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On spaces $C^b(X)$ weakly K-analytic

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A subset Y of the dual closed unit ball B_{E^*} of a Banach space E is called a Rainwater set for E if every bounded sequence of E that converges pointwise on Y converges weakly in E. In this paper, topological properties of Rainwater sets for the Banach space $C^b(X)$ of the real-valued continuous and bounded functions defined on a completely regular space X equipped with the supremum-norm are studied. This applies to characterize the weak K-analyticity of $C^b(X)$ in terms of certain Rainwater sets for $C^b(X)$. Particularly, we show that $C^b(X)$ is weakly K-analytic if and only if there exists a Rainwater set Y for $C^b(X)$ such that $\left(C^b(X), \sigma_Y\right)$ is both K-analytic and angelic, where σ_Y denotes the topology on $C^b(X)$ of the pointwise convergence on Y. For the case when X is compact, one gets classic Talagrand's theorem. As an application we show that if X is a compact space and Y is a G_δ-dense subspace, then X is Talagrand compact, i. e., $C_p(X)$ is K-analytic, if and only if the space $(C(X), \sigma_Y)$ is K-analytic.

1 Preliminaries

If X is a completely regular (Hausdorff) space, $C(X)$ denotes the linear space of all real-valued continuous functions defined on X and $C^b(X)$ the linear subspace of $C(X)$ consisting of all those *bounded* functions. We represent by $C_p(X)$ the space $C(X)$ equipped with the *pointwise* topology τ_p. The topological dual of $C_p(X)$ is denoted by $L(X)$, or by $L_p(X)$ when provided with the weak* topology. The linear space $C(X)$ equipped with the *compact-open* topology τ_c is represented by $C_c(X)$. The space $C^b(X)$ becomes a Banach space when equipped with the supremum-norm $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. If X is normal, the dual space $C^b(X)^*$ coincides with the space $rba(X)$ of regular bounded *finitely additive* real-valued measures on the Borel σ-algebra $\mathcal{B}(X)$ on X equipped with the variation norm $\|\mu\| = |\mu|(X)$. If X is compact then $C(X)^* = rca(\mathcal{B}(X))$ because of the Riesz representation theorem, where $rca(\mathcal{B}(X))$ consists of regular *countably additive* real-measures on $\mathcal{B}(X)$.

For a completely regular space X we shall currently identify X with its homeomorphic copy $\{\delta_x : x \in X\}$ in the weak* compact unit ball $B_{C^b(X)^*}$ of $C^b(X)^*$ endowed with the relatively weak* topology, where $\langle f, \delta_x\rangle = f(x)$, and the Stone-Čech compactification βX of X with the closure of $\{\delta_x : x \in X\}$ in $B_{C^b(X)^*}$ (weak*). The map $f \mapsto f^\beta$ is a linear isometry from $\left(C^b(X), \|\cdot\|_\infty\right)$ onto $(C(\beta X), \|\cdot\|_\infty)$, so we have that $C^b(X)^* = C(\beta X)^* = rca(\mathcal{B}(\beta X))$.

In what follows 'Banach space' means 'real Banach space' and, unless stated otherwise, $C^b(X)$, as well as $C(X)$ for X pseudocompact, are supposed to be equipped with the supremum-norm $\|\cdot\|_\infty$. If E is a Banach

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space, a subset X of the dual closed unit ball B_{E^*} is called a *Rainwater set* for E if every *bounded* sequence $\{x_n\}_{n=1}^{\infty}$ of E such that $x_n \to 0$ pointwise on X, satisfies that $x_n \to 0$ weakly in E (cf. [23]).

Classic Rainwater's theorem [24] asserts that the set of the extreme points of the closed dual unit ball is a Rainwater set for E. This is a consequence of Choquet's integral representation theorem (see [8] or [10, Chapter IX]). In fact, according to [25, Corollary 11], each *James boundary* of E (see [16, Definition 76]) is a Rainwater set for E. For recent results about the also called Rainwater-Simons theorem, see [1] and [17].

If X is completely regular we denote by υX the Hewitt realcompactification of X. A space X is called *pseudocompact* if $C(X) = C^b(X)$ or alternatively if $\upsilon X = \beta X$. It is well known that X is pseudocompact if and only if X is G_δ-dense in βX, i. e., that each nonempty G_δ-set of βX meets X (see [18]). A subset X of B_{E^*} is called a *norming set* (1-norming) for E if $\|u\| = \sup_{x \in X} |\langle u, x\rangle|$ (cf. [7, Chapter 6]) for each $u \in E$. A family $\{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of subsets of a set X will be called *ordered* if $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$, i. e., whenever $\alpha(i) \leq \beta(i)$ for every $i \in \mathbb{N}$. A completely regular space X is called K*-analytic* if there is an upper semi-continuous compact-valued map T from the product space $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}$ is equipped with the discrete topology, into X such that $\bigcup\{T(\alpha) : \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$. A completely regular space X is called *Lindelöf* Σ if there is a set $\Sigma \subseteq \mathbb{N}^{\mathbb{N}}$ and an upper semi-continuous map $T : \Sigma \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ stands for the family of all compact subspaces of X, such that $\bigcup\{T(\alpha) : \alpha \in \Sigma\} = X$ (see [3, Chapter II] or [19, Chapter 3]). The well known fact that the set of adherent points of a sequence is closed implies that the upper semi-continuity of T is equivalent to the following condition: If $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in Σ converging to α and $x_n \in T(\alpha_n)$, for each $n \in \mathbb{N}$, then the sequence $\{x_n\}_{n=1}^{\infty}$ has an adherent point $x \in T(\alpha)$ (see other proof in [19, Proposition 3.2], changing $\mathbb{N}^{\mathbb{N}}$ by Σ). For the definitions and properties of Talagrand, Corson and Valdivia compact sets, we refer the reader to [9]. A Banach space is called *weakly countably determined* (WCD for short) or a Vašák space if it is a Lindelöf Σ-space in its weak topology. Clearly, every K-analytic space is Lindelöf Σ and hence every weakly K-analytic Banach space is WCD. For the definition of weakly Lindelöf determined (WLD for short) Banach space, see [19, Section 19.12]. Let us mention that the class of WLD spaces contains the class of WCD spaces.

Main results of the paper are Theorems 3.2 and 4.1. In addition, Proposition 3.5 below implies classic Talagrand's theorem [26, Theorem 3.4] if $X = Y$ is a compact space.

2 Topological properties of some Rainwater sets for $C^b(X)$

Let X be a *compact space* and $K = \operatorname{Ext} B_{C(X)^*}$ be the set of the extreme points of the compact subset $B_{C(X)^*}$ of $C(X)^*$ (weak*). According to the Arens-Kelly theorem $K = \{\pm\,\delta_x : x \in X\}$ (see [4]). By the Lebesgue dominated convergence theorem, if $\{f_n\}_{n=1}^{\infty}$ is a norm-bounded sequence in $C(X)$ (with respect to the supremum-norm) then $f_n \to f$ weakly in $C(X)$ if and only if $f_n(x) \to f(x)$ for every $x \in X$, that is, $\langle f_n, \mu\rangle \to \langle f, \mu\rangle$ for every $\mu \in C(X)^*$ if and only if $\langle f_n, \delta_v\rangle \to \langle f, \delta_v\rangle$ for each $v \in K$ (see [11, IV.6.4 Corollary]). This is Rainwater's theorem for $C(X)$. We start by characterizing which subsets Y of a compact space X are Rainwater sets for $C(X)$.

Proposition 2.1 *Let X be a compact space and be $Y \subseteq X$. The following are equivalent*

1. *Y is a Rainwater set for $C(X)$.*

2. *Y is G_δ-dense in X.*

3. *Y is a James boundary for $C(X)$.*

Proof. $2 \Rightarrow 3$. If $f \in C(X)$ then $\{x \in X : |f(x)| = \|f\|_\infty\}$ is a nonempty G_δ-subset of X, hence it meets Y.

$2 \Rightarrow 1$. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $C(X)$ such that $f_n \to 0$ pointwise on Y. If $x \in X$ then $Z := \{y \in X : |f_n(x)| = |f_n(y)|\ \forall n \in \mathbb{N}\}$ is a nonempty G_δ-subset of X, hence $Z \cap Y \neq \emptyset$. Choosing $z \in Z \cap Y$, one has that $|f_n(x)| = |f_n(z)| \to 0$. Consequently $f_n \to 0$ pointwise on X, which implies that $f_n \to 0$ weakly in $C(X)$ by virtue of Rainwater's theorem for $C(X)$.

$\neg 2 \Rightarrow \neg 1 \wedge \neg 3$. Let Z be a nonempty G_δ-subset of X which does not meet Y. Since X is regular, we may assume without loss of generality that Z is closed. As compact spaces are normal, any closed G_δ-set is a zero set. Hence there is a continuous function $f : X \to [0,1]$ with $Z = f^{-1}(\{1\})$. Then f does not attain its norm on Y, which means that Y is not a James boundary for $C(X)$. Moreover, $\{f^n\}_{n=1}^{\infty}$ is a bounded sequence in $C(X)$ such that $f^n \to 0$ pointwise on Y which does not converges to zero weakly due to $\delta_z \in C(X)^*$ and $\langle f^n, \delta_z \rangle \to 1$ for each $z \in Z$. □

Corollary 2.2 *If X is completely regular, the following conditions hold*

1. *X is a Rainwater set for $C^b(X)$ if and only if X is pseudocompact.*
2. *If $Y \subseteq X$ is a Rainwater set for $C^b(X)$, then X is pseudocompact and Y is G_δ-dense in X.*

Proof. For the first statement note that, by Proposition 2.1, $X \subseteq \beta X$ is a Rainwater set for $C^b(X) = C(\beta X)$ if and only if X is a James boundary for $C(\beta X)$, i. e., if and only if X is pseudocompact. For the second statement, if $Y \subseteq X$ is a Rainwater set for $C(\beta X)$ then X is pseudocompact because of the previous statement. In addition, Proposition 2.1 ensures that Y is G_δ-dense in βX, hence in X. □

Remark 2.3 *If Y is a Rainwater set for a Banach space E, then Y separates the points of E.* If not there is a nonzero $x \in Y_\perp$, so the constant sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n = x$ for every $n \in \mathbb{N}$ converges pointwise to zero on Y but not weakly in E. Therefore Y cannot be a Rainwater set for E.

Example 2.4 If I is an infinite discrete space, the first statement of Corollary 2.2 prevents I to be a Rainwater set for $\ell_\infty(I)$.

Proposition 2.5 *Let Y be a Rainwater set for $C^b(X)$. If Z is a C-embedded and dense subset of Y then Z is a Rainwater set for $C^b(X)$. Consequently, if υZ is homeomorphic to a Rainwater set for $C^b(X)$, then Z is also homeomorphic to a Rainwater set for $C^b(X)$.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $C^b(X)$ such that $f_n(z) \to f(z)$ for every $z \in Z$, where $f \in C(X)$. Given $y \in Y$, set $f_0 := f$ and define $g_n(\omega) = f_n(\omega) - f_n(y)$ *for each* $\omega \in Y$ and every $n \in \mathbb{N}_0$. Since $g_n \in C(Y)$ and $g_n(y) = 0$ for all $n \in \mathbb{N}_0$, by setting $G_n := \{\omega \in Y : g_n(\omega) = 0\}$ one has that $y \in \bigcap_{z=0}^{\infty} G_n$, so that $G := \bigcap_{z=0}^{\infty} G_n$ is a nonempty intersection of countably many zero-sets of Y. Since Z is C-embedded and dense in Y, [15, 8.6 Theorem] yields that Y meets Z. So there exists $z_y \in G \cap Z$, which means that $f_n(z_y) = f_n(y)$ for every $n \in \mathbb{N}_0$. Since $f_n(z) \to f(z)$ for each $z \in Z$, it follows that $f_n(y) \to f(y)$ for every $y \in Y$. Now, putting together the facts that $\{f_n\}_{n=1}^{\infty}$ is bounded in $C^b(X)$ and Y is a Rainwater set for $C^b(X)$, we get that $f_n \to f$ weakly in $C^b(X)$. Hence Z is a Rainwater set for $C^b(X)$. □

Proposition 2.6 *Let Y be a norming set of a Banach space E and Z be a subspace of E^* with $Y \subseteq Z \subseteq B_{E^*}$. If Y is a Rainwater set for $C^b(Z)$, then Y is a Rainwater set for E.*

Proof. Let $\{u_n\}_{n=1}^{\infty}$ be a bounded sequence of E such that $\langle u_n, y \rangle \to 0$ for every $y \in Y$. Consider the restriction map $T : E \to C^b(Z)$ defined by $Tu = u|_Z$ and set $f_n := Tu_n$ for every $n \in \mathbb{N}$. Note that

$$\|u\| = \sup_{z \in B_{E^*}} |\langle u, z \rangle| \geq \sup_{z \in Z} |u|_Z(z)| = \|Tu\|_\infty$$
$$\|Tu\|_\infty \geq \sup_{y \in Y} |u|_Z(y)| = \|u\|$$

for each $u \in E$, since Y is norming for E, so that $\|Tu\|_\infty = \|u\|$. Hence T is a linear isometry from E into $\left(C^b(Z), \|\cdot\|_\infty\right)$ and consequently a weak-to-weak homeomorphism onto its image.

Since $f_n(y) = \langle u_n, y \rangle \to 0$ for $y \in Y$, one has that $f_n \to 0$ pointwise on Y. As in addition $\|f_n\|_\infty = \|u_n\|$ for each $n \in \mathbb{N}$, we have that $\{f_n\}_{n=1}^{\infty}$ is a bounded sequence of the Banach space $\left(C^b(Z), \|\cdot\|_\infty\right)$ converging pointwise on a Rainwater set Y for $C^b(Z)$. Consequently, $\langle f_n, \mu \rangle \to 0$ for all $\mu \in C^b(Z)^*$, i. e., $f_n \to 0$ weakly in $\operatorname{Im} T$. This implies that $u_n \to 0$ weakly in E. Hence Y is a Rainwater set for E. □

Corollary 2.7 *Let X be a completely regular space. If Y is a pseudocompact subset of $B_{C^b(X)^*}$ (weak*) that contains X, then Y is a Rainwater set for $C^b(X)$.*

Proof. Clearly Y is a James boundary for $C^b(X)$, so we can apply Proposition 2.1 to get the conclusion. Alternatively, the statement easily follows from [6, Proposition 2]. One can also use Proposition 2.6. Indeed, since $X \subseteq Y$ one has that Y is a norming set for the Banach space $E = C^b(X)$. Given that Y is a Rainwater set for $C^b(Y)$ due to Corollary 2.2, we can apply from Proposition 2.6 by setting $E = C^b(X)$ and $Z = Y$. □

Example 2.8 *If D is a dense subset of $\beta\mathbb{N}\setminus\mathbb{N}$, then $Y=\mathbb{N}\cup D$ is a Rainwater set for ℓ_∞.* Indeed, it is known that under these circumstances $\mathbb{N}\cup D$ is pseudocompact (cf. [14, 3.1 Theorem]). Since $\mathbb{N}\subseteq Y$, Corollary 2.7 ensures that Y is a Rainwater set for ℓ_∞.

Example 2.9 *A Rainwater set for $C^b(X)$, even contained in X, need not be pseudocompact.* If X is the one-point compactification of an uncountable discrete space Y, then Y is a non pseudocompact Rainwater set for $C(X)$, according to Proposition 2.1.

Example 2.10 *If X is a Valdivia compact set which is not Corson's, there is a non compact $Y\subsetneq X$ which is a Rainwater set for $C(X)$.* Since X is Valdivia compact, there exists a Σ-dense, hence countably compact, subset Y of X. Consequently Y is a Rainwater set for $C^b(Y)$ by Corollary 2.2. As $\beta Y=X$ implies that $C^b(Y)=C(X)$, we have that Y is a Rainwater set for $C(X)$. Since X is not Corson compact, $Y\neq X$ so that Y is not compact (see also [20, Section 1.2]).

3 Rainwater sets and weak K-analyticity of $C^b(X)$

The following lemma will be required for the proof of the subsequent theorem, where we characterize the weak K-analyticity or the WCD property of the Banach space $C^b(X)$ in terms of certain Rainwater sets for $C^b(X)$.

Lemma 3.1 *If $C^b(X)$ is WLD, then X is pseudocompact, so a Rainwater set for $C^b(X)$.*

Proof. First observe that if βX if Fréchet-Urysohn, then X is pseudocompact (this follows for instance from [21, Theorem 1]). Now if $C^b(X)$ is WLD then $C(\beta X)$ is WLD, so that βX is a Corson compact space. Since each Corson compact is a Fréchet-Urysohn space, the conclusion follows. □

If Y is a subset of $C^b(X)^*$ that separates the functions of $C^b(X)$, we denote by σ_Y the weak topology $\sigma\left(C^b(X),\operatorname{span}(Y)\right)$ on $C^b(X)$, where $\operatorname{span}(Y)$ stands for the linear span of Y. If $Y=X$, then clearly $\sigma_Y=\tau_p|_{C^b(X)}$.

Theorem 3.2 *Let X be completely regular. The following are equivalent*

1. *There exists a Rainwater set Y for $C^b(X)$ such that $\left(C^b(X),\sigma_Y\right)$ is K-analytic (resp. a Lindelöf Σ-space) and $C_p(Y)$ is angelic.*
2. *There exists a Rainwater set Y for $C^b(X)$ such that $\left(C^b(X),\sigma_Y\right)$ is both K-analytic (resp. a Lindelöf Σ-space) and angelic.*
3. *$C^b(X)$ is weakly K-analytic (resp. WCD).*

Proof. $1\Rightarrow 2$. Let Y be a Rainwater set for $C^b(X)$ enjoying the conditions of the first statement. Since, according to Remark 2.3, Y separates the functions of $C^b(X)$, the map $\varphi:\left(C^b(X),\sigma_Y\right)\to C_p(Y)$ given by $(\varphi f)(y)=\langle f,y\rangle$, where on the right side f must be regarded as a continuous linear functional on $B_{C^b(X)^*}$ (weak*), embeds homeomorphically $\left(C^b(X),\sigma_Y\right)$ onto a subspace of $C_p(Y)$. Hence the space $\left(C^b(X),\sigma_Y\right)$ is angelic.

$2\Rightarrow 3$. Assume first that $\left(C^b(X),\sigma_Y\right)$ is K-analytic. We use some the ideas of the proof of implication $(i)\Rightarrow(ii)$ of [13, Theorem 7.1.8]. Let us denote by Δ the closed unit ball of $C^b(X)$, i. e., $\Delta=B_{C^b(X)}$. Since $\left(C^b(X),\sigma_Y\right)$ is K-analytic, there is a covering $\left\{K_\alpha:\alpha\in\mathbb{N}^\mathbb{N}\right\}$ of $C^b(X)$ consisting of σ_Y-compact sets, such that $K_\alpha\subseteq K_\beta$ whenever $\alpha\leq\beta$ (see [26] or [19, Theorem 3.2]). Given that the weak topology of $C^b(X)$ is stronger than σ_Y, each K_α is weakly closed. Consequently, the family $\left\{K_\alpha\cap\Delta:\alpha\in\mathbb{N}^\mathbb{N}\right\}$ is an ordered covering of Δ consisting of weakly closed sets (some of them possibly empty). Let us show that each $K_\alpha\cap\Delta$ is a weakly compact set of the Banach space $C^b(X)$.

According to the Eberlein-Šmulian theorem it suffices to show that each nonempty $K_\alpha\cap\Delta$ is weakly sequentially compact. Thus, let $\{f_n\}_{n=1}^\infty$ be a sequence in $K_\alpha\cap\Delta$. Since $\left(C^b(X),\sigma_Y\right)$ is angelic, each σ_Y-compact set K_α is sequentially compact under the topology σ_Y. Hence there is a subsequence $\{f_{n_i}\}_{i=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ that converges under σ_Y to some $f\in K_\alpha\subseteq C^b(X)$. Particularly, $\langle f_{n_i},y\rangle\to\langle f,y\rangle$ for every $y\in Y$. Bearing in mind that $\|f_n\|_\infty\leq 1$ for every $n\in\mathbb{N}$ and that Y is a Rainwater set for $C^b(X)$, we get that $f_{n_i}\to f$ weakly in $C^b(X)$. Given that $K_\alpha\cap\Delta$ is weakly closed and $\{f_{n_i}\}_{i=1}^\infty\subseteq K_\alpha\cap\Delta$, it follows that $f\in K_\alpha\cap\Delta$. All this shows that $K_\alpha\cap\Delta$ is a weakly compact subset of $C^b(X)$, so that $\left\{K_\alpha\cap\Delta:\alpha\in\mathbb{N}^\mathbb{N}\right\}$ is an ordered covering of Δ

consisting of weakly compact sets. As in addition each Banach space in its weak topology is angelic and angelicity is inherited by subspaces, it turns out that Δ is weakly angelic too. So [19, Corollary 3.6] assures that Δ is K-analytic when equipped with the relative weak topology of $C^b(X)$. Finally, the fact that $C^b(X) = \bigcup_{n=1}^{\infty} n\Delta$ ensures that $C^b(X)$ is weakly K-analytic.

If $\left(C^b(X), \sigma_Y\right)$ is a Lindelöf Σ-space, there exists a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and an upper semi-continuous map $T : \Sigma \to \mathcal{K}\left(\left(C^b(X), \sigma_Y\right)\right)$ such that $\mathcal{A} = \{T(\alpha) : \alpha \in \Sigma\}$ is a σ_Y-compact covering of $C^b(X)$. So we can work with the family $\mathcal{A}$ as we did with the family $\left\{K_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\right\}$ in order to show that $\{T(\alpha) \cap \Delta : \alpha \in \Sigma\}$ is a covering of Δ consisting of weakly compact sets. We claim that the map $S : \Sigma \to \mathcal{K}((\Delta, \mathrm{weak}|_\Delta))$ defined by $S(\alpha) = T(\alpha) \cap \Delta$ is upper semi-continuous and, consequently, that $(\Delta, \mathrm{weak}|_\Delta)$ is a Lindelöf Σ-space. Indeed, let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in Σ such that $\alpha_n \to \alpha$ in Σ and take $f_n \in S(\alpha_n)$ for each $n \in \mathbb{N}$. Since T is upper σ_Y-semi-continuous and the topology σ_Y is angelic there is a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ that σ_Y-converges to some $f \in T(\alpha)$. Using the fact that Y is a Rainwater set and that $f_{n_i} \in \Delta$ for all $i \in \mathbb{N}$, we can assure that $f_{n_i} \to f$ weakly in $C^b(X)$. Since the set Δ is weakly closed, this means that $f_{n_i} \to f$ weakly in Δ with $f \in S(\alpha)$, which shows that S is upper-semicontinuous.

Now the fact that $C^b(X) = \bigcup_{n=1}^{\infty} n\Delta$ guarantees that $C^b(X)$ is a weakly Lindelöf Σ-space, i. e., that the Banach space $C^b(X)$ is WCD.

$3 \Rightarrow 1$. According to Lemma 3.1 the space X is pseudocompact, therefore a Rainwater set for $C^b(X) = C(X)$. Moreover, the pseudocompactness of X ensures that the space $C_p(X)$ is angelic (see [19, Theorem 4.5]). Thus we can take $Y = X$. On the other hand, the fact that $\sigma_X \leq \sigma\left(C(X), C(X)^*\right)$ guarantees that the space $(C(X), \sigma_X) = C_p(X)$ is K-analytic (resp. Lindelöf Σ). □

Corollary 3.3 *Let X be completely regular and Y be a Rainwater set for $C^b(X)$. If Y is weak* pseudocompact, are equivalent*

1. *$\left(C^b(X), \sigma_Y\right)$ is K-analytic (resp. a Lindelöf Σ-space).*
2. *$C^b(X)$ is weakly K-analytic (resp. WCD).*

Proof. If Y is weak* pseudocompact, then $C_p(Y)$ is angelic by [19, Theorem 4.5]. So the implication $1 \Rightarrow 2$ is a direct consequence of Theorem 3.2. The converse is obvious. □

Corollary 3.4 *If X is pseudocompact, then $C_p(X)$ is K-analytic (resp. a Lindelöf Σ-space) if and only if $C(X)$ is weakly K-analytic (resp. WCD).*

Proof. If X is pseudocompact, Corollary 2.2 says that X a Rainwater set for $C(X)$. The conclusion follows from the previous corollary. □

Proposition 3.5 *Let X be completely regular and let $X \subseteq Y \subseteq B_{C^b(X)^*}$ (weak*). If X is a Rainwater set for $C^b(Y)$, then the following conditions are equivalent:*

1. *$C_p(Y)$ is K-analytic (resp. a Lindelöf Σ-space).*
2. *$C^b(X)$ is weakly K-analytic (resp. WCD).*

Proof. $1 \Rightarrow 2$. Assume that $C_p(Y)$ is K-analytic or a Lindelöf Σ-space. Since X is a Rainwater set for $C^b(Y)$ and $X \subseteq Y$, Corollary 2.2 ensures that Y is pseudocompact and X is dense in Y. Since $X \subseteq Y \subseteq B_{C^b(X)^*}$ one has that $C^b(X) = C^b(Y)$ isometrically. Now the assumption that $C_p(Y)$ is K-analytic or Lindelöf Σ, together with the fact that Y is pseudocompact, impels Corollary 3.4 to ensure that $C(Y)$ is either weakly K-analytic or Lindelöf Σ. Hence $C^b(X)$ is weakly K-analytic or Lindelöf Σ, respectively, and we are done.

$2 \Rightarrow 1$. Assume conversely that $C^b(X)$ is weakly K-analytic or WCD and note that in this case X must be pseudocompact by Lemma 3.1. Notice that the map $T : C^b(X) \to C(Y)$ defined by $Tf := f^\beta|_Y$ is a linear isometry from $C^b(X)$ onto $C(Y)$, which implies that $T : C^b(X)(\mathrm{weak}) \to C_p(Y)$ is continuous. Hence if $C^b(X)$ is weakly K-analytic or WCD, then $C_p(Y)$ is K-analytic or Lindelöf Σ, respectively. □

4 A characterization of Talagrand and Gul'ko compact sets

In this section we provide a characterization of Talagrand and Gul'ko compactness based on Theorem 3.2

Theorem 4.1 *Let X be a compact space and Y be a G_δ-dense subspace. Then X is a Talagrand compact set (resp. Gul'ko compact) if and only if the space $(C(X), \sigma_Y)$ is K-analytic (resp. a Lindelöf Σ-space).*

P r o o f. If X is Talagrand compact then $C_p(X)$ is K-analytic, consequently $(C(X), \sigma_Y)$ is K-analytic. For the converse, if Y is a G_δ-dense subspace of X, we claim that Y is a Rainwater set for $C(X)$. In fact, let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence in $C(X)$ such that $f_n(y) \to f_0(y)$ for every Y. Let us show that given $x \in X$ there exists $y_x \in Y$ such that $f_n(y_x) = f_n(x)$ for every $n \in \mathbb{N}_0$. This would imply that $f_n(x) \to f_0(x)$ for every $x \in X$, and since X is compact it follows that $f_n \to f_0$ weakly in $C(X)$. But, given $x \in X$ define $Z_n := \{z \in X : f_n(z) = f_n(x)\}$, $\forall n \in \mathbb{N}_0$, as usual. Clearly Z_n is a zero-set for every $n \in \mathbb{N}_0$. Setting $Z := \bigcap_{n=0}^{\infty} Z_n$, we see that Z is a G_δ-subset of X. Consequently, $Z \cap Y \neq \emptyset$ and we can choose any $y_x \in Z \cap Y$. Hence Y is a Rainwater set for $C(X)$ as stated.

Particularly, this shows that $(C(X), \sigma_Y)$ is angelic (see also [6, Theorem 5]). Indeed, if M is a countably compact set in $(C(X), \sigma_Y)$ and $\{g_n\}_{n=1}^{\infty}$ is a sequence in M, let g_0 be a σ_Y-cluster point in M. Since there exists $y_{x_0} \in Y$ such that $g_n(y_{x_0}) = g_n(x_0)$ for every $n \in \mathbb{N}_0$, this implies that g_0 is also a σ_X-cluster point in M. Given that $C_p(X)$ is angelic because X is a compact space, we get that M is a compact set in $C_p(X)$ and hence a σ_Y-compact set. As in addition each compact set in $C_p(X)$ is Fréchet-Urysohn, it follows that each compact subspace in $(C(X), \sigma_Y)$ is a Fréchet-Urysohn space, which shows that $(C(X), \sigma_Y)$ is angelic.

Since Y is a Rainwater set for $C(X)$ and $(C(X), \sigma_Y)$ is both K-analytic and angelic, Theorem 3.2 allows us to conclude that $C(X)$ is weakly K-analytic. Hence $C_p(X)$ is K-analytic, so that X is Talagrand compact.

For the Lindelöf Σ case, the proof is exactly the same. □

Let X be a completely regular space. If x_0 is a non-isolated point of X and $Y := X \setminus \{x_0\}$, then Y is dense in X and the topology σ_Y on $C(X)$ is Hausdorff.

Proposition 4.2 *Let X be a compact space and let x_0 be a non-isolated point of X. If $Y := X \setminus \{x_0\}$ and $(C(X), \sigma_Y)$ is K-analytic, the following properties hold*

1. *If $\{x_0\}$ is a G_δ-subset of X and Y is a k_R-space, then $C_p(Y)$ is K-analytic.*

2. *If $\{x_0\}$ is not a G_δ-subset of X then $C_p(X)$ is K-analytic.*

P r o o f. If $\{x_0\}$ is a G_δ-subset of X, using the regularity of X we may locate a decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that $\overline{U_{n+1}} \subseteq U_n$, for every $n \in \mathbb{N}$, and that $\{x_0\} = \bigcap_{n=1}^{\infty} U_n$. Setting $K_n = X \setminus U_n$ for each $n \in \mathbb{N}$, then $\{K_n : n \in \mathbb{N}\}$ is a increasing sequence of compact subsets of Y such that $Y = \bigcup_{n=1}^{\infty} K_n$. Hence Y is σ-compact. Moreover, if Q is a compact subset of Y, since the sequence $\{V_n : n \in \mathbb{N}\}$, where $V_n = X \setminus \overline{U_n}$ for each $n \in \mathbb{N}$, is increasing, covers Y and is formed by open subsets of Y, there exists $m \in \mathbb{N}$ such that $Q \subseteq V_m \subseteq K_m$. This shows that Y is hemicompact. Further Y is a k_R-space. Define $\varphi : (C(X), \sigma_Y) \to C_p(Y)$ by $\varphi(f) = f|_Y$. Since φ is continuous and $(C(X), \sigma_Y)$ is K-analytic, it follows that $M := \varphi(C(X))$ is a K-analytic subspace of $C_p(Y)$. Moreover, if $a, b \in Y$ with $a \neq b$, there is $f \in C(X)$ with $f(a) \neq f(b)$. But then $g := f|_Y \in M$ and $g(a) \neq g(b)$. So that M is a K-analytic subspace of $C_p(Y)$ that separates the points of Y. According to [22, Main Theorem] (which is proved only for K-analytic spaces) the space $C_p(Y)$ is K-analytic.

Now let us assume that $\{x_0\}$ is not a G_δ-subset of X. If G is a nonempty G_δ-subset of X, necessarily $G \cap Y \neq \emptyset$. Otherwise $G = \{x_0\}$, which is not the case. Consequently the set Y is a G_δ-dense subspace of X and the statement is a straightforward consequence of Theorem 4.1. □

We do not know if this Proposition is true in the Lindelöf-Σ setting.

Example 4.3 Let X be a Talagrand compact space that contains a point x_0 such that $\{x_0\}$ is a G_δ-set. Define $Y := X \setminus \{x_0\}$. Since $C_p(X)$ is K-analytic, $(C(X), \sigma_Y)$ is K-analytic. As in addition Y is Fréchet-Urysohn, we get that $C_p(Y)$ is K-analytic as an application of the first statement of Proposition 4.2.

Example 4.4 Let $X = [0, \omega_1]$ with the order topology and $Y = [0, \omega_1)$. Since $\{x_0\}$ is not a G_δ-set, the second statement of Proposition 4.2 yields that $(C(X), \sigma_Y)$ is not K-analytic since $C_p(X)$ is not K-analytic. In any case, note that $(C(X), \sigma_Y) = C_p(Y)$ homeomorphically due to the pseudocompactness of Y, consequently the fact that $(C(X), \sigma_Y)$ is not K-analytic also follows from Corollary 3.4.

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