On the separable quotient problem

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Abstract While the classic separable quotient problem remains open, we partially survey (and provide some new) general results related to this problem, and examine the existence of a particular infinite-dimensional separable quotient in some Banach spaces of vector-valued functions, linear operators and vector measures. Some of the results presented are consequence of well-known facts, several of them relative to the presence of complemented copies of the classic sequence spaces $c_0$ and $\ell_p$, for $1 \leq p \leq \infty$.

Keywords Banach space · barrelled space · separable quotient · vector-valued function space · linear operator space · vector measure space · tensor product · Radon-Nikodým property

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1 Introduction

2 Preliminaries

One of unsolved problems of Functional Analysis (posed by S. Mazur in 1932) asks:

**Problem 1** Does any infinite-dimensional Banach space have a separable (infinite dimensional) quotient?

An easy application of the open mapping theorem shows that an infinite dimensional Banach space $X$ has a separable quotient if and only if $X$ is mapped on a separable Banach space under a continuous linear map.

Seems that the first comments about Problem 1 are mentioned in [45] and [54]. It is already well known that all reflexive, or even all infinite-dimensional weakly compactly generated Banach spaces (WCG for short), have separable quotients. In [37, Theorem IV.1(i)] Johnson and Rosenthal proved that every infinite dimensional separable Banach space admits a quotient with a Schauder basis. This results provides another (equivalent) reformulation of Problem 1.

**Problem 2** Does any infinite dimensional Banach space admits an infinite dimensional quotient with a Schauder basis?

In [36, Theorem 2] is proved that if $Y$ is a separable closed subspace of an infinite-dimensional Banach space $X$ and the quotient $X/Y$ has a separable quotient, then $Y$ is quasi-complemented in $X$. So one gets the next equivalent condition to Problem 1.

**Problem 3** Does any infinite dimensional Banach space $X$ contains a separable closed subspace which has a proper quasi-complement in $X$?

Although Problem 1 is left open for Banach spaces, a corresponding question whether any infinite dimensional metrizable and complete non-normed topological vector space $X$ admits a separable quotient has been already solved. Indeed, if $X$ is locally convex (i.e., $X$ is a Fréchet space), a result of Eidelheit [20] ensures that $X$ has a quotient isomorphic to $K^N$, where $K \in \{\mathbb{R}, \mathbb{C}\}$.

In [16] Drewnowski posed a more general question: whether any infinite dimensional metrizable and complete topological vector space $X$ has a closed subspace $Y$ such that the dimension of the quotient $X/Y$ is continuum (in short $\dim(X/Y) = c$). The same paper contains an observation stating that any infinite dimensional Fréchet locally convex space admits a quotient of dimension $c := 2^\aleph_0$. Finally Drewnowski’s problem has been solved by M. Popov, see [53]. He showed, among others, that for $0 < p < 1$ the space $X := L_p([0,1]^2)$ does not admit a proper closed subspace $Y$ such that $\dim(X/Y) \leq c$. Consequently $X$ does not have a separable quotient.

The organization of the present paper goes as follows. In the second section we gather general selected results about the separable quotient problem and
some classic results on Banach spaces containing copies of sequence spaces, providing as well some straightforward consequences.

In the second part, among others, we exhibit how the weak*-compactness of the dual unit ball is related to the existence of quotients isomorphic to $c_0$ or $\ell_2$.

Next section contains three classic results about complete tensor products of Banach spaces and their applications to the separable quotient problem. This section provides also some new material (as we hope), namely Theorems 11 and 12.

Last parts are devoted to examine the existence of separable quotients for many concrete classes of ‘big’ Banach spaces, as Banach spaces of vector-valued functions, bounded linear operators and vector measures. For these classes of spaces we fix particular separable quotients by means of the literature on the subject and the previous results.

This line of research has been also continued in a more general setting for the class of topological vector spaces, particularly for spaces $C(X)$ of real-valued continuous functions endowed with the pointwise and compact-open topology, see [38], [39] and [40].

3 A few results for general Banach spaces

Let us start with the following remarkable, possible the most general, concrete result related to Problem 1.

**Theorem 1 (Argyros-Dodos-Kanellopoulos)** Every infinite-dimensional dual Banach space has a separable quotient.

Hence the space $L(X,Y)$ of bounded linear operators between Banach spaces $X$ and $Y$ equipped with the operator norm has a separable quotient provided $Y \neq \{0\}$. Indeed, it follows from from the fact that $X^*$ is complemented in $L(X,Y)$, see Theorem 15 below for details.

There are several equivalent conditions to Problem 1. Let’s select a few of them. For the equivalence (2) and (3) below (and applications) in the class of locally convex spaces we refer the reader to [39], [61] and [62]. The equivalence between (1) and (3) is due to Saxon-Wilansky [64]. Recall that a locally convex space $E$ is barreled if every barrel (an absolutely convex closed and absorbing set in $E$) is a neighborhood of zero. We refer also to [5] and [37] for some partial results related to the next theorem.

**Theorem 2 (Saxon-Wilansky)** The following assertions are equivalent for an infinite-dimensional Banach space $X$.

1. $X$ contains a dense non-barreled linear subspace.
2. $X$ admits a strictly increasing sequence of closed subspaces of $X$ whose union is dense in $X$.
3. $X^*$ admits a strictly decreasing sequence of weakly*-closed subspaces whose intersection consists only of zero element.
4. $X$ has a separable quotient.

**Proof** We prove only the equivalence between (2) and (4) which holds for any locally convex space $X$.

(4) $\Rightarrow$ (2) Note that every separable Banach space has the above property (as stated in (2)) and this property is preserved under preimages of surjective linear operators (which clearly are open maps).

(2) $\Rightarrow$ (4) Let $\{X_n : n \in \mathbb{N}\}$ be such a sequence. We may assume that $\dim(X_{n+1}/X_n) \geq n$ for all $n \in \mathbb{N}$. Let $x_1 \in X_2 \setminus X_1$. There exists $x_1^* \in X^*$ such that $x_1^*x_1 = 1$ and $x_1^*$ vanishes on $X_1$. Assume that we have constructed $(x_1, x_1^*), \ldots, (x_n, x_n^*)$ in $X \times X^*$ with $x_j \in X_{j+1}$ such that $x_j^*x_1 = 1$ and $x_j^*$ vanishes on $X_j$ for $1 \leq j \leq n$. Choose $x_{n+1} \in X_{n+2} \cap (x_1)^{-1}(0) \cap \cdots \cap (x_n^*)^{-1}(0) \setminus X_{n+1}$. Then there exists $x_{n+1}^* \in X^*$ with $x_{n+1}^*x_{n+1} = 1$ vanishing on $X_{n+1}$. Since $X_{n+1} \subseteq \text{span}\{x_1, x_2, \ldots, x_n\} + \bigcap_{k=1}^\infty \text{ker} x_k^*$, $n \in \mathbb{N}$, we conclude that $\text{span}\{x_n : n \in \mathbb{N}\} + \bigcap_n \ker x_n^*$ is dense in $X$. Then $X/Y$ is separable for $Y := \bigcap_n \ker x_n^*$.

Hence, every Banach space whose weak*-dual is separable has a separable quotient. Theorem 2 applies to show that every infinite dimensional WCG Banach space has a separable quotient. Indeed, if $X$ is reflexive we apply Theorem 1. If $X$ is not reflexive, choose a weakly compact absolutely convex set $K$ in $X$ such that $\text{span}(K) = X$. Since $K$ is a barrel of $Y := \text{span}(K)$ and $X$ is not reflexive, it turns out that $Y$ is a dense non-barrelled linear subspace of $X$.

The class of WCG Banach spaces, introduced in [4], provides a quite successful generalization of reflexive and separable spaces. As proved in [4] there are many bounded projection operators with separable ranges on such spaces, so many separable complemented subspaces. For example, if $X$ is WCG and $Y$ is a separable subspace, there exists a separable closed subspace $Z$ with $Y \subseteq Z \subseteq X$ together with a contractive projection. This shows that every infinite-dimensional WCG Banach space admits many separable complemented subspaces, so separable quotients.

The Josefson-Nissenzweig theorem states that the dual of any infinite-dimensional Banach space contains a normal sequence, i.e., a normalized weak*-null sequence [51].

Recall (cf. [65]) that a sequence $\{y_n^*\}$ in the sphere $S(X^*)$ of $X^*$ is strongly normal if the subspace $\{x \in X : \sum_{n=1}^\infty |y_n^*x| < \infty\}$ is dense in $X$. Clearly every strongly normal sequence is normal. Having in mind Theorem 3 the following question is of interest.

**Problem 4** Does every normal sequence in $X^*$ contains a strongly normal subsequence?

By [65, Theorem 1], any strongly normal sequence in $X^*$ contains a subsequence $(y_n)$ which is a Schauder basic sequence in the weak*-topology, i.e., $(y_n)$ is a Schauder basis in its closed linear span in the weak*-topology. Conversely, any normalized Schauder basic sequence in $(X^*, w^*)$ is strongly normal, [65, Proposition 1].
Theorem 3 (Śliwa) Let $X$ be an infinite-dimensional Banach space. The following conditions are equivalent:

1. $X$ has a separable quotient.
2. $X^*$ has a strongly normal sequence.
3. $X^*$ has a basic sequence in the weak* topology.
4. $X^*$ has a pseudobounded sequence, i.e., a sequence $\{x_n^*\}$ in $X^*$ that is pointwise bounded on a dense subspace of $X$ and $\sup_n \|x_n^*\| = \infty$.

Proof We prove only (1) $\iff$ (2) and (1) $\iff$ (4). The equivalence between (2) and (3) follows from the preceding remark.

(1) $\implies$ (2) By Theorem 2 the space $X$ contains a dense non-barrelled subspace. Hence there exists a closed absolutely convex set $D$ in $X$ such that $H := \text{span}(D)$ is a proper dense subspace of $X$. For each $n \in \mathbb{N}$ choose $x_n \in X$ so that $\|x_n\| \le n^{-2}$ and $x_n \notin D$. Then select $x_n^* \in X^*$ such that $x_n^* x_n > 1$ and $|x_n^* x| \le 1$ for all $x \in D$. Set $y_n^* := \|x_n^*\|^{-1} x_n^*$ for all $n \in \mathbb{N}$. Since $\|x_n^*\| \ge n^2$, $y_n^* \in S(X^*)$ and $\sum_{n=1}^{\infty} |y_n^* x| < \infty$ for all $x \in D$, the sequence $\{y_n^*\}$ is as required.

(2) $\implies$ (1) Assume that $X^*$ contains a strongly normal sequence $\{y_n^*\}$. By [65, Theorem 1] (see also [37, Theorem III.1 and Remark III.1]) there exists a subsequence of $\{y_n^*\}$ which is a weak*-basic sequence in $X^*$. This implies that $X$ admits a strictly increasing sequence of closed subspaces whose union is dense in $X$. Indeed, since $(X^*, w^*)$ contains a basic sequence, and hence there exists a strictly decreasing sequence $\{U_n : n \in \mathbb{N}\}$ of closed subspaces in $(X^*, w^*)$ with $\bigcap_{n=1}^{\infty} U_n = \{0\}$, the space $X$ has a sequence as required. This provides a biorthogonal sequence as in the proof of (2) $\implies$ (3), Theorem 2.

(4) $\implies$ (1) Let $\{y_n^*\}$ be a pseudobounded sequence in $X^*$. Set $Y := \{x \in X : \sup_{n \in \mathbb{N}} |y_n^* x| < \infty\}$. The Banach-Steinhaus theorem applies to deduce that $Y$ is a proper and dense subspace of $X$. Note that $Y$ is not barrelled since $V := \{x \in X : \sup_{n=1}^{\infty} |y_n^* x| \le 1\}$ is a barrel in $H$ which is not a neighborhood of zero in $H$. Now apply Theorem 2.

(1) $\implies$ (4) Assume $X$ contains a dense non-barrelled subspace $Y$. Let $W$ be a barrel in $Y$ which is not a neighborhood of zero in $Y$. If $V$ is the closure of $W$ in $X$, the linear span $H$ of $V$ is a dense proper subspace of $X$. So, for every $n \in \mathbb{N}$ there is $x_n \in X \setminus V$ with $\|x_n\| \le n^{-2}$. Choose $z_n^* \in X^*$ so that $|z_n^* x_n| > 1$ and $|z_n^* x| = 1$ for all $x \in V$ and $n \in \mathbb{N}$. Then $\|z_n^*\| \ge n^2$ and $\sup_n |z_n^* x| < \infty$ for $x \in H$.

A slightly stronger property than that of condition (2) of Theorem 3 is considered in the next proposition. As shown in the proof, this property turns out to be equivalent to the presence in $X^*$ of a basic sequence equivalent to the unit vector basis of $c_0$.

Proposition 1 An infinite-dimensional Banach space $X$ has a quotient isomorphic to $\ell_1$ if and only if $X^*$ contains a normal sequence $\{y_n^*\}$ such that $\sum_n |y_n^* x| < \infty$ for all $x \in X$. 

Proof If there is a bounded linear operator \( Q \) from \( X \) onto \( \ell_1 \), its adjoint map fixes a sequence \( \{x_n^*\} \) in \( X^* \) such that the formal series \( \sum_{n=1}^{\infty} x_n^* \) is weakly unconditionally Cauchy and \( \inf_{n \in \mathbb{N}} \|x_n^*\| > 0. \) Setting \( y_n^* := \|x_n^*\|^{-1} x_n^* \) for each \( n \in \mathbb{N} \), the sequence \( \{y_n^*\} \) is as required. Conversely, if there is a normal sequence \( \{y_n^*\} \) like that of the statement, it defines a weak* Cauchy series in \( X^* \). Since the series \( \sum_{n=1}^{\infty} y_n^* \) does not converge in \( X^* \), according to [14, Chapter V, Corollary 1] the space \( X^* \) must contain a copy of \( \ell_\infty \). Consequently, \( X \) has a complemented copy of \( \ell_1 \) by [14, Chapter V, Theorem 10].

We refer to the following large class of Banach spaces, for which Problem 4 has a positive answer.

**Theorem 4 (Śliwa)** If \( X \) is an infinite-dimensional WCG Banach space, every normal sequence in \( X^* \) contains a strongly normal subsequence.

An interesting consequence of Theorem 2 is that ‘small’ Banach spaces always have a separable quotient. We present another proof, different from the one presented in [63, Theorem 3], which depends on the concept of strongly normal sequences.

**Corollary 1 (Saxon–Sanchez Ruiz)** If the density character \( d(X) \) of a Banach space \( X \) satisfies that \( \aleph_0 \leq d(X) < b \) then \( X \) has a separable quotient.

Recall that the density character of a Banach space \( X \) is the smallest cardinal of the dense subsets of \( X \). The bounding cardinal \( b \) is referred to as the minimum size for an unbounded subset of the preordered space \( (\mathbb{N}^\mathbb{N}, \preceq^*) \), where \( \alpha \preceq^* \beta \) stands for the eventual dominance preorder, defined so that \( \alpha \preceq^* \beta \) if the set \( \{n \in \mathbb{N} : \alpha(n) > \beta(n)\} \) is finite. So we have \( b := \inf\{\|F\| : F \subseteq \mathbb{N}^\mathbb{N}, \forall \alpha \in \mathbb{N}^\mathbb{N} \exists \beta \in F \text{ with } \alpha \prec^* \beta\} \). It is well known that \( b \) is a regular cardinal and \( \aleph_0 < b \leq c \). It is consistent that \( b = c > \aleph_1 \); indeed, Martin’s Axiom implies that \( b = c \).

**Proof (Proof of Corollary 1)** Assume \( X \) has a dense subset \( D \) of cardinality less than \( b \). We show that \( X^* \) has a strongly normal sequence and then we apply Theorem 3. Choose a normalized weak*-null sequence \( \{y_n^*\} \) in \( X^* \). For \( x \in D \) choose \( \alpha_x \in \mathbb{N}^\mathbb{N} \) such that for each \( n \in \mathbb{N} \) and every \( k \geq \alpha_x(n) \) one has \( |y_k^* x| < 2^{-n} \). Then \( \sum_n |y_k^* x| < \infty \) if \( \alpha_x \preceq^* \beta \). Choose \( \gamma \in \mathbb{N}^\mathbb{N} \) with \( \alpha_x \preceq^* \gamma \) for each \( x \in D \). Then the sequence \( \{y_{\gamma(n)}^*\} \) is strongly normal and Theorem 3 applies.

On the other hand, note also that every infinite-dimensional Banach space \( X \) has always an infinite-dimensional quotient \( Y \) with \( d(Y) \leq 2^{\aleph_0} \) and \( Y^* \) being weak*-separable. Indeed, let \( W \) be a countable infinite linearly independent set in \( X^* \). Then its weak*-closed linear span \( F \) is separable. Let \( Y := X/\text{span} F \). Then \( Y^* \) is isomorphic to \( F \) and \( Y \) is as desired.

Another line of research related to Problem 1 deals with those Banach spaces which contain complemented copies of concrete separable sequence...
Lemma 1 Let $X$ be a Banach space such that $X^*$ contains an infinite dimensional reflexive subspace $Y$. Then $X$ has a quotient isomorphic to $Y^*$. Consequently $X$ has a separable quotient.

Proof Let $Q : X \to Y^*$ be defined by $Qx(y) = y(x)$ for $y \in Y$ and $x \in X$. Let $j : Y \to X^*$ and $\phi_X : X \to X^{**}$ be the inclusion maps. Clearly $Q = j^* \circ \phi_X$ and $Q^* = \phi_X^* \circ j^{**}$. Since $Y$ is reflexive, $Q^*$ is an embedding map and consequently $Q$ is surjective.

Theorem 5 (Mujica) If $X$ is a Banach space that contains an isomorphic copy of $\ell_1$, then $X$ has a quotient isomorphic to $\ell_2$.

Proof If $X$ contains a copy of $\ell_1$, the dual space $X^*$ contains a copy of $L_1[0,1]$, see [14]. It is well known that the space $L_1[0,1]$ contains a copy of $\ell_2$. We apply Lemma 1.

Concerning copies of $\ell_1$, let us recall that from classic Rosenthal-Dor’s $\ell_1$-dichotomy [14, Chapter 11] one easily gets the following general result.

Theorem 6 If $X$ is a non-reflexive weakly sequentially complete Banach space, then $X$ contains an isomorphic copy of $\ell_1$.

The previous results suggest also the following

Problem 5 Describe a possibly large class of non-reflexive Banach spaces $X$ not containing an isomorphic copy of $\ell_1$ and having a separable quotient.

We may summarize this section with the following

Corollary 2 Let $X$ be an infinite-dimensional Banach space. Assume that either $X$ or $X^*$ contains an isomorphic copy of $c_0$ or either $X$ or $X^*$ contains an isomorphic copy of $\ell_1$. Then $X$ has separable quotient.

It is interesting to remark that there exists an infinite-dimensional separable Banach space $X$ such that neither $X$ nor $X^*$ contains a copy of $c_0$, $\ell_1$ or an infinite-dimensional reflexive subspace (see [33]).

We refer to [32] for several results (and many references) concerning $X$ not containing an isomorphic copy of $\ell_1$.

From now onwards, unless otherwise stated, $X$ is an infinite-dimensional Banach space over the field $\mathbb{K}$ of real or complex numbers, as well as all linear spaces we shall consider. Every measurable space $(\Omega, \Sigma)$, as well as every measure space $(\Omega, \Sigma, \mu)$, are supposed to be non trivial, i.e., there are in $\Sigma$ infinitely many pairwise disjoints sets (of finite positive measure). If either $X$ contains or does not contain an isomorphic copy of a Banach space $Z$ we shall write $X \supset Z$ or $X \not\supset Z$, respectively.
4 Weak* compactness of $B_{X^*}$ and separable quotients

In many cases the separable quotient problem is related to the weak*-compactness of the dual unit ball, as the following theorem shows.

**Theorem 7** Let $X$ be a Banach space and let $B_{X^*}$ be the dual unit ball equipped with the weak*-topology.

1. If $B_{X^*}$ is not sequentially compact, then $X$ has a separable quotient which is either isomorphic to $c_0$ or to $\ell_2$.
2. If $B_{X^*}$ is sequentially compact, then $X$ has a copy of $c_0$ if and only if it has a complemented copy of $c_0$.

**Proof** (Sketch) For the first case, if $B_{X^*}$ is not sequentially compact, according to the classic Hagler-Johnson theorem [35, Corollary 1], $X$ either has a quotient isomorphic to $c_0$ or $X$ contains a copy of $\ell_1$. The later case implies that the dual space $X^*$ of $X$ contains a copy of $L_1[0,1]$, so that $X^*$ contains a copy of $\ell_2$. Hence $X$ has a quotient isomorphic to $\ell_2$.

For the second case we refer the reader to [26, Theorem 4.1]. We provide a brief account of the argument. Let $\{x_n\}$ be a normalized basic sequence in $X$ equivalent to the unit vector basis $\{e_n\}$ of $c_0$ and let $\{x^*_n\}$ denote the sequence of coordinate functionals of $\{x_n\}$ extended to $X$ via Hahn-Banach’s theorem. If $K > 0$ is the basis constant of $\{x_n\}$ then $\|x^*_n\| \leq 2K$, so that $x^*_n \in 2KB_{X^*}$ for every $n \in \mathbb{N}$. Since $B_{X^*}$ is sequentially compact, there is a subsequence $\{z^*_n\}$ of $\{x^*_n\}$ that converges to a point $z^* \in X^*$ under the weak*-topology. Let $\{z_n\}$ be the corresponding subsequence of $\{x_n\}$, still equivalent to the unit vector basis of $c_0$, and let $F$ be the closed linear span of $\{z_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ define the linear functional $u_n : X \to \mathbb{K}$ by $u_n(x) = (z^*_n - z^*)x$, so that $\|u_n(x)\| \leq 4K \|x\|$ for each $n \in \mathbb{N}$. Since $u_n(x) \to 0$ for all $x \in X$, the linear operator $P : X \to F$ given by $Px = \sum_{n=1}^{\infty} u_n(x) z_n$ is well defined. Due to the formal series $\sum_{n=1}^{\infty} z_n$ is weakly unconditionally Cauchy, there is a constant $C > 0$ such that $\|Px\| \leq 4CK \|x\|$. Now the fact that $z^*_n y \to 0$ for each $y \in F$ means that $z^* \in F^\perp$, which implies that $Pz_j = z_j$ for each $j \in \mathbb{N}$. Thus $P$ is a bounded linear projection operator from $X$ onto $F$.

It can be easily seen that if $X^*$ contains an isomorphic copy of $\ell_1$ but $X$ does not, then $X$ has a quotient isomorphic to $c_0$. If $X^*$ has a copy of $\ell_1$, according to the first part of Theorem 7, then $X$ has a separable quotient isomorphic to $c_0$ or $\ell_2$.

On the other hand, the first statement of the previous theorem also implies that each Banach space that contains an isomorphic copy of $\ell_1(\mathbb{R})$ has a quotient isomorphic to $c_0$ or $\ell_2$. Particularly, each Banach space $X$ containing an isomorphic copy of $\ell_\infty$ enjoys this property. However, since $\ell_\infty$ is an injective Banach space and it has a separable quotient isomorphic to $\ell_2$ (as follows, for instance, from Theorem 5), one derives that $X$ has a separable quotient.
isomorphic to $\ell_2$ provided $X$ contains an isomorphic copy of $\ell_\infty$. Useful characterizations of Banach spaces containing a copy of $\ell_\infty$ can be found in the classic paper [55].

The class of Banach spaces for which $B_X$ (weak*) is sequentially compact is rich. This happens, for example if $X$ is a WCG Banach space. Of course, no WCG Banach space contains a copy of $\ell_\infty$. Another class with weak* sequentially compact dual balls is that of Asplund spaces. Note that the second statement of Theorem 7 applies in particular to each Banach space whose weak*-dual unit ball is Corson’s (a fact first observed in [49]) since, as is well-known, each Corson compact is Fréchet-Urysohn. So, one has the following corollary, where a Banach space $X$ is called weakly Landlőf determined (WLD for short) if there is a set $M \subseteq X$ with $\text{span}(M) = X$ enjoying the property that for each $x^* \in X^*$ the set $\{x \in M : x^*x \neq 0\}$ is countable.

**Corollary 3** If $X$ is a WLD Banach space, then $X$ contains a complemented copy of $c_0$ if and only if it contains a copy of $c_0$.

**Proof** If $X$ is a WLD Banach space, the dual unit ball $B_X$ (weak*) of $X$ is Corson (see [2, Proposition 1.2]), so the second statement of Theorem 7 applies.

If $K$ is an infinite Gul’ko compact space, then $C(K)$ is weakly countable determined (see [3]), hence WLD. Since $C(K)$ has plenty of copies of $c_0$, it must have many complemented copies of $c_0$. It must be pointed out that if $K$ is Corson compact then $C(K)$ need not be WLD. On the other hand, if a Banach space $X \supset c_0$ then $X^* \supset \ell_1$, so surely $X$ has $c_0$ or $\ell_2$ as a quotient (a general characterization of Banach spaces containing a copy of $c_0$ is provided in [56]). This fact can be sharpened, as the next corollary shows.

**Corollary 4** If a Banach space $X$ contains a copy of $c_0$, then $X$ has either an infinite-dimensional separable quotient isomorphic to $c_0$ or $\ell_2$, or a complemented copy of $c_0$.

**Proof** If $B_X$ (weak*) is not sequentially compact, $X$ has a separable quotient isomorphic to $c_0$ or $\ell_2$ as a consequence of the first part of Theorem 7. If $B_X$ (weak*) is sequentially compact, by the second part $X$ has a complemented copy of $c_0$.

**Corollary 5** (cf. [45] and [54]) If $K$ is an infinite compact Hausdorff space, then $C(K)$ always has a quotient isomorphic to $c_0$ or $\ell_2$. In case that $K$ is scattered, then $c_0$ embeds in $C(K)$ complementably.

**Proof** The first statement is clear. The second is due to in this case $C(K)$ is an Asplund space (see [34, Theorem 296]).

An extension of the previous corollary to all barreled spaces $C_k(X)$ with the compact-open topology has been obtained in [39].
5 Separable quotients in tensor products

We quote three classic results about the existence of copies of $c_0$, $\ell_\infty$ and $\ell_1$ in injective and projective tensor products which will be frequently used henceforth and provide a result concerning the existence of a separable quotient in $X \hat{\otimes} Y$. We complement these classic facts with other two results of our own. In the following theorem $c_{00}$ stands for the linear subspace of $c_0$ consisting of all those sequence of finite range.

**Theorem 8** (cf. [30, Theorem 2.3]) *Let $X$ be an infinite-dimensional normed space and let $Y$ be a Hausdorff locally convex space. If $Y \supset c_{00}$ then $X \hat{\otimes} Y$ contains a complemented subspace isomorphic to $c_0$.***

Particularly, if $X$ and $Y$ are infinite-dimensional Banach spaces and $X \supset c_0$ or $Y \supset c_0$, then $X \hat{\otimes} Y$ contains a complemented copy of $c_0$, (cf. [60]). On the other hand, if either $X \supset \ell_\infty$ or $Y \supset \ell_\infty$ then $X \hat{\otimes} Y \supset \ell_\infty$ and consequently $X \hat{\otimes} Y$ also has a separable quotient isomorphic to $\ell_2$. If $X \hat{\otimes} Y \supset \ell_\infty$, the converse statement also holds, as the next theorem asserts.

**Theorem 9** (cf. [18, Corollary 2]) *Let $X$ and $Y$ be Banach spaces. $X \hat{\otimes} Y \supset \ell_\infty$ if and only if $X \supset \ell_\infty$ or $Y \supset \ell_\infty$.***

This also implies that if $X \hat{\otimes} Y \supset \ell_\infty$ then $c_0$ embeds complementably in $X \hat{\otimes} Y$. Concerning projective tensor products, we have the following well-known fact.

**Theorem 10** (cf. [8, Corollary 2.6]) *Let $X$ and $Y$ be Banach spaces. If both $X \supset \ell_1$ and $Y \supset \ell_1$, then $X \hat{\otimes} Y$ has a complemented subspace isomorphic to $\ell_1$.***

Next we observe that if $X \hat{\otimes} Y$ is not a quotient of $X \hat{\otimes} Y$, then $X \hat{\otimes} Y$ has a separable quotient.

**Theorem 11** *Let $J : X \otimes Y \rightarrow X \hat{\otimes} Y$ be the identity map and consider the continuous linear extension $\tilde{J} : X \hat{\otimes} Y \rightarrow X \hat{\otimes} Y$. If $\tilde{J}$ is not a quotient map, then $X \hat{\otimes} Y$ has a separable quotient.*

**Proof** Observe that $X \otimes Y \subset \text{Im} \tilde{J} \subset X \hat{\otimes} Y$. Two cases are in order.

Assume first that $X \hat{\otimes} Y$ is a barrelled space. In this case, since $X \otimes Y$ is dense in $\text{Im} \tilde{J}$, we have that the range space $\text{Im} \tilde{J}$ is a barrelled subspace of $X \hat{\otimes} Y$. Given that the graph of $\tilde{J}$ is closed in $(X \hat{\otimes} Y) \times (X \hat{\otimes} Y)$ and $\text{Im} \tilde{J}$ is barrelled, it follows from [68, Theorem 19] that $\text{Im} \tilde{J}$ is a closed subspace of $X \hat{\otimes} Y$. Of course, this means that $\text{Im} \tilde{J} = X \hat{\otimes} Y$. Hence, the open map theorem shows that $\tilde{J}$ is an open map from $X \hat{\otimes} Y$ onto $X \hat{\otimes} Y$, so that $X \hat{\otimes} Y$ is a quotient of $X \hat{\otimes} Y$.

Assume now that $X \hat{\otimes} Y$ is not barrelled. In this case $X \hat{\otimes} Y$ is a non barrelled dense subspace of the Banach space $X \hat{\otimes} Y$, so we may apply Theorem 2 to get that $X \hat{\otimes} Y$ has a separable quotient.
Recall that the dual of $X \otimes_\pi Y$ coincides with the space of bounded linear operators from $X$ into $Y^*$, whereas the dual of $X \otimes_\varepsilon T$ may be identified with the subspace of those operators which are integral, see [58, Section 3.5].

**Proposition 2** Let $X$ and $Y$ be Banach spaces. If $X$ has the bounded approximation property and there is a bounded linear operator $T: X \to Y^*$ which is not integral, then $X \otimes_\varepsilon Y$ has a separable quotient.

**Proof** Since does exist a bounded not integral linear operator between $X$ and $Y^*$, the $\pi$-topology and $\varepsilon$-topology does not coincide on $X \otimes Y$, see [58]. Assume $X \otimes_\varepsilon Y$ is barrelled. Since $X$ has the bounded approximation property, [6, Theorem] applies to get that $X \otimes_\varepsilon Y = X \otimes_\pi Y$, which contradicts the assumption that $(X \otimes_\varepsilon Y)^* \neq (X \otimes_\pi Y)^*$. Thus $X \otimes_\varepsilon Y$ must be a non barrelled dense linear subspace of $X \otimes_\varepsilon Y$, which according to Theorem 2 ensures that $X \otimes_\varepsilon Y$ has a separable quotient.

For the next theorem, recall that a Banach space $X$ is called weakly compactly determined (WCD for short) if $X$ (weak) is $K$-countably determined, i.e., a Lindel"of $\Sigma$-space.

**Theorem 12** Let $X$ and $Y$ be WCD Banach spaces. If $X \otimes_\varepsilon Y \supset c_0$, then $c_0$ embeds complementably in $X \otimes_\varepsilon Y$.

**Proof** Since both $X$ and $Y$ are WCD Banach spaces, their dual unit balls $B_{X^*}$ (weak*) and $B_{X^*}$ (weak*) are Gul’ko compact. Given that the countable product of Gul’ko compact spaces is Gul’ko compact, the product space $K := B_{X^*}$ (weak*) $\times B_{X^*}$ (weak*) is Gul’ko compact. Consequently $C(K)$ is a WCD Banach space, which implies in turn that its weak*-dual unit ball $B_{C(K)^*}$ is Gul’ko compact. Particularly $B_{C(K)^*}$ (weak*) is angelic and consequently sequentially compact. Let $Z$ stand for the isometric copy of $X \otimes_\varepsilon Y$ in $C(K)$ and $P$ for the isomorphic copy of $c_0$ in $Z$. From the proof of the second statement of Theorem 7 it follows that $C(K)$ has a complemented copy $Q$ of $c_0$ contained in $P$. This implies that $Z$, hence $X \otimes_\varepsilon Y$, contains a complemented copy $Q$ of $c_0$.

### 6 Separable quotients in spaces of vector-valued functions

If $(\Omega, \Sigma, \mu)$ is a non trivial arbitrary measure space, we denote by $L_p(\mu, X)$, $1 \leq p \leq \infty$, the Banach space of all $X$-valued $p$-Bochner $\mu$-integrable ($\mu$-essentially bounded when $p = \infty$) classes of functions equipped with its usual norm. If $K$ is an infinite compact Hausdorff space, then $C(K, X)$ stands for the Banach space of all continuous functions $f: K \to X$ equipped with the supremum norm. By $B(\Sigma, X)$ we represent the Banach space of those bounded functions $f: \Omega \to X$ that are the uniform limit of a sequence of $\Sigma$-simple and $X$-valued functions, equipped with the supremum norm. The space of all $X$-valued bounded functions $f: \Omega \to X$ endowed with the supremum norm is written as $\ell_\infty(\Omega, X)$. Clearly $\ell_\infty(X) = \ell_\infty(\mathbb{N}, X)$. By $\ell_\infty(\Sigma)$ we denote the
completion of the space $\ell_\infty^\Sigma$ of scalarly-valued $\Sigma$-simple functions, endowed with the supremum norm.

On the other hand, if $(\Omega, \Sigma, \mu)$ is a (complete) finite measure space we represent by $P_1(\mu, X)$ the normed space consisting of all those [classes of] strongly $\mu$-measurable $X$-valued Pettis integrable functions $f$ defined on $\Omega$ provided with the semivariation norm

$$
\|f\|_{P_1(\mu, X)} = \sup \left\{ \int \Omega |x^* f(\omega)| \, d\mu(\omega) : x^* \in X^*, \|x^*\| \leq 1 \right\}.
$$

As is well known, in general $P_1(\mu, X)$ is not a Banach space if $X$ is infinite-dimensional, but it is always a barrelled space (see [19, Theorem 2] and [29, Remark 10.5.5]).

Our first result collects together a number of statements concerning Banach spaces of vector-valued functions related to the existence of separable quotients, most of them easily derived from well known facts relative to the presence of complemented copies of $c_0$ and $\ell_p$ for $1 \leq p \leq \infty$. We denote by $ca^+(\Sigma)$ the set of positive and finite measures on $\Sigma$.

**Theorem 13** The following statements on spaces of vector-valued functions hold.

1. $C(K, X)$ always has a complemented copy of $c_0$.
2. $C(K, X)$ has a quotient isomorphic to $\ell_1$ if and only if $X$ has $\ell_1$ as a quotient.
3. $L_p(\mu, X)$, with $1 \leq p < \infty$, has a complemented copy of $\ell_p$. In particular, the vector sequence space $\ell_p(X)$ has a complemented copy of $\ell_p$.
4. $L_p(\mu, X)$, with $1 < p < \infty$, has a quotient isomorphic to $\ell_1$ if and only if $X$ has $\ell_1$ as a quotient. Particularly $\ell_p(X)$ has a quotient isomorphic to $\ell_1$ if and only if the same happens to $X$.
5. $L_\infty(\mu, X)$ has a quotient isomorphic to $\ell_2$. Hence, so does $\ell_\infty(X)$.
6. If $\mu$ is purely atomic and $1 \leq p < \infty$, then $L_p(\mu, X)$ has a complemented copy of $c_0$ if and only if $X$ has a complemented copy of $c_0$. Particularly, the space $\ell_p(X)$ has a complemented copy of $c_0$ if and only if so does $X$.
7. If $\mu$ is not purely atomic and $1 \leq p < \infty$, then $L_p(\mu, X)$ has a complemented copy of $c_0$ if $X \supset c_0$.
8. If $\mu \in ca^+(\Sigma)$ is purely atomic and $1 < p < \infty$, then $L_p(\mu, X)$ has a quotient isomorphic to $c_0$ if and only if $X$ contains a quotient isomorphic to $c_0$.
9. If $\mu \in ca^+(\Sigma)$ is not purely atomic and $1 < p < \infty$, then $L_p(\mu, X)$ has a quotient isomorphic to $c_0$ if and only if $X$ contains a quotient isomorphic to $c_0$ or $X \supset \ell_1$.
10. If $\mu$ is $\sigma$-finite, then $L_\infty(\mu, X)$ has a quotient isomorphic to $\ell_1$ if and only if $\ell_\infty(X)$ has $\ell_1$ as a quotient.
11. $B(\Sigma, X)$ has a complemented copy of $c_0$ and a quotient isomorphic to $\ell_2$.
12. $\ell_\infty(\Omega, X)$ has a quotient isomorphic to $\ell_2$. 
13. If the cardinality of $\Omega$ is less than the first real-valued measurable cardinal, then $\ell_\infty(\Omega, X)$ has a complemented copy of $c_0$ if and only if $X$ enjoys the same property. Particularly, $\ell_\infty(X)$ contains a complemented copy of $c_0$ if and only if $X$ enjoys the same property.

14. $c_0(X)$ has a complemented copy of $c_0$.

15. $\ell_\infty(X)^*$ has a quotient isomorphic to $\ell_1$.

Proof Let us proceed with the proofs of the statements.

1. This well-know fact can be found in [9, Theorem] and [30, Corollary 2.5] (or in [10, Theorem 3.2.1]).
2. This is because $C(K, X)$ contains a complemented copy of $\ell_1$ if and only if $X$ contains a complemented copy of $\ell_1$ (see [59] or [10, Theorem 3.1.4]).
3. If $1 \leq p < \infty$, each $L_p(\mu, X)$ space contains a norm one complemented isometric copy of $\ell_p$ (see [10, Proposition 1.4.1]). For the second affirmation note that if $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite purely atomic measure space, then $\ell_p(X) = L_p(\mu, X)$ isometrically.
4. If $1 < p < \infty$ then $L_p(\mu, X)$ contains complemented copy of $\ell_1$ if and only if $X$ has the same property. This fact, discovered by F. Bombal in [7], can also be seen in [10, Theorem 4.3.1].
5. If $\mu$ is purely atomic and $1 \leq p < \infty$, according to [21], the only fact that $X \supset c_0$ implies that $L_p(\mu, X)$ contains a complemented copy of $c_0$.
6. If $\mu$ is purely atomic, then $L_p(\mu, X)$ contains a complemented copy of $c_0$ if and only if $X$ has the same property. This fact, discovered by F. Bombal in [7], can also be seen in [10, Theorem 4.3.1].
7. If $\mu$ is not purely atomic and $1 \leq p < \infty$, according to [21], the only fact that $X \supset c_0$ implies that $L_p(\mu, X)$ contains a complemented copy of $c_0$.
8. If $(\Omega, \Sigma, \mu)$ is a purely atomic finite measure space and $1 < p < \infty$, the statement corresponds to the first statement of [13, Theorem 1.1].
9. If $(\Omega, \Sigma, \mu)$ is a not purely atomic finite measure space and $1 < p < \infty$, the statement corresponds to the second statement of [13, Theorem 1.1].
10. If $(\Omega, \Sigma, \mu)$ is a $\sigma$-finite measure space, the existence of a complemented copy of $\ell_1$ in $L_\infty(\mu, X)$ is related to the local theory of Banach spaces, a fact discovered by S. Díaz in [12]. The statement, in the way as it has been formulated above, can be found in [10, Theorem 5.2.3].
11. $B(\Sigma, X)$ coincides with the completion of the normed space $\ell_0^\infty(\Sigma, X)$ of all $\Sigma$-simple and $X$-valued functions provided with the supremum norm. Since $\ell_0^\infty(\Sigma, X) = \ell_0^\infty(\Sigma) \otimes_X X$ and $X$ is infinite-dimensional, then $\ell_0^\infty(\Sigma, X)$ is not barrelled by virtue of classic Freniche’s theorem (see [30, Corollary 1.5]). Given that $\ell_0^\infty(\Sigma, X)$ is a non barrelled dense subspace of $B(\Sigma, X)$, Theorem 2 guarantees that $B(\Sigma, X)$ has in fact a separable quotient. However, we can be more precise. Since $B(\Sigma, X) = \ell_0^\infty(\Sigma) \otimes_X X$ and $\ell_0^\infty(\Sigma) \supset c_0$ due to the non triviality of the $\sigma$-algebra $\Sigma$, Theorem 8 implies that $B(\Sigma, X)$ contains a complemented copy of $c_0$. On the other hand, since $\ell_\infty$ is isometrically embedded in $B(\Sigma, X)$, it turns out that $\ell_2$ is a quotient of $B(\Sigma, X)$.
12. Clearly $\ell_\infty(\Omega, X) \supset \ell_\infty(\Omega) \supset \ell_\infty$ since the set $\Omega$ is infinite.
13. This property can be found in [46].
14. Just note that \( c_0(X) = c_0 \otimes \varepsilon X \), so we may apply Theorem 8.

15. It suffices to note that \( \ell_1(X^*) \) is linearly isometric to a complemented subspace of \( \ell_\infty(X)^* \) (see [10, Section 5.1]).

**Remark 1** \( L_p(\mu, X) \) if \( 1 \leq p < \infty \), as well as \( C(K, X) \), need not contain a copy of \( \ell_\infty \). By [47, Theorem], one has that \( L_p(\mu, X) \supset \ell_\infty \) if and only if \( X \supset \ell_\infty \), whereas \( C(K, X) \supset \ell_\infty \) if and only if \( C(K) \supset \ell_\infty \) or \( X \supset \ell_\infty \), as shown in [18, Corollary 3].

**Remark 2** Complemented copies of \( c_0 \) in \( L_\infty(\mu, X) \). If \( (\Omega, \Sigma, \mu) \) is a \( \sigma \)-finite measure, according to [11, Theorem 1] a necessary condition for the space \( L_\infty(\mu, X) \) to contain a complemented copy of \( c_0 \) is that \( X \supset c_0 \). The same happens with the space \( \ell_\infty(\Omega, X) \) (see [26, Theorem 2.1 and Corollary 2.3]).

**Theorem 14** The following statements on the space \( \hat{P_1}(\mu, X) \) hold.

1. If the finite measure space \( (\Omega, \Sigma, \mu) \) is not purely atomic, the Banach space \( \hat{P_1}(\mu, X) \) has a separable quotient.

2. If the range of the positive finite measure \( \mu \) is infinite and \( X \supset c_0 \) then \( \hat{P_1}(\mu, X) \) has a complemented copy of \( c_0 \).

**Proof** Observe that \( L_1(\mu) \otimes_\pi X = L_1(\mu, X) \) and \( \hat{P_1}(\mu, X) = L_1(\mu) \otimes_\varepsilon X \) isometrically. On the other hand, from the algebraic viewpoint \( L_1(\mu, X) \) is a linear subspace of \( P_1(\mu, X) \), which is dense under the norm of \( P_1(\mu, X) \). If

\[
J : L_1(\mu) \otimes_\pi X \rightarrow L_1(\mu) \otimes_\varepsilon X
\]

is the identity map, \( R \) a linear isometry from \( L_1(\mu, X) \) onto \( L_1(\mu) \otimes_\pi X \) and \( S \) a linear isometry from \( L_1(\mu) \otimes_\varepsilon X \) onto \( P_1(\mu, X) \), the mapping

\[
S \circ \tilde{J} \circ R : L_1(\mu, X) \rightarrow P_1(\mu, X),
\]

where \( \tilde{J} \) denotes the (unique) continuous linear extension of \( J \) to \( L_1(\mu) \otimes_\pi X \), coincides with the natural inclusion map \( T \) of \( L_1(\mu, X) \) into \( P_1(\mu, X) \) over the dense subspace of \( L_1(\mu, X) \) consisting of the \( X \)-valued (classes of) \( \mu \)-simple functions, which implies that \( S \circ \tilde{J} \circ R = T \). Since \( X \) is infinite-dimensional and \( \mu \) is not purely atomic, the space \( P_1(\mu, X) \) is not complete [67]. So necessarily we have that \( \text{Im}T \neq P_1(\mu, X) \). This implies in particular that \( \text{Im}\tilde{J} \neq L_1(\mu) \otimes_\pi X \). According to Theorem 11, this means that \( P_1(\mu, X) = L_1(\mu) \otimes_\varepsilon X \) has a separable quotient.

The proof of the second statement can be found in [31, Corollary 2].
7 Separable quotients in spaces of linear operators

If $Y$ is also a Banach space, let us denote by $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators $T : X \to Y$ equipped with the operator norm $\|T\|$. By $\mathcal{K}(X,Y)$ we represent the closed linear subspace of $\mathcal{L}(X,Y)$ consisting of all those compact operators. We design by $\mathcal{L}_w^w(X^*,Y)$ the closed linear subspace of $\mathcal{L}(X^*,Y)$ formed by all weak*-weakly continuous operators and by $\mathcal{K}_w^w(X^*,Y)$ the closed linear subspace of $\mathcal{K}(X^*,Y)$ consisting of all weak*-weakly continuous operators. The closed subspace of $\mathcal{L}(X,Y)$ consisting of weakly compact linear operators is designed by $\mathcal{W}(X,Y)$. It is worthwhile to mention that $\mathcal{L}_w^w(X^*,Y) = \mathcal{L}_w^w(Y^*,X)$ isometrically, as well as $\mathcal{K}_w^w(X^*,Y) = \mathcal{K}_w^w(Y^*,X)$, by means of the linear mapping $T \mapsto T^*$. The Banach space of nuclear operators $T : X \to Y$ equipped with the so-called nuclear norm $\|T\|_N$ is denoted by $\mathcal{N}(X,Y)$. Let us recall that $\|T\| \leq \|T\|_N$.

Classic references for this section are the monographs [42] and [44].

The first statement of Theorem 15 answers a question of Prof. T. Dobrowolski posed during the 31st Summer Conference on Topology and its Applications at Leicester (2016).

**Theorem 15** The following conditions on $\mathcal{L}(X,Y)$ hold.

1. If $Y \neq \{0\}$, then $\mathcal{L}(X,Y)$ always has a separable quotient.
2. If $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{L}(X,Y)$ has a quotient isomorphic to $\ell_2$.
3. If $X^* \supset \ell_q$ and $Y \supset \ell_p$, with $1 \leq p < \infty$ and $1/p + 1/q = 1$, then $\mathcal{L}(X,Y)$ has a quotient isomorphic to $\ell_2$.

**Proof** Let us prove each of these statements.

1. First observe that $X^*$ is complemented in $\mathcal{L}(X,Y)$. Indeed, choose $y_0 \in Y$ with $\|y_0\| = 1$ and apply the Hahn-Banach theorem to get $y_0^* \in X^*$ such that $\|y_0^*\| = 1$ and $y_0^*y_0 = 1$. The map $\varphi : X^* \to \mathcal{L}(X,Y)$ defined by $(\varphi x^*)(x) = x^*x$, $y_0$ for every $x \in X$ is a linear isometry into $\mathcal{L}(X,Y)$ (see [44, 39.1.(2)]), and the operator $P : \mathcal{L}(X,Y) \to \mathcal{L}(X,Y)$ given by $PT = \varphi(y_0^* \circ T)$ is a norm one linear projection operator from $\mathcal{L}(X,Y)$ onto $\text{Im} \varphi$. Hence $X^*$ is linearly isometric to a norm one complemented linear subspace of $\mathcal{L}(X,Y)$. Since $X^*$ is a dual Banach space, it has a separable quotient by Theorem 1. Hence the operator space $\mathcal{L}(X,Y)$ has a separable quotient.

2. Since $X^* \otimes c_0 Y$ is isometrically embedded in $\mathcal{L}(X,Y)$ and both $X^*$ and $Y$ are isometrically embedded in $X^* \otimes c_0 Y$, if either $X^* \supset c_0$ or $Y \supset c_0$, then $\mathcal{L}(X,Y) \supset c_0$. In this case, according to [25, Corollary 1], $\mathcal{L}(X,Y)$ contains an isomorphic copy of $\ell_\infty$. This ensures that $\mathcal{L}(X,Y)$ has a separable quotient isomorphic to $\ell_2$.

3. If $\{e_n : n \in \mathbb{N}\}$ is the unit vector basis of $\ell_p$, define $T_n : \ell_p \to \ell_p$ by $T_n \xi = \xi_n e_n$ for each $n \in \mathbb{N}$. Since

$$\left\| \sum_{i=1}^{n} a_i T_i \right\| = \sup_{\|\xi\|_p \leq 1} \left( \sum_{i=1}^{n} |a_i| |\xi_i|^p \right)^{1/p} \leq \sup_{1 \leq i \leq n} |a_i|$$


for any scalars $a_1, \ldots, a_n$, we can see that $\{T_n : n \in \mathbb{N}\}$ is a basic sequence in $\mathcal{K}(\ell_p, \ell_p)$ equivalent to the unit vector basis of $c_0$. Since it holds in general that $E^* \hat{\otimes}_z F = \mathcal{K}(E, F)$ for Banach spaces $E$ and $F$ whenever $E^*$ has the approximation property, if $1/p + 1/q = 1$ one has that

$$\ell_q \hat{\otimes}_z \ell_p = \ell_p \hat{\otimes}_z \ell_p = \mathcal{K}(\ell_p, \ell_p)$$

isometrically. So we have $\ell_q \hat{\otimes}_z \ell_p \supseteq c_0$. As in addition $\ell_q \hat{\otimes}_z \ell_p$ is isometrically embedded in $X^* \hat{\otimes}_z Y$, which in turn is also isometrically embedded in $\mathcal{L}(X, Y)$, we conclude that $\mathcal{L}(X, Y) \supseteq c_0$. So, we use again [25, Corollary 1] to conclude that $\mathcal{L}(X, Y) \supseteq \ell_\infty$. Thus $\mathcal{L}(X, Y)$ has a quotient isomorphic to $\ell_2$.

The Banach space $\mathcal{L}(X, Y)$ need not contain a copy of $\ell_\infty$ in order to have a separable quotient, as the following example shows.

Let $1 < p, q < \infty$ with conjugated indices $p', q'$, i.e., $1/p + 1/p' = 1/q + 1/q' = 1$.

**Example 1** If $p > q'$ then $\mathcal{L}(\ell_p, \ell_{q'})$ does not contain an isomorphic copy of $c_0$.

**Proof** Since it holds in general that $\mathcal{L}(X, Y^*) = (X \hat{\otimes}_\pi Y)^*$ isometrically for arbitrary Banach spaces $X$ and $Y$ (see for instance [58, Section 2.2]), the fact that $\ell_{q'}^* = \ell_q$ assures that $\mathcal{L}(\ell_p, \ell_{q'}) = (\ell_p \hat{\otimes}_z \ell_q)^*$ isometrically. Now let us assume by contradiction that $\mathcal{L}(\ell_p, \ell_{q'}) \supseteq c_0$, which implies that $\ell_p \hat{\otimes}_z \ell_q$ contains a complemented copy of $\ell_1$ (see [14, Chapter 5, Theorem 10]). Since $p > q'$, according to [58, Corollary 4.24] or [15, Chapter 8, Corollary 5], the space $\ell_p \hat{\otimes}_z \ell_q$ is reflexive, which contradicts the fact that it has a quotient isomorphic to the non reflexive space $\ell_1$. So we must conclude that $\mathcal{L}(\ell_p, \ell_{q'}) \not\supseteq c_0$.

On the other hand, since $\mathcal{L}(\ell_p, \ell_{q'})$ is a dual Banach space, Theorem 1 shows that $\mathcal{L}(\ell_p, \ell_{q'})$ has a separable quotient. Alternatively, we can also apply the first statement of Theorem 15.

**Proposition 3** If $X^*$ has the approximation property, then the Banach space $\mathcal{N}(X, Y)$ of nuclear operators has a separable quotient.

**Proof** Since $X^*$ enjoys the approximation property, it follows that $\mathcal{N}(X, Y) = X^* \hat{\otimes}_\pi Y^*$ isometrically. Hence $X^*$ is linearly isometric to a complemented subspace of $\mathcal{N}(X, Y)$. Since $X^*$, as a dual Banach space, has a separable quotient, the transitivity of the quotient map yields that $\mathcal{N}(X, Y)$ has a separable quotient.

**Theorem 16** The following statements hold.

1. If $X \supseteq c_0$ and $Y \supseteq c_0$, then $\mathcal{L}_{w^*}(X^*, Y)$ has a quotient isomorphic to $\ell_2$.
2. If $X$ has a separable quotient isomorphic to $\ell_1$, then $\mathcal{L}_{w^*}(X^*, Y)$ enjoys the same property.
3. If $(\Omega, \Sigma, \mu)$ is an arbitrary measure space and $Y \neq \{0\}$, then the space $\mathcal{L}_{w^*}(L_\infty(\mu), Y)$ has a quotient isomorphic to $\ell_1$. 

4. If \( X^* \ni c_0 \) or \( Y \ni \ell_\infty \), then \( \mathcal{K}(X,Y) \) has a quotient isomorphic to \( \ell_2 \).
5. If either \( X^* \ni c_0 \) or \( Y \ni \ell_\infty \), then \( \mathcal{K}(X,Y) \) contains a complemented copy of \( c_0 \).
6. If \( X \ni \ell_\infty \) or \( Y \ni \ell_\infty \), then \( \mathcal{K}_{w*}(X^*,Y) \) has a quotient isomorphic to \( \ell_2 \).
7. The space \( W(X,Y) \) always has a separable quotient.
8. If \( X \ni c_0 \) and \( Y \ni c_0 \), then \( W(X,Y) \) contains a complemented copy of \( c_0 \).

**Proof** In many cases it suffices to show that the corresponding Banach space contains an isomorphic copy of \( \ell_\infty \).

1. By [27, Theorem 1.5] if \( X \ni c_0 \) and \( Y \ni c_0 \) then \( L_{w*}(X^*,Y) \ni \ell_\infty \).
2. Choose \( y_0 \in Y \) with \( \|y_0\| = 1 \) and select \( y_0^* \in Y^* \) such that \( \|y_0^*\| = 1 \) and \( y_0^*y_0 = 1 \). The map \( \psi : X \to L_{w*}(X^*,Y) \) given by \( \psi(x)(x^*) = x^*x \cdot y_0 \), for \( x^* \in X^* \), is well-defined and if \( x_0^* \to x^* \) under the weak*-topology of \( X^* \) then \( \psi(x)(x_0^*) \to \psi(x)(x^*) \) weakly in \( Y \), so that \( \psi \) embeds \( X \) isometrically in \( L_{w*}(X^*,Y) \). On the other hand, the operator \( Q : L_{w*}(X^*,Y) \to L_{w*}(X^*,Y) \) given by \( QT = \psi(y_0^* \circ T) \), which is also well-defined since \( y_0^* \circ T \in X \) whenever \( T \) is weak*-weakly continuous, is a bounded linear projection operator from \( L_{w*}(X^*,Y) \) onto \( \text{Im} \psi \). Since we are assuming that \( \ell_1 \) is a quotient of \( X \), it follows that \( \ell_1 \) is also isomorphic to a quotient of \( L_{w*}(X^*,Y) \).
3. This statement is consequence of the previous one, since \( \ell_1 \) embeds complementably in \( L_1(\mu) \).
4. According to [43, if \( X^* \ni c_0 \) or \( Y \ni \ell_\infty \), then \( \mathcal{K}(X,Y) \ni \ell_\infty \).
5. This property has been shown in [57, Corollary 1].
6. The map \( \psi : X \to L_{w*}(X^*,Y) \) defined above by \( \psi(x)(x^*) = x^*x \cdot y_0 \), for every \( x^* \in X^* \), yields a finite-rank (hence compact) operator \( \psi(x) \), so that \( \text{Im} \psi \ni \mathcal{K}_{w*}(X^*,Y) \). On the other hand, if \( x_0 \in X \) with \( \|x_0\| = 1 \) and \( x_0^* \in X^* \) verifies that \( \|x_0^*\| = 1 \) and \( x_0^*x_0 = 1 \), the map \( \phi : Y \to \mathcal{K}_{w*}(X^*,Y) \) given by \( \phi(y)(x) = x_0^*x \cdot y \), for every \( x \in X \), is a linear isometry from \( Y \) into \( \mathcal{K}_{w*}(X^*,Y) \). Hence \( X \) and \( Y \) are isometrically embedded in \( \mathcal{K}_{w*}(X^*,Y) \).
7. Just note that \( W(X,Y) = L_{w*}(X^{**},Y) \) isometrically. Since \( X^* \) is complementably embedded in \( L_{w*}(X^{**},Y) \), the conclusion follows from Theorem 1.
8. According to [28, Theorem 2.5], under those conditions the space \( W(X,Y) \) contains a complemented copy of \( c_0 \).

**Remark 3** If neither \( X \) nor \( Y \) contains a copy of \( c_0 \), then \( L_{w*}(X^*,Y) \) cannot contain a complemented copy of \( c_0 \), as observed in [22].

The following result sharpens the first statement of Theorem 14.

**Corollary 6** If \( (\Omega, \Sigma, \mu) \) is a finite measure space, \( P_1(\mu,X) \) has a quotient isomorphic to \( \ell_1 \).

**Proof** This follows from the second statement of the previous theorem together with the fact that \( P_1(\mu,X) = L_{w*}(L_\infty(\mu),X) \) (see [15, Chapter 8, Theorem 5]).
Remark 4 The space $P_1 (\mu, X)$ need no contain a copy of $\ell_\infty$. It can be easily shown that $P_1 (\mu, X)$ embeds isometrically in the space $K_{w^*} \ (ca (\Sigma)^*, X)$, where $ca (\Sigma)$ denotes the Banach space of scalarly-valued countably additive measures equipped with the variation norm. Since $ca (\Sigma) \not\subset \ell_\infty$, it follows from [18, Theorem] that $P_1 (\mu, X) \supset \ell_\infty$ if and only if $X \supset \ell_\infty$.

8 Separable quotients in spaces of vector measures

In this section we denote by $ba (\Sigma, X)$ the Banach space of all $X$-valued bounded finitely additive measures $F : \Sigma \to X$ provided with the semivariation norm $\|F\|$. The closed linear subspace of $ba (\Sigma, X)$ consisting of those countably additive measures is represented by $ca (\Sigma, X)$, while $cca (\Sigma, X)$ stands for the (closed) linear subspace of $ca (\Sigma, X)$ of all measures with relatively compact range. It can be easily shown that $ca (\Sigma, X) = L_{w^*} (ca (\Sigma)^*, X)$ isometrically. We also design by $bvca (\Sigma, X)$ the Banach space of all $X$-valued countably additive measures $F : \Sigma \to X$ of bounded variation equipped with the variation norm $|F|$. Finally, following [58, page 107], we denote by $M_1 (\Sigma, X)$ the closed linear subspace of $bvca (\Sigma, X)$ consisting of all those $F \in bvca (\Sigma, X)$ that have the so-called Radon-Nikodým property, i.e., such that for each $\lambda \in ca^+ (\Sigma)$ with $F \ll \lambda$ there exists $f \in L_1 (\lambda, X)$ with $F (E) = \int_E f \, d\lambda$ for every $E \in \Sigma$. For this section, our main references are [15] and [58].

Theorem 17 The following statements hold. In the first case $X$ need not be infinite-dimensional.

1. If $X \neq \{0\}$, then $ba (\Sigma, X)$ always has a separable quotient.
2. If $X \supset c_0$, then $ba (\Sigma, X)$ has a quotient isomorphic to $\ell_2$.
3. If $X \supset c_0$ but $X \nsubseteq \ell_\infty$, then $ba (\Sigma, X)$ has a complemented copy of $c_0$.
4. If $\Sigma$ admits no atomless probability measure, then $ca (\Sigma, X)$ has a quotient isomorphic to $\ell_1$.
5. If $X \supset c_0$ and $\Sigma$ admits a nonzero atomless $\lambda \in ca^+ (\Sigma)$, then $ca (\Sigma, X)$ has a quotient isomorphic to $\ell_2$.
6. If there exists some $F \in cca (\Sigma, X)$ of unbounded variation, then $cca (\Sigma, X)$ has a separable quotient.
7. If $X \supset c_0$, then $cca (\Sigma, X)$ contains a complemented copy of $c_0$.
8. If $X \supset \ell_1$, then $M_1 (\Sigma, X)$ has a quotient isomorphic to $\ell_1$.

Proof In cases 2 and 3 it suffices to show that the corresponding Banach space contains an isomorphic copy of $\ell_\infty$.

1. This happens because $ba (\Sigma, X) = \mathcal{L} (\ell_\infty (\Sigma), X)$ isometrically. Since $\ell_\infty (\Sigma)$ is infinite-dimensional by virtue of the non triviality of the $\sigma$-algebra $\Sigma$, the statement follows from the first statement of Theorem 15.
2. By point 2 of Theorem 15, if $X \supset c_0$ then $\mathcal{L} (\ell_\infty (\Sigma), X)$ has a quotient isomorphic to $\ell_2$. The statement follows from the fact that $ba (\Sigma, X) = \mathcal{L} (\ell_\infty (\Sigma), X)$. 

3. \( ba (\Sigma, X) \) has a complemented copy of \( c_0 \) by virtue of [28, Corollary 3.2].
4. If the non trivial \( \sigma \)-algebra \( \Sigma \) admits no atomless probability measure, it can be shown that \( ca (\Sigma, X) \) is linearly isometric to \( \ell_1 (\Gamma, X) \) for some infinite set \( \Gamma \). Since \( \ell_1 (\Gamma, X) = L_1 (\mu, X) \), where \( \mu \) is the counting measure on \( 2\Gamma \), the conclusion follows from the third statement of Theorem 13.
5. Since \( ca (\Sigma) \otimes_\pi X = cca (\Sigma, X) \) isometrically, if \( X \supset c_0 \) then \( cca (\Sigma, X) \supset c_0 \) and hence \( ca (\Sigma, X) \supset c_0 \). If \( \Sigma \) admits a nonzero atomless \( \lambda \in ca^+ (\Sigma) \), then one has \( ca (\Sigma, X) \supset \ell_\infty \) by virtue of [17, Theorem 1].
6. Observe that \( cca (\Sigma, X) = ca (\Sigma) \otimes_\pi X \) and \( M_1 (\Sigma, X) = ca (\Sigma) \otimes_\pi X \) isometrically (see [58, Theorem 5.22]) but, at the same time, from the algebraic point of view, \( M_1 (\Sigma, X) \) is a linear subspace of \( cca (\Sigma, X) \) since every Bochner indefinite integral has a relatively compact range, [15, Chapter II, Corollary 9 (c)]. If there exists some \( F \in cca (\Sigma, X) \) of unbounded variation, then \( M_1 (\Sigma, X) \neq cca (\Sigma, X) \), so the statement follows from Theorem 11.
7. Since \( X \supset c_0 \) and \( ca (\Sigma) \) is infinite-dimensional, then \( X \otimes_\pi ca (\Sigma) \) contains a complemented copy of \( c_0 \) by [30, Theorem 2.3].
8. Since \( M_1 (\Sigma, X) = ca (\Sigma) \otimes_\pi X \) and \( ca (\Sigma) \supset \ell_1 \), if \( X \supset \ell_1 \) then \( M_1 (\Sigma, X) \) has a quotient isomorphic to \( \ell_1 \) as follows from Theorem 10.

**Remark 5** If \( \omega \in \Omega \) and \( E (\Sigma, X) \) is either \( ba (\Sigma, X) \), \( ca (\Sigma, X) \) or \( bvca (\Sigma, X) \), the map \( P_{\omega} : E (\Sigma, X) \rightarrow E (\Sigma, X) \) defined by \( P_{\omega} (F) = F (\omega) \delta_\omega \) is a bounded linear projection operator onto the copy \( \{ x \delta_\omega : x \in X \} \) of \( X \) in \( E (\Sigma, X) \). Hence, if \( X \) has a separable quotient isomorphic to \( Z \), then \( E (\Sigma, X) \) also has a separable quotient isomorphic to \( Z \).

**Remark 6** The space \( P_1 (\mu, X) \) need not contain a copy of \( \ell_\infty \). Since \( cca (\Sigma, X) = K_{ca^+ (\Sigma)^0, X} \), according to [18, Theorem or Corollary 4], \( cca (\Sigma, X) \supset \ell_\infty \) if and only if \( X \supset \ell_\infty \).

**Remark 7** Concerning the space \( M_1 (\Sigma, X) \), it is worthwhile to mention that it follows from [24, Theorem] that \( M_1 (\Sigma, X) \supset \ell_\infty \) if and only if \( X \supset \ell_\infty \).

**References**