On the separable quotient problem

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Abstract. While the classic *separable quotient problem* remains open, we survey general results related to this problem and examine the existence of a particular infinite-dimensional separable quotient in some Banach spaces of vector-valued functions, linear operators and vector measures. Most of the results presented are consequence of known facts, some of them relative to the presence of complemented copies of the classic sequence spaces c_0 and ℓ_p, for $1 \leq p \leq \infty$. This makes our presentation supplementary to a previous survey (1997) due to Mujica.

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1. Preliminaries

One of unsolved problems of Functional Analysis (posed by S. Mazur in 1932) asks:

Problem 1. *Does any infinite-dimensional Banach space have a separable* (*infinite dimensional*) *quotient?*

An easy application of the open mapping theorem shows that an infinite dimensional Banach space X has a separable quotient if and only if X is mapped on a separable Banach space under a continuous linear map.

Seems that the first comments about Problem 1 are mentioned in [**45**] and [**54**]. It is already well known that all reflexive, or even all infinite-dimensional weakly compactly generated Banach spaces (WCG for short), have separable quotients. In [**37**, Theorem IV.1(i)] Johnson and Rosenthal proved that every infinite dimensional separable Banach space admits a quotient with a Schauder basis. The latter result provides another (equivalent) reformulation of Problem 1.

Problem 2. *Does any infinite dimensional Banach space admits an infinite dimensional quotient with a Schauder basis?*

In [**36**, Theorem 2] is proved that if Y is a separable closed subspace of an infinite-dimensional Banach space X and the quotient X/Y has a separable quotient, then Y is quasi-complemented in X. So one gets the next equivalent condition to Problem 1.

Problem 3. *Does any infinite dimensional Banach space X contains a separable closed subspace which has a proper quasi-complement in X?*

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Although Problem 1 is left open for Banach spaces, a corresponding question whether any infinite dimensional metrizable and complete non-normed topological vector space X admits a separable quotient has been already solved. Indeed, if X is locally convex (i. e., X is a Fréchet space), a result of Eidelheit [**19**] ensures that X has a quotient isomorphic to $\mathbb{K}^{\mathbb{N}}$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

In [**15**] Drewnowski posed a more general question: whether any infinite dimensional metrizable and complete topological vector space X has a closed subspace Y such that the dimension of the quotient X/Y is the continuum (in short $\dim(X/Y) = \mathfrak{c}$). The same paper contains an observation stating that any infinite dimensional Fréchet locally convex space admits a quotient of dimension $\mathfrak{c} := 2^{\aleph_0}$. Finally Drewnowski's problem has been solved by M. Popov, see [**53**]. He showed, among others, that for $0 < p < 1$ the space $X := L_p([0,1]^{2^{\mathfrak{c}}})$ does not admit a proper closed subspace Y such that $\dim(X/Y) \leq \mathfrak{c}$. Consequently X does not have a separable quotient.

The organization of the present paper goes as follows. In the second section we gather general selected results about the separable quotient problem and some classic results on Banach spaces containing copies of sequence spaces, providing as well some straightforward consequences. In the third section, among others, we exhibit how the weak*-compactness of the dual unit ball is related to the existence of quotients isomorphic to c_0 or ℓ_2. Next section contains three classic results about complete tensor products of Banach spaces and their applications to the separable quotient problem. Last sections are devoted to examine the existence of separable quotients for many concrete classes of 'big' Banach spaces, as Banach spaces of vector-valued functions, bounded linear operators and vector measures. For these classes of spaces we fix particular separable quotients by means of the literature on the subject and the previous results. This line of research has been also continued in a more general setting for the class of topological vector spaces, particularly for spaces $C(X)$ of real-valued continuous functions endowed with the pointwise and compact-open topology, see [**39**], [**40**] and [**41**].

This paper intends to supplement Mujica's survey article [**50**] by collecting together some results not mentioned there, adding new facts published afterwards and gathering some results being consequence of known facts, some of them relative to the presence of complemented copies of the classic spaces c_0 and ℓ_p, for $1 \leq p \leq \infty$, hoping this will be useful for researchers interested in this area.

2. A few results for general Banach spaces

Let us start with the following remarkable concrete result related to Problem 1.

Theorem 4 (Argyros-Dodos-Kanellopoulos)**.** *Every infinite-dimensional dual Banach space has a separable quotient.*

Hence the space $\mathcal{L}(X,Y)$ of bounded linear operators between Banach spaces X and Y equipped with the operator norm has a separable quotient provided $Y \neq \{\mathbf{0}\}$. Indeed, it follows from the fact that X^* is complemented in $\mathcal{L}(X,Y)$, see Theorem 30 below for details.

There are several equivalent conditions to Problem 1. Let's select a few of them. For the equivalence (2) and (3) below (and applications) in the class of locally convex spaces we refer the reader to [**40**], [**61**] and [**62**]. The equivalence between (1) and (3) is due to Saxon-Wilansky [**64**]. Recall that a locally convex space E is *barrelled* if every barrel (an absolutely convex closed and absorbing set in E) is a neighborhood of zero. We refer also to [**4**] and [**37**] for some partial results related to the next theorem.

Theorem 5 (Saxon-Wilansky). *The following assertions are equivalent for an infinite-dimensional Banach space X.*

1. *X contains a dense non-barrelled linear subspace.*
2. *X admits a strictly increasing sequence of closed subspaces of X whose union is dense in X.*
3. *X^* admits a strictly decreasing sequence of weak*-closed subspaces whose intersection consists only of the zero element.*
4. *X has a separable quotient.*

Proof. We prove only the equivalence between (2) and (4) which holds for any locally convex space X.

(4) $\Rightarrow$ (2) Note that every separable Banach space has the above property (as stated in (2)) and this property is preserved under preimages of surjective linear operators (which clearly are open maps).

(2) $\Rightarrow$ (4) Let $\{X_n : n \in \mathbb{N}\}$ be such a sequence. We may assume that $\dim(X_{n+1}/X_n) \geq n$ for all $n \in \mathbb{N}$. Let $x_1 \in X_2 \setminus X_1$. There exists x_1^* in X^* such that $x_1^* x_1 = 1$ and x_1^* vanishes on X_1. Assume that we have constructed $(x_1, x_1^*), \ldots, (x_n, x_n^*)$ in $X \times X^*$ with $x_j \in X_{j+1}$ such that $x_j^* x_j = 1$ and x_j^* vanishes on X_j for $1 \leq j \leq n$. Choose $x_{n+1} \in X_{n+2} \cap (x_1^*)^{-1}(\mathbf{0}) \cap \cdots \cap (x_n^*)^{-1}(\mathbf{0}) \setminus X_{n+1}$. Then there exists $x_{n+1}^* \in X^*$ with $x_{n+1}^* x_{n+1} = 1$ vanishing on X_{n+1}. Since $X_{n+1} \subseteq \operatorname{span}\{x_1, x_2, \ldots x_n\} + \bigcap_{k=1}^{\infty} \ker x_k^*, n \in \mathbb{N}$, we conclude that $\operatorname{span}\{x_n : n \in \mathbb{N}\} + \bigcap_n \ker x_n^*$ is dense in X. Then X/Y is separable for $Y := \bigcap_n \ker x_n^*$. ∎

Particularly every Banach space whose weak*-dual is separable has a separable quotient. Theorem 5 applies to show that every infinite dimensional WCG Banach space has a separable quotient. Indeed, if X is reflexive we apply Theorem 4. If X is not reflexive, choose a weakly compact absolutely convex set K in X such that $\overline{\operatorname{span}(K)} = X$. Since K is a barrel of $Y := \operatorname{span}(K)$ and X is not reflexive, it turns out that Y is a dense non-barrelled linear subspace of X.

The class of WCG Banach spaces, introduced in [**3**], provides a quite successful generalization of reflexive and separable spaces. As proved in [**3**] there are many bounded projection operators with separable ranges on such spaces, so many separable complemented subspaces. For example, if X is WCG and Y is a separable subspace, there exists a separable closed subspace Z with $Y \subseteq Z \subseteq X$ together with a contractive projection. This shows that every infinite-dimensional WCG Banach space admits many separable complemented subspaces, so separable quotients.

The Josefson-Nissenzweig theorem states that the dual of any infinite-dimensional Banach space contains a *normal sequence*, i. e., a normalized weak*-null sequence [**51**].

Recall (cf. [**65**]) that a sequence $\{y_n^*\}$ in the sphere $S(X^*)$ of X^* is *strongly normal* if the subspace $\{x \in X : \sum_{n=1}^{\infty} |y_n^* x| < \infty\}$ is dense in X. Clearly every strongly normal sequence is normal. Having in mind Theorem 7, the following question is of interest.

Problem 6. *Does every normal sequence in X^* contains a strongly normal subsequence?*

By [**65**, Theorem 1], any strongly normal sequence in X^* contains a subsequence $\{y_n\}$ which is a Schauder basic sequence in the weak*-topology, i. e., $\{y_n\}$ is a Schauder basis in its closed linear span in the weak*-topology. Conversely, any normalized Schauder basic sequence in (X^*, w^*) is strongly normal, [**65**, Proposition 1].

The following theorem from [**65**] exhibits a connection between these concepts.

Theorem 7 (Śliwa)**.** *Let X be an infinite-dimensional Banach space. The following conditions are equivalent:*

1. *X has a separable quotient.*
2. *X^* has a strongly normal sequence.*
3. *X^* has a basic sequence in the weak* topology.*
4. *X^* has a pseudobounded sequence, i. e., a sequence $\{x_n^*\}$ in X^* that is pointwise bounded on a dense subspace of X and* $\sup_n \|x_n^*\| = \infty$.

Proof. We prove only (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (4). The equivalence between (2) and (3) follows from the preceding remark.

(1) $\Rightarrow$ (2) By Theorem 5 the space X contains a dense non-barrelled subspace. Hence there exists a closed absolutely convex set D in X such that $H := \operatorname{span}(D)$ is a proper dense subspace of X. For each $n \in \mathbb{N}$ choose $x_n \in X$ so that $\|x_n\| \le n^{-2}$ and $x_n \notin D$. Then select x_n^* in X^* such that $x_n^* x_n > 1$ and $|x_n^* x| \le 1$ for all $x \in D$. Set $y_n^* := \|x_n^*\|^{-1} x_n^*$ for all $n \in \mathbb{N}$. Since $\|x_n^*\| \ge n^2$, $y_n^* \in S(X^*)$ and $\sum_{n=1}^{\infty} |y_n^* x| < \infty$ for all $x \in D$, the sequence $\{y_n^*\}$ is as required.

(2) $\Rightarrow$ (1) Assume that X^* contains a strongly normal sequence $\{y_n^*\}$. By [**65**, Theorem 1] (see also [**37**, Theorem III.1 and Remark III.1]) there exists a subsequence of $\{y_n^*\}$ which is a weak*-basic sequence in X^*. This implies that X admits a strictly increasing sequence of closed subspaces whose union is dense in X. Indeed, since (X^*, w^*) contains a basic sequence, and hence there exists a strictly decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of closed subspaces in (X^*, w^*) with $\bigcap_{n=1}^{\infty} U_n = \{\mathbf{0}\}$, the space X has a sequence as required. This provides a biorthogonal sequence as in the proof of (2) $\Rightarrow$ (3), Theorem 5.

(4) $\Rightarrow$ (1) Let $\{y_n^*\}$ be a pseudobounded sequence in X^*. Set $Y := \{x \in X : \sup_{n \in \mathbb{N}} |y_n^* x| < \infty\}$. The Banach-Steinhaus theorem applies to deduce that Y is a proper and dense subspace of X. Note that Y is not barrelled since $V := \{x \in X : \sup_{n=1}^{\infty} |y_n^* x| \le 1\}$ is a barrel in H which is not a neighborhood of zero in H. Now apply Theorem 5.

(1) $\Rightarrow$ (4) Assume X contains a dense non-barrelled subspace Y. Let W be a barrel in Y which is not a neighborhood of zero in Y. If V is the closure of W in X, the linear span H of V is a dense proper subspace of X. So, for every $n \in \mathbb{N}$ there is $x_n \in X \setminus V$

with $\|x_n\| \leq n^{-2}$. Choose $z_n^* \in X^*$ so that $|z_n^* x_n| > 1$ and $|z_n^* x| = 1$ for all $x \in V$ and $n \in \mathbb{N}$. Then $\|z_n^*\| \geq n^2$ and $\sup_n |z_n^* x| < \infty$ for $x \in H$. □

A slightly stronger property than that of condition (2) of Theorem 7 is considered in the next proposition. As shown in the proof, this property turns out to be equivalent to the presence in X^* of a basic sequence equivalent to the unit vector basis of c_0.

Proposition 8. *An infinite-dimensional Banach space X has a quotient isomorphic to ℓ_1 if and only if X^* contains a normal sequence $\{y_n^*\}$ such that $\sum_n |y_n^* x| < \infty$ for all $x \in X$.*

Proof. If there is a bounded linear operator Q from X onto ℓ_1, its adjoint map fixes a sequence $\{x_n^*\}$ in X^* such that the formal series $\sum_{n=1}^{\infty} x_n^*$ is weakly unconditionally Cauchy and $\inf_{n \in \mathbb{N}} \|x_n^*\| > 0$. Setting $y_n^* := \|x_n^*\|^{-1} x_n^*$ for each $n \in \mathbb{N}$, the sequence $\{y_n^*\}$ is as required. Conversely, if there is a normal sequence $\{y_n^*\}$ like that of the statement, it defines a weak* Cauchy series in X^*. Since the series $\sum_{n=1}^{\infty} y_n^*$ does not converge in X^*, according to [**13**, Chapter V, Corollary 11] the space X^* must contain a copy of ℓ_∞. Consequently, X has a complemented copy of ℓ_1 by [**13**, Chapter V, Theorem 10]. □

We refer to the following large class of Banach spaces, for which Problem 6 has a positive answer.

Theorem 9 (Śliwa)**.** *If X is an infinite-dimensional WCG Banach space, every normal sequence in X^* contains a strongly normal subsequence.*

An interesting consequence of Theorem 5 is that 'small' Banach spaces always have a separable quotient. We present another proof, different from the one presented in [**63**, Theorem 3], which depends on the concept of strongly normal sequences.

Corollary 10 (Saxon–Sanchez Ruiz)**.** *If the density character $d(X)$ of a Banach space X satisfies that $\aleph_0 \leq d(X) < \mathfrak{b}$ then X has a separable quotient.*

Recall that the *density character* of a Banach space X is the smallest cardinal of the dense subsets of X. The *bounding cardinal* $\mathfrak{b}$ is referred to as the minimum size for an unbounded subset of the preordered space $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, where $\alpha \leq^* \beta$ stands for the *eventual dominance preorder*, defined so that $\alpha \leq^* \beta$ if the set $\{n \in \mathbb{N} : \alpha(n) > \beta(n)\}$ is finite. So we have $\mathfrak{b} := \inf\{|F| : F \subseteq \mathbb{N}^{\mathbb{N}}, \forall \alpha \in \mathbb{N}^{\mathbb{N}}\ \exists \beta \in F \text{ with } \alpha <^* \beta\}$. It is well known that $\mathfrak{b}$ is a regular cardinal and $\aleph_0 < \mathfrak{b} \leq \mathfrak{c}$. It is consistent that $\mathfrak{b} = \mathfrak{c} > \aleph_1$; indeed, Martin's Axiom implies that $\mathfrak{b} = \mathfrak{c}$. Note that Todorcevic showed in [**69**] that under Martin's maximal axiom every space of density character $\aleph_1$ has a quotient space with an uncountable monotone Schauder basis, and thus a separable quotient.

Proof of Corollary 10. Assume X has a dense subset D of cardinality less than $\mathfrak{b}$. We show that X^* has a *strongly normal sequence* and then we apply Theorem 7. Choose a normalized weak*-null sequence $\{y_n^*\}$ in X^*. For $x \in D$ choose $\alpha_x \in \mathbb{N}^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$ and every $k \geq \alpha_x(n)$ one has $|y_k^* x| < 2^{-n}$. Then $\sum_n |y_{\beta(n)}^* x| < \infty$ if $\alpha_x \leq^* \beta$. Choose $\gamma \in \mathbb{N}^{\mathbb{N}}$ with $\alpha_x \leq^* \gamma$ for each $x \in D$. Then the sequence $\{y_{\gamma(n)}^*\}$ is strongly normal and Theorem 7 applies. □

Another line of research related to Problem 1 deals with those Banach spaces which contain complemented copies of concrete separable sequence spaces. Recall the following important result found in Mujica's survey paper (see [**50**, Theorem 4.1]). We need the following result due to Rosenthal, see [**54**, Corollary 1.6, Proposition 1.2].

Lemma 11. *Let X be a Banach space such that X^* contains an infinite-dimensional reflexive subspace Y. Then X has a quotient isomorphic to Y^*. Consequently X has a separable quotient.*

Proof. Let $Q : X \to Y^*$ be defined by $Qx(y) = y(x)$ for $y \in Y$ and $x \in X$. Let $j : Y \to X^*$ and $\phi_X : X \to X^{**}$ be the inclusion maps. Clearly $Q = j^* \circ \phi_X$ and $Q^* = \phi_X^* \circ j^{**}$. Since Y is reflexive, Q^* is an embedding map and consequently Q is surjective. □

Theorem 12 (Mujica)**.** *If X is a Banach space that contains an isomorphic copy of ℓ_1, then X has a quotient isomorphic to ℓ_2.*

Proof. If X contains a copy of ℓ_1, the dual space X^* contains a copy of $L_1[0,1]$, see [**13**]. It is well known that the space $L_1[0,1]$ contains a copy of ℓ_2. We apply Lemma 11. □

Concerning copies of ℓ_1, let us recall that from classic Rosenthal-Dor's ℓ_1-dichotomy [**13**, Chapter 11] one easily gets the following general result.

Theorem 13. *If X is a non-reflexive weakly sequentially complete Banach space, then X contains an isomorphic copy of ℓ_1.*

The previous results suggest also the following

Problem 14. *Describe a possibly large class of non-reflexive Banach spaces X not containing an isomorphic copy of ℓ_1 and having a separable quotient.*

We may summarize this section with the following

Corollary 15. *Let X be an infinite-dimensional Banach space. Assume that either X or X^* contains an isomorphic copy of c_0 or either X or X^* contains an isomorphic copy of ℓ_1. Then X has separable quotient.*

It is interesting to remark that there exists an infinite-dimensional separable Banach space X such that neither X nor X^* contains a copy of c_0, ℓ_1 or an infinite-dimensional reflexive subspace (see [**33**]).

We refer to [**32**] for several results (and many references) concerning X not containing an isomorphic copy of ℓ_1.

From now onwards, unless otherwise stated, X is an infinite-dimensional Banach space over the field $\mathbb{K}$ of real or complex numbers, as well as all linear spaces we shall consider. Every measurable space (Ω, Σ), as well as every measure space (Ω, Σ, μ), are supposed to be non trivial, i. e., there are in Σ infinitely many pairwise disjoints sets (of finite positive measure). If either X contains or does not contain an isomorphic copy of a Banach space Z we shall frequently write $X \supset Z$ or $X \not\supset Z$, respectively.

3. Weak* compactness of B_{X^*} and separable quotients

In many cases the separable quotient problem is related to the weak*-compactness of the dual unit ball, as the following theorem shows.

Theorem 16. *Let X be a Banach space and let B_{X^*} (weak*) be the dual unit ball equipped with the weak*- topology.*

1. *If B_{X^*} (weak*) is not sequentially compact, then X has a separable quotient which is either isomorphic to c_0 or to ℓ_2.*
2. *If B_{X^*} (weak*) is sequentially compact, then X has a copy of c_0 if and only if it has a complemented copy of c_0.*

Proof. (Sketch) For the first case, if B_{X^*} (weak*) is not sequentially compact, according to the classic Hagler-Johnson theorem [**35**, Corollary 1], X either has a quotient isomorphic to c_0 or X contains a copy of ℓ_1. The later case implies that the dual space X^* of X contains a copy of $L_1[0,1]$, so that X^* contains a copy of ℓ_2. Hence X has a quotient isomorphic to ℓ_2.

The second statement follows from [**20**], where the Gelfand-Phillips property is used. For a direct proof we refer the reader to [**26**, Theorem 4.1]. We provide a brief account of the argument. Let $\{x_n\}$ be a normalized basic sequence in X equivalent to the unit vector basis $\{e_n\}$ of c_0 and let $\{x_n^*\}$ denote the sequence of coordinate functionals of $\{x_n\}$ extended to X via Hahn-Banach's theorem. If $K > 0$ is the basis constant of $\{x_n\}$ then $\|x_n^*\| \leq 2K$, so that $x_n^* \in 2KB_{X^*}$ for every $n \in \mathbb{N}$. Since B_{X^*} (weak*) is sequentially compact, there is a subsequence $\{z_n^*\}$ of $\{x_n^*\}$ that converges to a point $z^* \in X^*$ under the weak*-topology. Let $\{z_n\}$ be the corresponding subsequence of $\{x_n\}$, still equivalent to the unit vector basis of c_0, and let F be the closed linear span of $\{z_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ define the linear functional $u_n : X \to \mathbb{K}$ by $u_n(x) = (z_n^* - z^*)x$, so that $|u_n(x)| \leq 4K\|x\|$ for each $n \in \mathbb{N}$. Since $u_n(x) \to 0$ for all $x \in X$, the linear operator $P : X \to F$ given by $Px = \sum_{n=1}^{\infty} u_n(x) z_n$ is well defined. Due to the formal series $\sum_{n=1}^{\infty} z_n$ is weakly unconditionally Cauchy, there is a constant $C > 0$ such that $\|Px\| \leq 4CK\|x\|$. Now the fact that $z_n^* y \to 0$ for each $y \in F$ means that $z^* \in F^{\perp}$, which implies that $Pz_j = z_j$ for each $j \in \mathbb{N}$. Thus P is a bounded linear projection operator from X onto F. □

It can be easily seen that if X^* contains an isomorphic copy of ℓ_1 but X does not, then X has a quotient isomorphic to c_0. If X^* has a copy of ℓ_1, according to the first part of Theorem 16, then X has a separable quotient isomorphic to c_0 or ℓ_2.

On the other hand, the first statement of the previous theorem also implies that each Banach space that contains an isomorphic copy of $\ell_1(\mathbb{R})$ has a quotient isomorphic to c_0 or ℓ_2. Particularly, each Banach space X containing an isomorphic copy of ℓ_∞ enjoys this property. However, since ℓ_∞ is an injective Banach space and it has a separable quotient isomorphic to ℓ_2 (as follows, for instance, from Theorem 12), one derives that X has a separable quotient isomorphic to ℓ_2 provided X contains an isomorphic copy of ℓ_∞. Useful characterizations of Banach spaces containing a copy of ℓ_∞ can be found in the classic paper [**55**]. It is also shown in [**67**] that ℓ_∞ is a quotient of a Banach space X if and only if B_{X^*} contains a weak*-homeomorphic copy of $\beta\mathbb{N}$. Hence such a space X has in particular a separable quotient isomorphic to ℓ_2.

The class of Banach spaces for which B_{X^*} (weak*) is sequentially compact is rich. This happens, for example if X is a WCG Banach space. Of course, no WCG Banach space contains a copy of ℓ_∞. Another class with weak* sequentially compact dual balls is that of *Asplund* spaces. Note that the second statement of Theorem 16 applies in particular to each Banach space whose weak*-dual unit ball is Corson's (a fact first observed in [**49**]) since, as is well-known, each Corson compact is Fréchet-Urysohn. So, one has the following corollary, where a Banach space X is called *weakly Lindelöf determined* (WLD for short) if there is a set $M \subseteq X$ with $\overline{\operatorname{span}(M)} = X$ enjoying the property that for each $x^* \in X^*$ the set $\{x \in M : x^*x \neq 0\}$ is countable.

Corollary 17. *If X is a WLD Banach space, then X contains a complemented copy of c_0 if and only if it contains a copy of c_0.*

Proof. If X is a WLD Banach space, the dual unit ball B_{X^*} (weak*) of X is Corson (see [**1**, Proposition 1.2]), so the second statement of Theorem 16 applies. □

If K is an infinite Gul'ko compact space, then $C(K)$ is weakly countable determined (see [**2**]), hence WLD. Since $C(K)$ has plenty of copies of c_0, it must have many complemented copies of c_0. It must be pointed out that if K is Corson compact then $C(K)$ need not be WLD. On the other hand, if a Banach space $X \supset c_0$ then $X^* \supset \ell_1$, so surely X has c_0 or ℓ_2 as a quotient (a general characterization of Banach spaces containing a copy of c_0 is provided in [**56**]). This fact can be sharpened, as the next corollary shows.

Corollary 18. *If a Banach space X contains a copy of c_0, then X has either an infinite-dimensional separable quotient isomorphic to c_0 or ℓ_2, or a complemented copy of c_0.*

Proof. If B_{X^*} (weak*) is not sequentially compact, X has a separable quotient isomorphic to c_0 or ℓ_2 as a consequence of the first part of Theorem 16. If B_{X^*} (weak*) is sequentially compact, by the second part X has a complemented copy of c_0. □

Corollary 19. (cf. [**45**] and [**54**]) *If K is an infinite compact Hausdorff space, then $C(K)$ always has a quotient isomorphic to c_0 or ℓ_2. In case that K is scattered, then c_0 embeds in $C(K)$ complementably.*

Proof. The first statement is clear. The second is due to in this case $C(K)$ is an Asplund space (see [**34**, Theorem 296]). □

An extension of the previous corollary to all barrelled spaces $C_k(X)$ with the compact-open topology has been obtained in [**40**].

4. Separable quotients in tensor products

We quote three classic results about the existence of copies of c_0, ℓ_∞ and ℓ_1 in injective and projective tensor products which will be frequently used henceforth and provide a result concerning the existence of a separable quotient in $X \widehat{\otimes}_\pi Y$. We complement these classic facts with other results of our own. In the following theorem c_{00} stands for the linear subspace of c_0 consisting of all those sequence of finite range.

Theorem 20. (cf. [**30**, Theorem 2.3]) *Let X be an infinite-dimensional normed space and let Y be a Hausdorff locally convex space. If $Y \supset c_{00}$ then $X \widehat{\otimes}_\varepsilon Y$ contains a complemented subspace isomorphic to c_0.*

Particularly, if X and Y are infinite-dimensional Banach spaces and $X \supset c_0$ or $Y \supset c_0$, then $X \widehat{\otimes}_\varepsilon Y$ contains a complemented copy of c_0, (cf. [**60**]). On the other hand, if either $X \supset \ell_\infty$ or $Y \supset \ell_\infty$ then $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$ and consequently $X \widehat{\otimes}_\varepsilon Y$ also has a separable quotient isomorphic to ℓ_2. If $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$, the converse statement also holds, as the next theorem asserts.

Theorem 21. (cf. [**17**, Corollary 2]) *Let X and Y be Banach spaces. $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$ if and only if $X \supset \ell_\infty$ or $Y \supset \ell_\infty$.*

This also implies that if $X \widehat{\otimes}_\varepsilon Y \supset \ell_\infty$ then c_0 embeds complementably in $X \widehat{\otimes}_\varepsilon Y$. Concerning projective tensor products, we have the following well-known fact.

Theorem 22. (cf. [**7**, Corollary 2.6]) *Let X and Y be Banach spaces. If both $X \supset \ell_1$ and $Y \supset \ell_1$, then $X \widehat{\otimes}_\pi Y$ has a complemented subspace isomorphic to ℓ_1.*

Next we observe that if $X \widehat{\otimes}_\varepsilon Y$ is not a quotient of $X \widehat{\otimes}_\pi Y$, then $X \widehat{\otimes}_\varepsilon Y$ has a separable quotient.

Theorem 23. *Let $J : X \otimes_\pi Y \to X \otimes_\varepsilon Y$ be the identity map and consider the continuous linear extension $\widetilde{J} : X \widehat{\otimes}_\pi Y \to X \widehat{\otimes}_\varepsilon Y$. If $\widetilde{J}$ is not a quotient map, then $X \widehat{\otimes}_\varepsilon Y$ has a separable quotient.*

Proof. Observe that $X \otimes_\varepsilon Y \subset \mathrm{Im}\widetilde{J} \subset X \widehat{\otimes}_\varepsilon Y$. Two cases are in order.

Assume first that $X \otimes_\varepsilon Y$ is a barrelled space. In this case, since $X \otimes_\varepsilon Y$ is dense in $\mathrm{Im}\widetilde{J}$, we have that the range space $\mathrm{Im}\widetilde{J}$ is a barrelled subspace of $X \widehat{\otimes}_\varepsilon Y$. Given that the graph of $\widetilde{J}$ is closed in $(X \widehat{\otimes}_\pi Y) \times (X \widehat{\otimes}_\varepsilon Y)$ and $\mathrm{Im}\widetilde{J}$ is barrelled, it follows from [**70**, Theorem 19] that $\mathrm{Im}\widetilde{J}$ is a closed subspace of $X \widehat{\otimes}_\varepsilon Y$. Of course, this means that $\mathrm{Im}\widetilde{J} = X \widehat{\otimes}_\varepsilon Y$. Hence, the open map theorem shows that $\widetilde{J}$ is an open map from $X \widehat{\otimes}_\pi Y$ onto $X \widehat{\otimes}_\varepsilon Y$, so that $X \widehat{\otimes}_\varepsilon Y$ is a quotient of $X \widehat{\otimes}_\pi Y$.

Assume now that $X \otimes_\varepsilon Y$ is not barrelled. In this case $X \otimes_\varepsilon Y$ is a non barrelled dense subspace of the Banach space $X \widehat{\otimes}_\varepsilon Y$, so we may apply Theorem 5 to get that $X \widehat{\otimes}_\varepsilon Y$ has a separable quotient. □

Recall that the dual of $X \otimes_\pi Y$ coincides with the space of bounded linear operators from X into Y^*, whereas the dual of $X \otimes_\epsilon T$ may be identified with the subspace of those operators which are integral, see [**58**, Section 3.5].

Proposition 24. *Let X and Y be Banach spaces. If X has the bounded approximation property and there is a bounded linear operator $T : X \to Y^*$ which is not integral, then $X \widehat{\otimes}_\varepsilon Y$ has a separable quotient.*

Proof. Since does exist a bounded not integral linear operator between X and Y^*, the π-topology and ϵ-topology does not coincide on $X \otimes Y$, see [**58**]. Assume $X \otimes_\varepsilon Y$ is barrelled. Since X has the bounded approximation property, [**5**, Theorem] applies to get that $X \otimes_\varepsilon Y = X \otimes_\pi Y$, which contradicts the assumption that $(X \otimes_\varepsilon Y)^* \neq (X \otimes_\pi Y)^*$. Thus $X \otimes_\varepsilon Y$ must be a non barrelled dense linear subspace of $X \widehat{\otimes}_\varepsilon Y$, which according to Theorem 5 ensures that $X \widehat{\otimes}_\varepsilon Y$ has a separable quotient. □

For the next theorem, recall that a Banach space X is called *weakly countably determined* (WCD for short) if X (weak) is a Lindelöf Σ-space.

Theorem 25. *Let X and Y be WCD Banach spaces. If $X \widehat{\otimes}_\varepsilon Y \supset c_0$, then c_0 embeds complementably in $X \widehat{\otimes}_\varepsilon Y$.*

Proof. Since both X and Y are WCD Banach spaces, their dual unit balls B_{X^*} (weak*) and B_{X^*} (weak*) are Gul'ko compact. Given that the countable product of Gul'ko compact spaces is Gul'ko compact, the product space $K := B_{X^*}$ (weak*) $\times$ B_{X^*} (weak*) is Gul'ko compact. Consequently $C(K)$ is a WCD Banach space, which implies in turn that its weak*-dual unit ball $B_{C(K)^*}$ is Gul'ko compact. Particularly $B_{C(K)^*}$ (weak*) is angelic and consequently sequentially compact. Let Z stand for the isometric copy of $X \widehat{\otimes}_\varepsilon Y$ in $C(K)$ and P for the isomorphic copy of c_0 in Z. From the proof of the second statement of Theorem 16 it follows that $C(K)$ has a complemented copy Q of c_0 *contained* in P. This implies that Z, hence $X \widehat{\otimes}_\varepsilon Y$, contains a complemented copy Q of c_0. □

5. Separable quotients in spaces of vector-valued functions

If (Ω, Σ, μ) is a non trivial arbitrary measure space, we denote by $L_p(\mu, X)$, $1 \leq p \leq \infty$, the Banach space of all X-valued p-Bochner μ-integrable (μ-essentially bounded when $p = \infty$) classes of functions equipped with its usual norm. If K is an infinite compact Hausdorff space, then $C(K, X)$ stands for the Banach space of all continuous functions $f : K \to X$ equipped with the supremum norm. By $B(\Sigma, X)$ we represent the Banach space of those bounded functions $f : \Omega \to X$ that are the uniform limit of a sequence of Σ-simple and X-valued functions, equipped with the supremum norm. The space of *all* X-valued bounded functions $f : \Omega \to X$ endowed with the supremum norm is written as $\ell_\infty(\Omega, X)$. Clearly $\ell_\infty(X) = \ell_\infty(\mathbb{N}, X)$. By $\ell_\infty(\Sigma)$ we denote the completion of the space $\ell_0^\infty(\Sigma)$ of scalarly-valued Σ-simple functions, endowed with the supremum norm.

On the other hand, if (Ω, Σ, μ) is a (complete) finite measure space we represent by $P_1(\mu, X)$ the normed space consisting of all those [classes of] strongly μ-measurable X-valued Pettis integrable functions f defined on Ω provided with the semivariation norm

$$\|f\|_{P_1(\mu,X)} = \sup \left\{ \int_\Omega |x^* f(\omega)| \, d\mu(\omega) : x^* \in X^*, \, \|x^*\| \leq 1 \right\}.$$

As is well known, in general $P_1(\mu, X)$ is not a Banach space if X is infinite-dimensional, but it is always a barrelled space (see [**18**, Theorem 2] and [**29**, Remark 10.5.5]).

Our first result collects together a number of statements concerning Banach spaces of vector-valued functions related to the existence of separable quotients, most of them easily derived from well known facts relative to the presence of complemented copies of c_0 and ℓ_p for $1 \leq p \leq \infty$. We denote by $ca^+(\Sigma)$ the set of positive and finite measures on Σ.

Theorem 26. *The following statements on spaces of vector-valued functions hold.*

1. *$C(K,X)$ always has a complemented copy of c_0.*
2. *$C(K,X)$ has a quotient isomorphic to ℓ_1 if and only if X has ℓ_1 as a quotient.*
3. *$L_p(\mu,X)$, with $1 \leq p < \infty$, has a complemented copy of ℓ_p. In particular, the vector sequence space $\ell_p(X)$ has a complemented copy of ℓ_p.*
4. *$L_p(\mu,X)$, with $1 < p < \infty$, has a quotient isomorphic to ℓ_1 if and only if X has ℓ_1 as a quotient. Particularly $\ell_p(X)$ has a quotient isomorphic to ℓ_1 if and only if the same happens to X.*
5. *$L_\infty(\mu,X)$ has a quotient isomorphic to ℓ_2. Hence, so does $\ell_\infty(X)$.*
6. *If μ is purely atomic and $1 \leq p < \infty$, then $L_p(\mu,X)$ has a complemented copy of c_0 if and only if X has a complemented copy of c_0. Particularly, the space $\ell_p(X)$ has a complemented copy of c_0 if and only if so does X.*
7. *If μ is not purely atomic and $1 \leq p < \infty$, then $L_p(\mu,X)$ has complemented copy of c_0 if $X \supset c_0$.*
8. *If $\mu \in ca^+(\Sigma)$ is purely atomic and $1 < p < \infty$, then $L_p(\mu,X)$ has a quotient isomorphic to c_0 if and only if X contains a quotient isomorphic to c_0.*
9. *If $\mu \in ca^+(\Sigma)$ is not purely atomic and $1 < p < \infty$, then $L_p(\mu,X)$ has a quotient isomorphic to c_0 if and only if X contains a quotient isomorphic to c_0 or $X \supset \ell_1$.*
10. *If μ is σ-finite, then $L_\infty(\mu,X)$ has a quotient isomorphic to ℓ_1 if and only if $\ell_\infty(X)$ has ℓ_1 as a quotient.*
11. *$B(\Sigma,X)$ has a complemented copy of c_0 and a quotient isomorphic to ℓ_2.*
12. *$\ell_\infty(\Omega,X)$ has a quotient isomorphic to ℓ_2.*
13. *If the cardinality of Ω is less than the first real-valued measurable cardinal, then $\ell_\infty(\Omega,X)$ has a complemented copy of c_0 if and only if X enjoys the same property. Particularly, $\ell_\infty(X)$ contains a complemented copy of c_0 if and only if X enjoys the same property.*
14. *$c_0(X)$ has a complemented copy of c_0.*
15. *$\ell_\infty(X)^*$ has a quotient isomorphic to ℓ_1.*

Proof. Let us proceed with the proofs of the statements.

1. This well-know fact can be found in [**8**, Theorem] and [**30**, Corollary 2.5] (or in [**9**, Theorem 3.2.1]).
2. This is because $C(K,X)$ contains a complemented copy of ℓ_1 if and only if X contains a complemented copy of ℓ_1 (see [**59**] or [**9**, Theorem 3.1.4]).
3. If $1 \leq p < \infty$, each $L_p(\mu,X)$ space contains a norm one complemented isometric copy of ℓ_p (see [**9**, Proposition 1.4.1]). For the second affirmation note that if (Ω,Σ,μ) is a σ-finite purely atomic measure space, then $\ell_p(X) = L_p(\mu,X)$ isometrically.
4. If $1 < p < \infty$ then $L_p(\mu,X)$ contains complemented copy of ℓ_1 if and only if X does (see [**48**] or [**9**, Theorem 4.1.2]).

5. The space ℓ_∞ is isometrically embedded in $L_\infty(\mu)$, which is in turn isometric to a norm one complemented subspace of $L_\infty(\mu, X)$.
6. If μ is purely atomic, then $L_p(\mu, X)$ contains a complemented copy of c_0 if and only if X has the same property. This fact, discovered by F. Bombal in [**6**], can also be seen in [**9**, Theorem 4.3.1].
7. If μ is not purely atomic and $1 \leq p < \infty$, according to [**21**], the only fact that $X \supset c_0$ implies that $L_p(\mu, X)$ contains a complemented copy of c_0.
8. If (Ω, Σ, μ) is a purely atomic finite measure space and $1 < p < \infty$, the statement corresponds to the first statement of [**12**, Theorem 1.1].
9. If (Ω, Σ, μ) is a not purely atomic finite measure space and $1 < p < \infty$, the statement corresponds to the second statement of [**12**, Theorem 1.1].
10. If (Ω, Σ, μ) is a σ-finite measure space, the existence of a complemented copy of ℓ_1 in $L_\infty(\mu, X)$ is related to the local theory of Banach spaces, a fact discovered by S. Díaz in [**11**]. The statement, in the way as it has been formulated above, can be found in [**9**, Theorem 5.2.3].
11. Since $\ell_0^\infty(\Sigma, X) = \ell_0^\infty(\Sigma) \otimes_\varepsilon X$ and X is infinite-dimensional, then $\ell_0^\infty(\Sigma, X)$ is not barrelled by virtue of classic Freniche's theorem (see [**30**, Corollar 1.5]). Given that $\ell_0^\infty(\Sigma, X)$ is a non barrelled dense subspace of $B(\Sigma, X)$, Theorem 5 guarantees that $B(\Sigma, X)$ has in fact a separable quotient. However, we can be more precise. Since $B(\Sigma, X) = \ell_0^\infty(\Sigma) \widehat{\otimes}_\varepsilon X$ and $\ell_0^\infty(\Sigma) \supset c_{00}$ due to the non triviality of the σ-algebra Σ, Theorem 20 implies that $B(\Sigma, X)$ contains a complemented copy of c_0. On the other hand, since ℓ_∞ is isometrically embedded in $B(\Sigma, X)$, it turns out that ℓ_2 is a quotient of $B(\Sigma, X)$.
12. Clearly $\ell_\infty(\Omega, X) \supset \ell_\infty(\Omega) \supset \ell_\infty$ since the set Ω is infinite.
13. This property can be found in [**46**].
14. Just note that $c_0(X) = c_0 \widehat{\otimes}_\varepsilon X$, so we may apply Theorem 20.
15. It suffices to note that $\ell_1(X^*)$ is linearly isometric to a complemented subspace of $\ell_\infty(X)^*$ (see [**9**, Section 5.1]). $\square$

Remark 27. *$L_p(\mu, X)$ if $1 \leq p < \infty$, as well as $C(K, X)$, need not contain a copy of ℓ_∞.* By [**47**, Theorem], one has that $L_p(\mu, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$, whereas $C(K, X) \supset \ell_\infty$ if and only if $C(K) \supset \ell_\infty$ or $X \supset \ell_\infty$, as shown in [**17**, Corollary 3].

Remark 28. *Complemented copies of c_0 in $L_\infty(\mu, X)$.* If (Ω, Σ, μ) is a σ-finite measure, according to [**10**, Theorem 1] a necessary condition for the space $L_\infty(\mu, X)$ to contain a complemented copy of c_0 is that $X \supset c_0$. The same happens with the space $\ell_\infty(\Omega, X)$ (see [**26**, Theorem 2.1 and Corollary 2.3]).

Theorem 29. *The following statements on the space $\widehat{P_1(\mu, X)}$ hold.*

1. *If the finite measure space (Ω, Σ, μ) is not purely atomic, the Banach space $\widehat{P_1(\mu, X)}$ has a separable quotient.*
2. *If the range of the positive finite measure μ is infinite and $X \supset c_0$ then $\widehat{P_1(\mu, X)}$ has a complemented copy of c_0.*

Proof. Observe that $L_1(\mu)\,\widehat{\otimes}_\pi X = L_1(\mu, X)$ and $\widehat{P_1(\mu, X)} = L_1(\mu)\,\widehat{\otimes}_\varepsilon X$ isometrically. On the other hand, from the algebraic viewpoint $L_1(\mu, X)$ is a linear subspace of $P_1(\mu, X)$, which is dense under the norm of $P_1(\mu, X)$. If

$$J : L_1(\mu) \otimes_\pi X \to L_1(\mu) \otimes_\varepsilon X$$

is the identity map, R a linear isometry from $L_1(\mu, X)$ onto $L_1(\mu)\,\widehat{\otimes}_\pi X$ and S a linear isometry from $L_1(\mu)\,\widehat{\otimes}_\varepsilon X$ onto $\widehat{P_1(\mu, X)}$, the mapping

$$S \circ \widetilde{J} \circ R : L_1(\mu, X) \to \widehat{P_1(\mu, X)},$$

where $\widetilde{J}$ denotes the (unique) continuous linear extension of J to $L_1(\mu)\,\widehat{\otimes}_\pi X$, coincides with the natural inclusion map T of $L_1(\mu, X)$ into $P_1(\mu, X)$ over the dense subspace of $L_1(\mu, X)$ consisting of the X-valued (classes of) μ-simple functions, which implies that $S \circ \widetilde{J} \circ R = T$. Since X is infinite-dimensional and μ is not purely atomic, the space $P_1(\mu, X)$ is not complete [**68**]. So necessarily we have that $\mathrm{Im}T \neq \widehat{P_1(\mu, X)}$. This implies in particular that $\mathrm{Im}\widetilde{J} \neq L_1(\mu)\,\widehat{\otimes}_\pi X$. According to Theorem 23, this means that $\widehat{P_1(\mu, X)} = L_1(\mu)\,\widehat{\otimes}_\varepsilon X$ has a separable quotient.

The proof of the second statement can be found in [**31**, Corollary 2]. □

6. Separable quotients in spaces of linear operators

If Y is also a Banach space, let us denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators $T : X \to Y$ equipped with the operator norm $\|T\|$. By $\mathcal{K}(X, Y)$ we represent the closed linear subspace of $\mathcal{L}(X, Y)$ consisting of all those compact operators. We design by $\mathcal{L}_{w^*}(X^*, Y)$ the closed linear subspace of $\mathcal{L}(X^*, Y)$ formed by all weak*-weakly continuous operators and by $\mathcal{K}_{w^*}(X^*, Y)$ the closed linear subspace of $\mathcal{K}(X^*, Y)$ consisting of all weak*-weakly continuous operators. The closed subspace of $\mathcal{L}(X, Y)$ consisting of weakly compact linear operators is designed by $\mathcal{W}(X, Y)$. It is worthwhile to mention that $\mathcal{L}_{w^*}(X^*, Y) = \mathcal{L}_{w^*}(Y^*, X)$ isometrically, as well as $\mathcal{K}_{w^*}(X^*, Y) = \mathcal{K}_{w^*}(Y^*, X)$, by means of the linear mapping $T \mapsto T^*$. The Banach space of nuclear operators $T : X \to Y$ equipped with the so-called nuclear norm $\|T\|_N$ is denoted by $\mathcal{N}(X, Y)$. Let us recall that $\|T\| \leq \|T\|_N$. Classic references for this section are the monographs [**38**] and [**44**].

The first statement of Theorem 30 answers a question of Prof. T. Dobrowolski posed during the 31st Summer Conference on Topology and its Applications at Leicester (2016).

Theorem 30. *The following conditions on $\mathcal{L}(X, Y)$ hold.*

1. *If $Y \neq \{\mathbf{0}\}$, then $\mathcal{L}(X, Y)$ always has a separable quotient.*
2. *If $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{L}(X, Y)$ has a quotient isomorphic to ℓ_2.*
3. *If $X^* \supset \ell_q$ and $Y \supset \ell_p$, with $1 \leq p < \infty$ and $1/p + 1/q = 1$, then $\mathcal{L}(X, Y)$ has a quotient isomorphic to ℓ_2.*

Proof. Let us prove each of these statements.

1. First observe that X^* is complemented in $\mathcal{L}(X, Y)$. Indeed, choose $y_0 \in Y$ with $\|y_0\| = 1$ and apply the Hahn-Banach theorem to get $y_0^* \in Y^*$ such that $\|y_0^*\| = 1$ and $y_0^* y_0 = 1$. The map $\varphi : X^* \to \mathcal{L}(X, Y)$ defined by $(\varphi x^*)(x) = x^* x \cdot y_0$ for

every $x \in X$ is a linear isometry into $\mathcal{L}(X,Y)$ (see [**44**, 39.1.(2')]), and the operator $P : \mathcal{L}(X,Y) \to \mathcal{L}(X,Y)$ given by $PT = \varphi(y_0^* \circ T)$ is a norm one linear projection operator from $\mathcal{L}(X,Y)$ onto $\operatorname{Im}\varphi$. Hence X^* is linearly isometric to a norm one complemented linear subspace of $\mathcal{L}(X,Y)$. Since X^* is a dual Banach space, it has a separable quotient by Theorem 4. Hence the operator space $\mathcal{L}(X,Y)$ has a separable quotient.

2. Since $X^* \otimes_\varepsilon Y$ is isometrically embedded in $\mathcal{L}(X,Y)$ and both X^* and Y are isometrically embedded in $X^* \otimes_\varepsilon Y$, if either $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{L}(X,Y) \supset c_0$. In this case, according to [**25**, Corollary 1], $\mathcal{L}(X,Y)$ contains an isomorphic copy of ℓ_∞. This ensures that $\mathcal{L}(X,Y)$ has a separable quotient isomorphic to ℓ_2.
3. If $\{e_n : n \in \mathbb{N}\}$ is the unit vector basis of ℓ_p, define $T_n : \ell_p \to \ell_p$ by $T_n\xi = \xi_n e_n$ for each $n \in \mathbb{N}$. Since
$$\left\| \sum_{i=1}^n a_i T_i \right\| = \sup_{\|\xi\|_p \le 1} \left(\sum_{i=1}^n |a_i \xi_i|^p \right)^{1/p} \le \sup_{1 \le i \le n} |a_i|$$
for any scalars $a_1, \ldots, a_n$, we can see that $\{T_n : n \in \mathbb{N}\}$ is a basic sequence in $\mathcal{K}(\ell_p, \ell_p)$ equivalent to the unit vector basis of c_0. Since it holds in general that $E^* \widehat{\otimes}_\varepsilon F = \mathcal{K}(E,F)$ for Banach spaces E and F whenever E^* has the approximation property, if $1/p + 1/q = 1$ one has that
$$\ell_q \widehat{\otimes}_\varepsilon \ell_p = \ell_p^* \widehat{\otimes}_\varepsilon \ell_p = \mathcal{K}(\ell_p, \ell_p)$$
isometrically. So we have $\ell_q \widehat{\otimes}_\varepsilon \ell_p \supset c_0$. As in addition $\ell_q \widehat{\otimes}_\varepsilon \ell_p$ is isometrically embedded in $X^* \widehat{\otimes}_\varepsilon Y$, which in turn is also isometrically embedded in $\mathcal{L}(X,Y)$, we conclude that $\mathcal{L}(X,Y) \supset c_0$. So, we use again [**25**, Corollary 1] to conclude that $\mathcal{L}(X,Y) \supset \ell_\infty$. Thus $\mathcal{L}(X,Y)$ has a quotient isomorphic to ℓ_2. □

The Banach space $\mathcal{L}(X,Y)$ need not contain a copy of ℓ_∞ in order to have a separable quotient, as the following example shows.

Let $1 < p, q < \infty$ with conjugated indices p', q', i. e., $1/p + 1/p' = 1/q + 1/q' = 1$.

Example 31. *If $p > q'$ then $\mathcal{L}(\ell_p, \ell_{q'})$ does not contain an isomorphic copy of c_0.*

Proof. Since it holds in general that $\mathcal{L}(X, Y^*) = (X \widehat{\otimes}_\pi Y)^*$ isometrically for arbitrary Banach spaces X and Y (see for instance [**58**, Section 2.2]), the fact that $\ell_{q'}^* = \ell_q$ assures that $\mathcal{L}(\ell_p, \ell_{q'}) = (\ell_p \widehat{\otimes}_\pi \ell_q)^*$ isometrically. Now let us assume by contradiction that $\mathcal{L}(\ell_p, \ell_{q'}) \supset c_0$, which implies that $\ell_p \widehat{\otimes}_\pi \ell_q$ contains a complemented copy of ℓ_1 (see [**13**, Chapter 5, Theorem 10]). Since $p > q'$, according to [**58**, Corollary 4.24] or [**14**, Chapter 8, Corollary 5], the space $\ell_p \widehat{\otimes}_\pi \ell_q$ is reflexive, which contradicts the fact that it has a quotient isomorphic to the non reflexive space ℓ_1. So we must conclude that $\mathcal{L}(\ell_p, \ell_{q'}) \not\supset c_0$.

On the other hand, since $\mathcal{L}(\ell_p, \ell_{q'})$ is a dual Banach space, Theorem 4 shows that $\mathcal{L}(\ell_p, \ell_{q'})$ has a separable quotient. Alternatively, we can also apply the first statement of Theorem 30. □

Proposition 32. *If X^* has the approximation property, the Banach space $\mathcal{N}(X,Y)$ of nuclear operators has a separable quotient.*

Proof. Since X^* enjoys the approximation property, it follows that $\mathcal{N}(X,Y) = X^*\widehat{\otimes}_\pi Y$ isometrically. Hence X^* is linearly isometric to a complemented subspace of $\mathcal{N}(X,Y)$. Since X^*, as a dual Banach space, has a separable quotient, the transitivity of the quotient map yields that $\mathcal{N}(X,Y)$ has a separable quotient. □

Theorem 33. *The following statements hold.*

1. *If $X \supset c_0$ and $Y \supset c_0$, then $\mathcal{L}_{w^*}(X^*,Y)$ has a quotient isomorphic to ℓ_2.*
2. *If X has a separable quotient isomorphic to ℓ_1, then $\mathcal{L}_{w^*}(X^*,Y)$ enjoys the same property.*
3. *If (Ω,Σ,μ) is an arbitrary measure space and $Y \neq \{\mathbf{0}\}$, then $\mathcal{L}_{w^*}(L_\infty(\mu),Y)$ has a quotient isomorphic to ℓ_1.*
4. *If $X^* \supset c_0$ or $Y \supset \ell_\infty$, then $\mathcal{K}(X,Y)$ has a quotient isomorphic to ℓ_2.*
5. *If either $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{K}(X,Y)$ contains a complemented copy of c_0.*
6. *If $X \supset \ell_\infty$ or $Y \supset \ell_\infty$, then $\mathcal{K}_{w^*}(X^*,Y)$ has a quotient isomorphic to ℓ_2.*
7. *The space $\mathcal{W}(X,Y)$ always has a separable quotient.*
8. *If $X \supset c_0$ and $Y \supset c_0$, then $\mathcal{W}(X,Y)$ contains a complemented copy of c_0.*

Proof. In many cases it suffices to show that the corresponding Banach space contains an isomorphic copy of ℓ_∞.

1. By [**27**, Theorem 1.5] if $X \supset c_0$ and $Y \supset c_0$ then $\mathcal{L}_{w^*}(X^*,Y) \supset \ell_\infty$.
2. Choose $y_0 \in Y$ with $\|y_0\| = 1$ and select $y_0^* \in Y^*$ such that $\|y_0^*\| = 1$ and $y_0^* y_0 = 1$. The map $\psi : X \to \mathcal{L}_{w^*}(X^*,Y)$ given by $\psi(x)(x^*) = x^*x \cdot y_0$, for $x^* \in X^*$, is well-defined and if $x_d^* \to x^*$ under the weak*- topology of X^* then $\psi(x)(x_d^*) \to \psi(x)(x^*)$ weakly in Y, so that ψ embeds X isometrically in $\mathcal{L}_{w^*}(X^*,Y)$. On the other hand, the operator $Q : \mathcal{L}_{w^*}(X^*,Y) \to \mathcal{L}_{w^*}(X^*,Y)$ given by $QT = \psi(y_0^* \circ T)$, which is also well-defined since $y_0^* \circ T \in X$ whenever T is weak*-weakly continuous, is a bounded linear projection operator from $\mathcal{L}_{w^*}(X^*,Y)$ onto $\mathrm{Im}\psi$. Since we are assuming that ℓ_1 is a quotient of X, it follows that ℓ_1 is also isomorphic to a quotient of $\mathcal{L}_{w^*}(X^*,Y)$.
3. This statement is consequence of the previous one, since ℓ_1 embeds complementably in $L_1(\mu)$.
4. According to [**43**], if $X^* \supset c_0$ or $Y \supset \ell_\infty$, then $\mathcal{K}(X,Y) \supset \ell_\infty$.
5. This property has been shown in [**57**, Corollary 1].
6. The map $\psi : X \to \mathcal{L}_{w^*}(X^*,Y)$ defined above by $\psi(x)(x^*) = x^*x \cdot y_0$, for every $x^* \in X^*$, yields a finite-rank (hence compact) operator $\psi(x)$, so that $\mathrm{Im}\psi \subset \mathcal{K}_{w^*}(X^*,Y)$. On the other hand, if $x_0 \in X$ with $\|x_0\| = 1$ and $x_0^* \in X^*$ verifies that $\|x_0^*\| = 1$ and $x_0^* x_0 = 1$, the map $\phi : Y \to \mathcal{K}_{w^*}(X^*,Y)$ given by $\phi(y)(x) = x_0^* x \cdot y$, for every $x \in X$, is a linear isometry from Y into $\mathcal{K}_{w^*}(X^*,Y)$. Hence X and Y are isometrically embedded in $\mathcal{K}_{w^*}(X^*,Y)$.
7. Just note that $\mathcal{W}(X,Y) = \mathcal{L}_{w^*}(X^{**},Y)$ isometrically. Since X^* is complementably embedded in $\mathcal{L}_{w^*}(X^{**},Y)$, the conclusion follows from Theorem 4.
8. According to [**28**, Theorem 2.5], under those conditions the space $\mathcal{W}(X,Y)$ contains a complemented copy of c_0. □

Remark 34. If neither X nor Y contains a copy of c_0, then $\mathcal{L}_{w^*}(X^*, Y)$ cannot contain a complemented copy of c_0 as observed in [**22**].

The following result sharpens the first statement of Theorem 29.

Corollary 35. *If (Ω, Σ, μ) is a finite measure space, $\widehat{P_1(\mu, X)}$ has a quotient isomorphic to ℓ_1.*

Proof. This follows from the second statement of the previous theorem together with the fact that $\widehat{P_1(\mu, X)} = \mathcal{L}_{w^*}(L_\infty(\mu), X)$ (see [**14**, Chapter 8, Theorem 5]).

Remark 36. *The space $P_1(\mu, X)$ need no contain a copy of ℓ_∞.* It can be easily shown that $P_1(\mu, X)$ embeds isometrically in the space $\mathcal{K}_{w^*}(ca(\Sigma)^*, X)$, where $ca(\Sigma)$ denotes the Banach space of scalarly-valued countably additive measures equipped with the variation norm. Since $ca(\Sigma) \not\supset \ell_\infty$, it follows from [**17**, Theorem] that $P_1(\mu, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$.

7. Separable quotients in spaces of vector measures

In this section we denote by $ba(\Sigma, X)$ the Banach space of all X-valued bounded finitely additive measures $F : \Sigma \to X$ provided with the semivariation norm $\|F\|$. The closed linear subspace of $ba(\Sigma, X)$ consisting of those countably additive measures is represented by $ca(\Sigma, X)$, while $cca(\Sigma, X)$ stands for the (closed) linear subspace of $ca(\Sigma, X)$ of all measures with relatively compact range. It can be easily shown that $ca(\Sigma, X) = \mathcal{L}_{w^*}(ca(\Sigma)^*, X)$ isometrically. We also design by $bvca(\Sigma, X)$ the Banach space of all X-valued countably additive measures $F : \Sigma \to X$ of bounded variation equipped with the variation norm $|F|$. Finally, following [**58**, page 107], we denote by $\mathcal{M}_1(\Sigma, X)$ the closed linear subspace of $bvca(\Sigma, X)$ consisting of all those $F \in bvca(\Sigma, X)$ that have the so-called *Radon-Nikodým property*, i. e., such that for each $\lambda \in ca^+(\Sigma)$ with $F \ll \lambda$ there exists $f \in L_1(\lambda, X)$ with $F(E) = \int_E f \, d\lambda$ for every $E \in \Sigma$. For this section, our main references are [**14**] and [**58**].

Theorem 37. *The following statements hold. In the first case X need not be infinite-dimensional.*

1. *If $X \neq \{\mathbf{0}\}$, then $ba(\Sigma, X)$ always has a separable quotient.*
2. *If $X \supset c_0$, then $ba(\Sigma, X)$ has a quotient isomorphic to ℓ_2.*
3. *If $X \supset c_0$ but $X \not\supset \ell_\infty$, then $ba(\Sigma, X)$ has a complemented copy of c_0.*
4. *If Σ admits no atomless probability measure, then $ca(\Sigma, X)$ has a quotient isomorphic to ℓ_1.*
5. *If $X \supset c_0$ and Σ admits a nonzero atomless $\lambda \in ca^+(\Sigma)$, then $ca(\Sigma, X)$ has a quotient isomorphic to ℓ_2.*
6. *If there exists some $F \in cca(\Sigma, X)$ of unbounded variation, then $cca(\Sigma, X)$ has a separable quotient.*
7. *If $X \supset c_0$, then $cca(\Sigma, X)$ contains a complemented copy of c_0.*
8. *If $X \supset \ell_1$, then $\mathcal{M}_1(\Sigma, X)$ has a quotient isomorphic to ℓ_1.*

Proof. In cases 2 and 3 it suffices to show that the corresponding Banach space contains an isomorphic copy of ℓ_∞.

1. This happens because $ba(\Sigma, X) = \mathcal{L}(\ell_\infty(\Sigma), X)$ isometrically. Since $\ell_\infty(\Sigma)$ is infinite-dimensional by virtue of the non triviality of the σ-algebra Σ, the statement follows from the first statement of Theorem 30.
2. By point 2 of Theorem 30, if $X \supset c_0$ then $\mathcal{L}(\ell_\infty(\Sigma), X)$ has a quotient isomorphic to ℓ_2. The statement follows from the fact that $ba(\Sigma, X) = \mathcal{L}(\ell_\infty(\Sigma), X)$.
3. $ba(\Sigma, X)$ has a complemented copy of c_0 by virtue of [**28**, Corollary 3.2].
4. If the non trivial σ-algebra Σ admits no atomless probability measure, it can be shown that $ca(\Sigma, X)$ is linearly isometric to $\ell_1(\Gamma, X)$ for some infinite set Γ. Since $\ell_1(\Gamma, X) = L_1(\mu, X)$, where μ is the counting measure on 2^Γ, the conclusion follows from the third statement of Theorem 26.
5. Since $ca(\Sigma) \widehat{\otimes}_\varepsilon X = cca(\Sigma, X)$ isometrically, if $X \supset c_0$ then $cca(\Sigma, X) \supset c_0$ and hence $ca(\Sigma, X) \supset c_0$. If Σ admits a nonzero atomless $\lambda \in ca^+(\Sigma)$, then one has $ca(\Sigma, X) \supset \ell_\infty$ by virtue of [**16**, Theorem 1].
6. Observe that $cca(\Sigma, X) = ca(\Sigma) \widehat{\otimes}_\varepsilon X$ and $\mathcal{M}_1(\Sigma, X) = ca(\Sigma) \widehat{\otimes}_\pi X$ isometrically (see [**58**, Theorem 5.22]) but, at the same time, from the algebraic point of view, $\mathcal{M}_1(\Sigma, X)$ is a linear subspace of $cca(\Sigma, X)$ since every Bochner indefinite integral has a relatively compact range, [**14**, Chapter II, Corollary 9 (c)]. If there exists some $F \in cca(\Sigma, X)$ of unbounded variation, then $\mathcal{M}_1(\Sigma, X) \neq cca(\Sigma, X)$, so the statement follows from Theorem 23.
7. Since $X \supset c_0$ and $ca(\Sigma)$ is infinite-dimensional, then $X \widehat{\otimes}_\varepsilon ca(\Sigma)$ contains a complemented copy of c_0 by [**30**, Theorem 2.3].
8. Since $\mathcal{M}_1(\Sigma, X) = ca(\Sigma) \widehat{\otimes}_\pi X$ and $ca(\Sigma) \supset \ell_1$, if $X \supset \ell_1$ then $\mathcal{M}_1(\Sigma, X)$ has a quotient isomorphic to ℓ_1 as follows from Theorem 22. □

Remark 38. If $\omega \in \Omega$ and $E(\Sigma, X)$ is either $ba(\Sigma, X)$, $ca(\Sigma, X)$ or $bvca(\Sigma, X)$, the map $P_\omega : E(\Sigma, X) \to E(\Sigma, X)$ defined by $P_\omega(F) = F(\Omega)\,\delta_\omega$ is a bounded linear projection operator onto the copy $\{x\,\delta_\omega : x \in X\}$ of X in $E(\Sigma, X)$. Hence, if X has a separable quotient isomorphic to Z, then $E(\Sigma, X)$ also has a separable quotient isomorphic to Z.

Remark 39. *$cca(\Sigma, X)$ may not have a copy of ℓ_∞.* Since $cca(\Sigma, X) = \mathcal{K}_{w^*}(ca(\Sigma)^*, X)$, according to [**17**, Theorem or Corollary 4], $cca(\Sigma, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$.

Remark 40. Concerning the space $\mathcal{M}_1(\Sigma, X)$, it is worthwhile to mention that it follows from [**24**, Theorem] that $\mathcal{M}_1(\Sigma, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$.

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