{i_k} are unit vectors of E_u). Set

$x_n := \sum_{k=1}^{n} e_{i_k}$ $(n \in \mathbb{N})$. Then, the balls $V_n := \left\{x \in E_u : ||x - x_n||_u \leq \left|\left|e_{i_{n+1}}\right|\right|_u\right\}$ form a shrinking sequence in E_u. But $\bigcap_{n\in\mathbb{N}} V_n = \varnothing$; hence, E_u is not spherically complete. Clearly $\widehat{E_u}$, as a spherical completion of E_u, is spherically complete, thus we imply that E_h $(=E_u)$ and $\widehat{E_u}$ are not isometrically isomorphic. □

Remark 4.9. Note that (see Remark 4.5) if $\mathbb{K}$ is non-spherically complete then the space ℓ^∞ is not of universal disposition for the class $\mathcal{U}_{FNA}$. Indeed, take $Y = \mathbb{K}_v^2$, $e_1 = (1, 0, 0, ...) \in \ell^\infty$ and define the isometric embedding $i : [e_1] \to \mathbb{K}_v^2 : e_1 \longmapsto (1, 0)$. On the other hand, by [8, Proposition 3.2], every two-dimensional linear subspace of ℓ^∞ containing e_1 has two non-zero orthogonal elements. Thus, there is no isometric embedding $f : Y \to \ell^\infty$ such that $f(1, 0) = e_1$.

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